Complement 3C

Laser light *vs.* incoherent light: energy density and number of photons per mode

The difference between laser light and the light emitted by an incoherent source can only be fully appreciated with reference to certain notions of energetic photometry to be spelt out in the first part of this complement. It will be shown in Section 3C.2 how the laws of photometry, presented in Section 3C.1, drastically limit the energy density that can be obtained from a conventional incoherent source (e.g., a heated filament, or a discharge lamp) in comparison with a laser source (Section 3C.3). Far from being merely circumstantial, the laws for classical sources given in Section 3C.1 are of a fundamental kind, that can be deduced from the basic principles of thermodynamics. Another way to relate these properties of light to the fundamental principles of physics is to examine them in the context provided by the statistical physics of photons, as will be discussed in Sections 3C.4 and 3C.5.

3C.1 Conservation of radiance for an incoherent source

3C.1.1 Étendue and radiance

An incoherent source comprises a large number of independent, elementary emitters, emitting electromagnetic waves with a random distribution of uncorrelated phases. It emits light in every direction. A light beam produced by this source can be decomposed into elementary pencils of light. Since the light is incoherent, the total power carried by the beam is the sum of the powers carried by the elementary pencils.

An elementary pencil is defined by the element $\mathrm{d}S$ of the source from

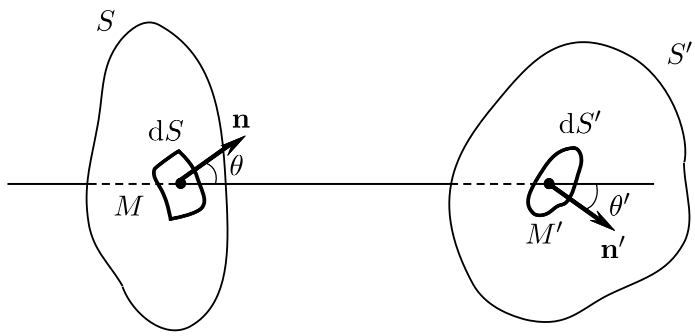

Figure 3C.1: Light beams and light pencils. The two surface elements dS and dS' determine a light pencil, i.e., the set of rays passing from dS to dS'. The whole set of pencils passing from S to S' makes up the beam.

which it originates, and a second surface element dS' as shown in Figure 3C.1. The *étendue* of the pencil (sometimes referred to as "geometric extent") is defined by

$$\mathrm{d}U = \frac{\mathrm{d}S \cos\theta \, \mathrm{d}S' \cos\theta'}{MM'^{\,2}} , \tag{3C.1}$$

where θ and θ' are the angles between the average direction MM' of the pencil and the normals $\mathbf{n}$ and $\mathbf{n}'$ to the two surface elements. Introducing solid angles

$$\mathrm{d}\Omega = \frac{\mathrm{d}S' \cos\theta'}{MM'^2} \tag{3C.2}$$

and

$$\mathrm{d}\Omega' = \frac{\mathrm{d}S \cos\theta}{MM'^2} , \tag{3C.3}$$

the étendue may be written in the form

$$\mathrm{d}U = \mathrm{d}S \cos\theta \, \mathrm{d}\Omega = dS' \cos\theta' \mathrm{d}\Omega' . \tag{3C.4}$$

The radiant flux dΦ carried by the radiation in the pencil, i.e., the transported power, is given by

$$\mathrm{d}\Phi = L(M,\theta) \; dU , \tag{3C.5}$$

where $L(M,\theta)$ is called the *radiance* at the point M. The radiance has units of power per surface per steradian ($\mathrm{W\,m^{-2}\,sr^{-1}}$). For many type of sources (and in particular, a black body with uniform temperature), the radiance depends on neither the point M nor the direction θ, and this is the case considered hereafter in order to simplify the notation.

A light beam is defined by two apertures (one of which can be identified with the source) and the power Φ it transports is clearly the sum of the powers $d\Phi$ carried by the elementary pencils making it up. If the radiance L is uniform and independent of the direction, it follows that

$$\Phi = LU \ , \tag{3C.6}$$

where the étendue U is a purely geometrical quantity, obtained by double integration of (3C.1) or (3C.4).

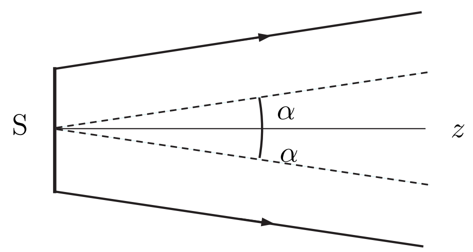

Figure 3C.2: Beam with a rotational angular invariance around the axis z perpendicular to the surface S. For $\alpha = \pi/2$, the étendue is $U = \pi S$

It is useful to know the expression of the étendue of a beam with angular rotational invariance around an axis perpendicular to the first surface S, and an angular diameter 2α (Figure 3C.2):

$$U = S \int_0^\alpha \mathrm{d}\alpha \cos\alpha \, 2\pi \sin\alpha = \pi S \sin^2\alpha \ . \tag{3C.7}$$

We shall also need to know the expression of the differential étendue dU of a pencil defined by an elementary surface $\mathrm{d}x\mathrm{d}y$ and elementary divergences $\mathrm{d}\alpha_x = \mathrm{d}k_x/k$ and $\mathrm{d}\alpha_y = \mathrm{d}k_y/k$ around the axis z (the quantity $k = \omega/c$ is the magnitude of the wave vector). It is

$$\mathrm{d}U = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\alpha_x \, \mathrm{d}\alpha_y = \frac{c^2}{\omega^2} \mathrm{d}x \, \mathrm{d}k_x \, \mathrm{d}y \, \mathrm{d}k_y \ . \tag{3C.8}$$

The definitions and properties discussed above have been given implicitly for a monochromatic beam. In the case of a polychromatic beam, a differential spectral radiance $\mathcal{L}(\omega)$ is defined. It is then understood that the above properties apply to each spectral element $d\omega$ characterized by the radiance $\mathcal{L}(\omega)d\omega$. Consequently, for a spectral interval $d\omega$ and a pencil with etendue dU,

$$\mathrm{d}\Phi = \mathcal{L}(\omega) d\omega \mathrm{d}U \ . \tag{3C.9}$$

Since the different spectral elements of an incoherent source are independent, their contributions to the total power are additive and the expression (3C.9) can be integrated over the variable ω.

3C.1.2 Conservation of radiance

If a light beam propagates through a non-absorbent medium, and if the dioptric components it encounters have been given an antireflective treatment, the power is conserved. It can also be shown that, for a perfect (aberration free) optical system, the étendue is conserved during propagation. More exactly nU is conserved, where n is the index of refraction, but here we consider only beams in vacuum. Equation (3C.6) then shows that the radiance is conserved during propagation. The beam remains characterized by an unchanging radiance, which makes it easy to calculate the photometric quantities required to understand an optical instrument, as we shall see below.

Note that if the beam encounters absorbent media, or partially reflecting surfaces, the radiance will decrease even if the optical system is aberration free. Moreover, if the system is not aberration free, the étendue can only grow, and this generally leads to a decrease in radiance.

To sum up, during the propagation of a beam produced by an incoherent source, the radiance is conserved if the optical systems it passes through are perfect. Otherwise it can only decrease.

Comment

(i) This property is closely related to the second law of thermodynamics which would be violated if it were possible to increase the specific intensity during propagation. There is also a connection with statistical mechanics which can be made clear if we observe that the conservation of throughput is nothing other than the analog of Liouville's theorem, i.e., the conservation of volume in phase space during a Hamiltonian evolution.[1]

(ii) The properties discussed above are no longer valid if the media involved have nonlinear behaviour. Note, for example, that in this case frequency changes may occur, and the spectral elements may no longer be independent. Also, non-linear processes as parametric amplification may increase irradiance.

3C.2 Maximal irradiance by an incoherent source

The irradiance is the radiant power received per unit area of a surface, viz.,

$$E = \frac{\mathrm{d}\Phi}{\mathrm{d}S'} \,. \tag{3C.10}$$

In many applications, it is desirable to have a large irradiance. We shall thus calculate the maximal irradiance that can be obtained from an incoherent source of radiance L.

[1] See, for example, C. Kittel, *Elementary Statistical Physics*, Dover (2004).

Consider first the case shown in Figure 3C.1, where the irradiated surface area $\mathrm{d}S'$ is directly opposite the source. Using (3C.4) and (3C.5), it can be seen that the irradiance due to the source element $\mathrm{d}S$ is given by

$$\mathrm{d}E = L\cos\theta'\,\mathrm{d}\Omega' \,. \tag{3C.11}$$

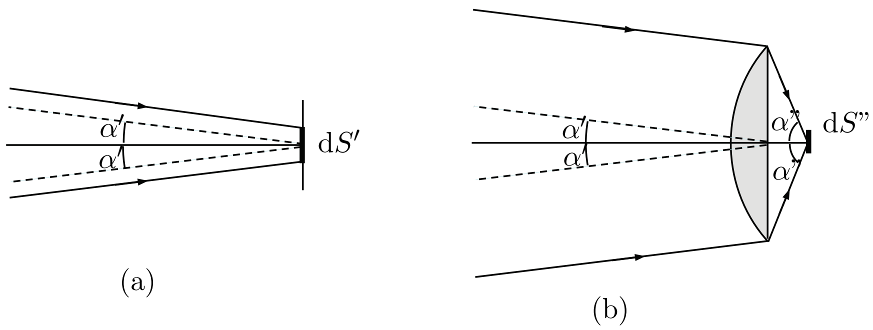

Figure 3C.3: Irradiance of a surface illuminated by a source with rotational symmetry, viewed under an angular diameter $2\alpha'$. (a) Direct illumination. (b) Illumination through a magnifying lens: the irradiance increases because of the larger apparent angular diameter, while the radiance L is conserved. With the sun as the source, one can obtain high temperatures at the focus of the lens. Note however than the irradiance cannot exceed the value $E_{\mathrm{max}} = \pi L$ for $\alpha" = \pi/2$, L being the radiance of the source.

We consider now the case where the surface $\mathrm{d}S'$ is illuminated by a source, which is a disk perpendicular to the direction to the source, and viewed from M' under an angular diameter $2\alpha'$ (Figure 3C.3.a). The surface $\mathrm{d}S'$ is perpendicular to the average direction to the source. A calculation analogous to the one of (3C.7), based on (3C.10-3C.11) then yields

$$E = \pi L \sin^2\alpha' \,. \tag{3C.12}$$

The maximum value of E, obtained for $\alpha' = \pi/2$, i.e., a source viewed under a solid angle of 2π sr, is

$$E_{\mathrm{max}} = \pi L \,. \tag{3C.13}$$

It is impossible to go beyond the maximum value of the irradiance of (3C.13). Whatever instrument is used, the radiance L thus determines the order of magnitude of the maximal irradiance that can be obtained from a given source, i.e., in the final count, the maximal electric field, which is the important quantity for the interaction between matter and radiation.

Comment

(i) Using systems of elliptical or parabolic mirrors, the point M' can be irradiated by a solid angle greater than 2π up to a limit of 4π. In this case, the bound given by (3C.13) must be doubled.

(ii) The above considerations show how it is possible to burn dry tinder by focussing the sun's rays using a magnifying glass: one is increasing the sun's angular diameter from its true value ($2\alpha' \simeq 10^{-2}$ radians) to an apparent value of 2α"(which can easily reach values larger than $\pi/2$) (see Figure 3C.3.b). The irradiance is then multiplied by $\sin^2 \alpha" / \sin^2 \alpha'$ (close to 10^4 in our example). The same principle is at work in solar ovens placed at the top of a tower and in sight of a large number of mirrors which reflect the sun's rays onto the target. The temperature obtained can reach several thousand kelvins, but never go above the temperature of the sun's surface (5000 K).

Orders of magnitude

According to Planck's law, a black body at 3000 K emits about 500 $\mathrm{W/cm^2}$ into a half-space, over a spectral band extending from the infrared ($1.5\,\mu$m) to the green ($0.5\,\mu$m), a range of some 4×10^{14} Hz. This is roughly the output of an incandescent lamp. The maximal irradiance that can be obtained is thus of the order of 1 $\mathrm{kW/cm^2}$. A high-pressure arc lamp or the sun will give at best ten times more, i.e., 10 $\mathrm{kW/cm^2}$.

The maximal irradiance cited here is the total power density received per unit area. Another important quantity is the power per unit area and per unit frequency. It is equal to about 2×10^{-12} $\mathrm{W\,cm^{-2}Hz^{-1}}$ in our example. To understand the relevance of this quantity, suppose for example that one wishes to excite an atomic transition. Only one frequency band, of the order of the natural width of the atomic transition will be used, i.e., a few MHz, and the useful irradiance is of the order of 10^{-5} $\mathrm{W/cm^2}$. Such a value is much too low to reach the strong-field regime, and it is even too low to saturate an atomic transition. Indeed, saturation intensities, leading to a saturation parameter s at resonance (see Equation 2.20 or 2.83) of the order of unity, are closer to the value 10^{-3} $\mathrm{W/cm^2}$.

3C.3 Maximal irradiance by laser light

Let us now consider a 10-W laser beam. Since it is spatially coherent, it can be focussed on an elementary diffraction spot, with size of the order of the wavelength, i.e., an area less than $1\,\mu\mathrm{m}^2$. We now have available some 10^9 $\mathrm{W/cm^2}$, rather than at most a few times 10^3 $\mathrm{W/cm^2}$ that can be obtained from an arc lamp. Moreover, the laser power is supplied in a spectral band that can be of the order of, or even less than 1 MHz. If one aims to excite a narrow atomic transition, all the power can be used and the strong field regime can be reached (saturation parameter 10^{12} for our example).

In the time domain, the technique of mode locking described in Section 5.3 can be used to concentrate the laser power in time in the form of pulses as short as 1 fs (10^{-15} s), repeated every 10 ns. The maximal power is 10^7 times greater than the average power and the irradiance reaches 10^{16} W/cm^2. Under these conditions, the electric field of the electromagnetic wave is *stronger than the Coulomb field* exerted by the proton on the electron of a hydrogen atom in its ground state. A new energy scale is thus attained in the interaction between light and matter.[2]

What characterizes laser light, as compared with thermal radiation, is thus the possibility of concentrating it both in space and in time (or in the spectrum), whereupon enormous energy densities can be obtained, as attested by many laser applications. This possibility stems from the fact that a laser mode contains a very large number of indistinguishable photons, while in the case of thermal radiation, the number of photons per elementary mode of the radiation is less than unity. We shall discuss this point first for cavity modes, then for freely propagating beams.

3C.4 Photon number per mode

3C.4.1 Thermal radiation in a cavity

The energy of the thermal radiation contained in a cavity, e.g., a cube of side L, at temperature T, is proportional to the volume L^3 of the cavity. Planck's law then gives the energy in the frequency band of width $d\omega$ around ω as

$$\mathrm{d}E = L^3 \frac{\hbar\omega^3}{\pi^2 c^2} \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1} \,\mathrm{d}\omega \,. \tag{3C.14}$$

In such a cavity, discrete modes can be defined, characterised by a wave vector $\mathbf{k}$ (or by a frequency $\omega = ck$ and a direction of propagation $\mathbf{k}/k$), and a polarisation. The density of modes in $\mathbf{k}$ space is also proportional to the volume L^3, and the number of modes with frequencies lying in the band of width $\mathrm{d}\omega$ about ω is equal to

$$\mathrm{d}N = L^3 \frac{\omega^2}{\pi^2 c^2} \mathrm{d}\omega \,. \tag{3C.15}$$

Dividing $\mathrm{d}E$ by $\hbar\omega\,\mathrm{d}N$ shows that the number of photons in a mode of frequency ω is given by the Bose–Einstein distribution[3]

$$\mathcal{N}(\omega) = \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1} \,. \tag{3C.16}$$

[2] Suitable amplification and compression of such laser pulses can lead to another increase of several orders of magnitude. See for instance http://extreme-light-infrastructure.com/.

[3] This well known result is interpreted by considering the heat radiation in each mode as an ensemble of bosons at thermodynamic equilibrium (see for example C. Kittel, *Elementary Statistical Physics*, Dover (2004)).

This number of photons per mode, independent of the volume and shape of the cavity, is called the *photon degeneracy parameter*, where the photons are treated as indistinguishable bosons, since photons in the same mode cannot be distinguished by either their frequency, their propagation direction, or their polarization. For a thermal source at 3000 K, the number of photons per mode $\mathcal{N}(\omega)$ is less than unity, of the order of 3×10^{-4} in the middle of the visible spectrum (5×10^{14} Hz, or $\lambda = 0.6\,\mu$m). For solar radiation (temperature 5800 K), $\mathcal{N}(\omega)$ is of the order of 10^{-2} in the middle of the visible spectrum. In both cases, *there are far fewer than one photon per mode.*

3C.4.2 Laser Cavity

Consider now a single-mode laser emitting a power Φ. If the output mirror has transmission coefficient T and the cavity a total length L_{cav} as shown in Figure 3.1, the number of photons contained in the laser cavity is equal to

$$\mathcal{N}_{\text{cav}} = \frac{\Phi}{\hbar\omega}\frac{1}{T}\frac{L_{\text{cav}}}{c}\,. \tag{3C.17}$$

As an example, consider a helium–neon laser delivering 1 mW, with a cavity of length 1 m and coefficient $T = 10^{-2}$. Equation (3C.17) shows that there are about 2×10^{9} photons in the mode of the cavity laser. *This very high degeneracy is an essential feature of laser light.* It is intimately related to the possibility of obtaining considerably higher power densities than from thermal radiation, where the degeneracy parameter is less than unity for practically accessible temperatures.

3C.5 Number of photons per mode for a free beam

3C.5.1 Free propagative mode

In the last section, we discussed the idea of a radiation mode trapped in a cavity. The definition of an elementary radiative mode is a little more subtle in the case of freely propagating beams. Consider a pencil of light emitted into free space, e.g., from a small hole in the wall of a closed box (black body radiation, see Figure 3C.4). How can the notion of a mode be generalized to this case where there is no longer a boundary condition to discretise the problem?

We define a *free mode* by first considering a pencil of *minimal étendue compatible with the wave nature of the radiation.* Such an elementary pencil, with average wavelength $\lambda = 2\pi c/\omega$ and cross-sectional area $\mathrm{d}S$, will have an angular divergence due to diffraction equal to $\lambda/\sqrt{S}$. Its etendue is given by

$$\mathrm{d}U_{\min} = \mathrm{d}S\left(\frac{\lambda}{\sqrt{S}}\right)^2 = \lambda^2 = 4\pi^2\frac{c^2}{\omega^2}\,. \tag{3C.18}$$

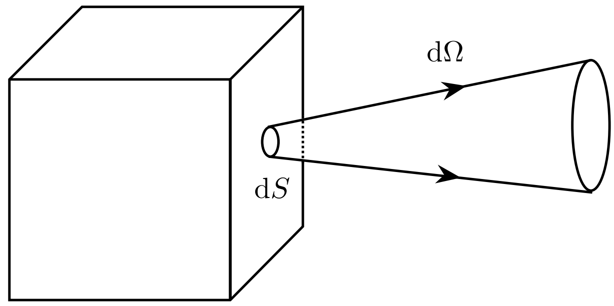

Figure 3C.4: **Pencil of radiation** obtained by making a small hole of cross-section dS in the wall of a cavity containing radiation at temperature T and considering only a solid angle dΩ. The radiance is that of a black body at temperature T. A free mode is associated with a minimal pencil, with étendue $\mathrm{d}U = \mathrm{d}S\mathrm{d}\Omega = \lambda^2$, where $\lambda = 2\pi c/\omega$, and a minimal wave packet of duration Δt and spectral width $\Delta\omega = 2\pi/\Delta t$.

Using (3C.8), this equation can be considered as a consequence of the relations describing the effects of diffraction

$$\Delta x\,\Delta k_x \geq 2\pi \ , \tag{3C.19}$$

$$\Delta y\,\Delta k_y \geq 2\pi \ . \tag{3C.20}$$

Regarding the direction of propagation z (longitudinal), we consider the conjugate variables time and frequency, and in particular, a wave packet of duration Δt. Its spectral width then satisfies the relation

$$\Delta t\,\Delta\omega \geq 2\pi \ , \tag{3C.21}$$

and for a minimal wave packet, we have

$$(\Delta t\,\Delta\omega)_{\min} = 2\pi \ . \tag{3C.22}$$

A dispersion

$$\Delta k_z = \frac{\Delta\omega}{c} \tag{3C.23}$$

in the component k_z of the wave vector is associated with the dispersion $\Delta\omega$, while the longitudinal extent

$$\Delta z = c\Delta t \tag{3C.24}$$

is associated with the duration Δt. The analogous relation to (3C.19) and (3C.20) is then deduced, viz.,

$$\Delta z\,\Delta k_z \geq 2\pi \ , \tag{3C.25}$$

with equality holding for a minimal wave packet.

From Equations (3C.19), (3C.20) and (3C.25), we conclude that a free space mode is characterised by a minimal extension in phase space (direct product of the real space and the wave vector space). This minimal unit is called an elementary cell in the phase space. It is not possible to specify an electromagnetic wave packet more precisely with respect to position and wave vector. The corresponding phase space volume is

$$(\Delta x\,\Delta k_x)_{\min}(\Delta y\,\Delta k_y)_{\min}(\Delta z\,\Delta k_z)_{\min} = (2\pi)^3\ . \tag{3C.26}$$

Comment

In the case of a material particle, the phase space is the direct product of the position and momentum spaces, and the elementary cell has volume h^3, the cube of Planck's constant. Equation (3C.26) can be recovered by using the relation $\mathbf{p} = h\mathbf{k}/2\pi$ for the photon.

3C.5.2 Pencil of thermal radiation

The spectral radiance $\mathcal{L}(\omega)$ of a black-body, or of a hole in the wall of a closed box containing heat radiation, is

$$\mathcal{L}(\omega) = \frac{c}{4\pi}\frac{1}{L^3}\frac{\mathrm{d}E}{\mathrm{d}\omega}\ , \tag{3C.27}$$

where the energy density $(1/L^3)\mathrm{d}E/\mathrm{d}\omega$ is given by Planck's law (3C.14).

The energy contained in a mode, characterised by (3C.18) and (3C.22), or (3C.26) is given by

$$\begin{aligned} E_{\text{mode}} &= \frac{1}{2}\mathcal{L}(\omega)\,\mathrm{d}U_{\min}(\Delta\omega\Delta t)_{\min} \\ &= 4\pi^3\frac{c^2}{\omega^2}\mathcal{L}(\omega)\ . \end{aligned} \tag{3C.28}$$

The factor of $1/2$ accounts for the existence of two orthogonal polarisations, and hence two distinguishable modes for a minimal wave packet. Using (3C.27) and (3C.14), this implies that

$$E_{\text{mode}} = \frac{\hbar\omega}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}\ , \tag{3C.29}$$

whereupon the *number of photons per free propagative mode of the black body radiation* $E_{\text{mode}}/\hbar\omega$ is still given by the Bose–Einstein distribution (3C.15).

3C.5.3 Beam emitted by a laser

The divergence of a transverse single-mode laser beam is entirely due to diffraction. It is thus a minimal pencil for transverse coordinates. Regarding longitudinal coordinates, we consider as above a wave packet with minimal duration compatible with the spectral width $\Delta\omega$ of the line, according to (3C.22), viz.,

$$\Delta t = \frac{2\pi}{\Delta\omega} . \tag{3C.30}$$

The number of photons per free mode is thus

$$\mathcal{N}_{\text{laser}} = \frac{\Phi}{\hbar\omega}\frac{2\pi}{\Delta\omega} . \tag{3C.31}$$

For a single mode laser the line width $\Delta\omega$ is usually much less than the separation $2\pi c/L_{\text{cav}}$ between two modes, and comparing (3C.31) with (3C.17), it can be seen that the number of photons in the laser beam mode is at least of the order of the number of photons in the laser cavity, i.e., very much greater than unity. If the Schawlow–Townes limit is taken for the line width (see Section 5.4), which is typically of the order of 10^{-3} Hz, the number of photons per mode can exceed 10^{15} for a laser of a few mW. Even if more realistic line widths are considered, of the order of the MHz, values greater than 10^9 photons per mode are found for cavities of length around 1 m.

For freely propagating beams, we find once again that *a laser beam has a much higher photon number per mode than unity, in contrast with a beam produced by a thermal source.*

3C.6 Conclusion

It has been shown that laser light offers the possibility of focussing in space as well as in time, or in a complementary way, in angle (highly collimated beam) and in spectrum (highly monochromatic beam). These possibilities are fundamentally related to the fact that all the photons are in the same mode: since they are indistinguishable bosons, they are spatially and temporally coherent, and they can be concentrated on the minimal dimensions allowed by the size $\times$ divergence relation (diffraction), and by the time $\times$ frequency relation.

Hence, we may say that what characterises laser light most fundamentally, as compared with ordinary light, is a photon number per mode much higher than unity.