Introduction to Martingale Methods in Option Pricing

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Lecture Notes in Mathematics 4

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Preface

This booklet is based on the lectures delivered by the author at the Department of Mathematics, City University of Hong Kong, in the autumn of 1997. It is aimed primarily at those people who are non-experts in, but willing to learn, the option pricing theory. The reader is assumed to have some knowledge in probability theory. But no prerequisities in finance are required.

As indicated by the title, the main objective of the booklet is to provide a short and comprehensive presentation of the main ideas and fundamental results on martingale methods in option pricing, although the PDE approach is also occasionally touched. The booklet is divided into seven chapters. Chapter 1 is devoted to introduce general characteristics of derivative securities and summarize some knowledge in probability theory necessary for further reading. In Chapter 2 the main concepts of the option pricing and hedging and the risk-neutral valuation principle are presented in the discrete-time market model. Chapter 3 provides a concise and rather complete summary on Itô calculus, an important tool for an advanced theory of finance. The celebrated Black-Scholes equation and valuation formulas for option pricing as well as the practical uses of Black-Scholes formulas are presented in Chapter 4. The martingale methods in option pricing are introduced in the Black-Scholes setting. Chapter 5 deals with the concrete pricing problem of path-dependent exotic options. Some explicit valuation formulas for barrier options, Asian options and lookback options are derived by using martingale methods or PDE approches. In Chapter 6 we introduce a general framework for a financial market: the Itô process and diffusion process models. The martingale methods in contingent claim pricing are fully exhibited and a brief discussion on the pricing of American contingent claims is made. Chapter 7 is devoted to introduce the bond market and term structure models for interest rates. The pricing of interest rate derivatives is briefly presented. In Chapter 8, a new look at the fundamental theorem of asset pricing, based on Yan (1997), is reported.

After the first version of the lectures was complete, the author learned that three excellent books on the same subject had been recently published. They are Karatzas (1997), Karatzas and Shreve (1997), Musiela and Rutkowski (1997). It is the author's hope that the present booklet provides an easy access to these books which contain abundant and much more advanced material. This booklet can be used as a textbook for graduate courses in financial mathematics, provided the instructor supplements the omitted proofs, for which the references are indicated. Proofs or skechy proofs for most of the results are provided, except Chapter 3, for which the reader can find the omitted proofs in Karatzas and Shreve (1988).

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Jia-An Yan

CHAPTER 1

Introduction

The option pricing theory, originated from options markets, has nowadays become a powerful and effective tool for quantitative analysis in many economical problems. For example, problems concerning complicated investments, decision-making and risk management in a company are closely related to the option pricing. In this chapter we introduce general characteristics of options and summarize some basic knowledge in probability theory.

1.1 Basic concepts and terminology of derivative securities

A *derivative security* or *contingent claim* is a financial contract written on an underlying asset (such as stock, Treasury bill, foreign currency, stock-index, or even another derivative security). Its value is derived from, or contingent on, some variables related to the underlying asset, hence the name. Best-known derivatives are options, futures, and forward contracts.

1.1.1 What is an option?

Broadly speaking, an *option* is a contract that entitles its owner with the right but no obligation to buy or sell a specified quantity of an underlying instrument such as a stock, currency, commodity, stock-index and index futures for a specified price at a particular time or within a specified time period. The value of an option depends on some underlying state variables, such as the price of the underlying asset. There are two basic types of option : call and put. "Call" refers to the right to buy and "put" refers to the right to sell. A *call* (*put*) option gives its owner the right with no obligation to buy (sell) a certain amount of the underlying asset by a certain date, known as the *expiration date* or *maturity*, for a certain price, known as the *exercise price* or *strike price*. If the option can be exercised at any time before maturity, it is called an *American* option. If it can only be exercised at maturity, it is called a *European* option.

The above described call or put option is called a *standard* or *vanilla* option. An option that is not a vanilla put or call is called a *non-standard* or *exotic* option. Most exotic options are *path-dependent* in the sense that their payoff depends on the current and past values of underlying state variables. Best-known examples of path-dependent exotic options are barrier options, Asian options, and lookback options. A *barrier option* can either come into existence or become worthless if the price of the underlying asset reaches a prescribed level (known as a *barrier*) before expiry. The payoff of an *Asian option* depends on a suitably defined average of the asset price over a certain time period. The payoff of an *lookback option* depends on asset price maximum or minimum. An option that has no expiry date (i.e. having an infinite time horizon) is called a *perpetual option.* A *Russian option* is a perpetual American-style option which, at any time chosen by the owner, pays the maximum realized asset price up to that date.

Note that in the above classification of options, the names of continents or countries have nothing to do with trading-places of options and refer merely to a technicality in the option contract.

1.1.2 Forward contracts and futures contracts

A forward contract is an agreement between two parties whereby the seller (known as in a short position) agrees to deliver to the buyer (known as in a long position) on a specified date and at a fixed price, known as *delivery price*, a specified quantity of an underlying asset. A long position in a forward contract is equivalent to a long position in a European call option and a short position in a European put option, both with the same expiry and exercise price as the forward contract. Here and henceforth, a *long position* refers to a position in an asset or contract which one has purchased or owned, a *short position* refers to a position in an asset (resp. contract) which one does not already own but has sold (resp. has written). A futures contract is similar to a forward contract. One distinction between them is that unlike forward contracts, futures contracts are not "tailored" to the specific needs of a particular buyer and seller, they are *standardized* with regard to the quality of the asset, time to maturity, price quotation, and delivery procedure. Another distinction is that futures contracts are traded on exchanges while forward contracts are traded in " over-the-counter "markets. But the most important distinction lies in their respective price settlement procedures. There is no cash transfer between the two parties of a forward contract until its delivery date. So a forward contract has a market value at any time t before its delivery date, the initial value being equal to zero. However, futures contracts have a *daily settlement* (or so-called *marking-to-market*) procedure that requires the buyer and seller to adjust their position daily according to the gains or losses due to the futures price changes. Here the *futures price* is the anticipated future unit price of the underlying asset. One should beware that the futures price is not the price or market value of the futures contract. Futures prices change continuously in such a way that they make the market value of a futures contract always equal to zero. Similarly, the *forward price* at time t of the underlying asset is not the market value at time t of the forward contract. It is defined as the derivery price of the forward contract such that the contract has zero value when it is initiated at time t. Due to the daily settlement the futures price and the forward price are generally not the same. We will study this problem in Chapter 7.

1.1.3 Option pricing and hedging

An option provides a non-negative payoff which is not identically null. It must have some value (called *price* or *market value*) at any time before maturity. The initial price of an option is usually called *premium.* There are two fundamental problems related to an option: pricing and hedging. *Pricing* an option is to determine its market value at any time t before maturity. The minimal value with which one can build a trading strategy to generate (resp. cover) the payoff of an option is called the *fair price* (resp. *upper price* or *selling price*) of the option. Such a trading strategy is called *hedging* or *replicating strategy* (resp. *super-hedging* or *super-replicating strategy*). Note that the term hedging has also other meanings in different contexts. For example, taking opposite positions in different financial instruments in order to reduce (but not necessary eliminate) a risk is also called hedging (see Section 1.2 below). To distinguish these two types of hedge, we sometimes call the previous one a “perfect hedge”.

In option pricing theory, one of the basic assumptions on the market is the absence of *arbitrage* opportunity. It means that there is no riskless and profitable opportunity in the market. This assumption implies the so-called *law of one price*, which states that two financial packages having identical payoffs must sell for the same price. Another basic assumption is the existence of a riskless investment that gives a guaranteed return with no chance of default. A good approximation to such an investment is a government bond or a deposit in a sound bank. The second one is called the *bank account* (or *money market account, savings account*). The bank account earns interest at a *riskless interest rate.*

In option pricing one often needs to choose one asset as a commom unit, on the basis of which the prices of other assets are expressed. The resulting relative prices are called the *deflated prices* or *discounted prices.* Such an asset is called a *numeraire asset* or *numeraire*, and its price process is called a *numeraire.* In most cases, one takes the bank account as a numeraire asset. In this case, an important concept concerning interest rates is the *present value.* If the bank account earns a constant interest rate r, then in continuous-time setting, the present value at time t of a value ξ at a future time T is defined as $e^{-r(T-t)}\xi$. If we deposit such amount of money in a bank at time t, we get exactly the value ξ at time T. Similarly, the present value at time t of a value η at a past time s is defined as $e^{r(t-s)}\eta$. For example, if a stock pays dividends with a time-dependent yield $(d_s, s \le t)$ (the ratio of the dividend payment to the stock price), then the present value of the dividends up to time t is equal to $\int_0^t e^{rs} d_s S_s ds$, where S_s is the stock price at time s.

1.1.4 Put-call relationships

From the above two basic assumptions on the market we can deduce an intrinsic relationship between the values of a European vanilla call and a vanilla put written on a same asset with the same maturity T and same strike price K. In fact, let S_t denote the asset price at time t and let C_t and P_t denote the values at time t of such a call and a put, respectively. We assume that the riskless interest rate of the bank account is a constant r. We consider a portfolio consisting of a long position in a share of non-dividend-paying stock and in one put option and a short position in one call option. The value of this portfolio at time T is given by

$$S_T + P_T - C_T = S_T + (K - S_T)^+ - (S_T - K)^+ = K.$$

Here and henceforth, for a real number a we denote max(a,0) by a^+. Thus this portfolio is riskless. According to the law of one price the wealth of this portfolio at time $t < T$ should be $Ke^{-r(T-t)}$, because if at time t we invest such money in the riskless security we get also money K at time T. Thus we obtain the following equality:

$$S_t + P_t - C_t = Ke^{-r(T-t)},$$

which is called *put-call parity.*

The put-call parity can be generalized to options on a dividend-paying stock and the result is

$$S_t + P_t - C_t = D_t + Ke^{-r(T-t)},$$

where D_t is the present value at time t of the dividends. For American vanilla options on a dividend-paying stock, we can only get the following two inequalities:

$$C_t + K + D_t - S_t \geq P_t \geq C_t - S_t + Ke^{-r(T-t)}.$$

1.1.5 Intrinsic value and time value

At any time t before maturity options are referred to as *in the money, at the money,* or *out of the money.* An in-the-money call (resp. put) option is one whose strike price is less (resp. greater) than the current price of the underlying asset. An at-the-money call or put option is one whose strike price is equal to the current price of the underlying asset. An out-the-money call (resp. put) option is one whose strike price is greater (resp. less) than the current price of the underlying asset.

Let S_t denote the price of an asset at time t and K the strike price of a call (resp. put) written on that asset. We call $(S_t - K)^+$ (resp. $(K - S_t)^+$) the *intrinsic value* of the call (resp. put) option. An in-the-money American option must be worth at least as much as its intrinsic value since the holder can realize a positive intrinsic value by exercising immediately. In this case the option is said to have *time value*, which is equal to the difference between the market value and the intrinsic value of the option.

1.2 What are options for?

Like futures contracts, three basic uses of options are speculating, spreading, and hedging. Each of these uses involves the management of risk, with each strategy changing risk in a different way.

The first use of options is speculating. To see this let us consider a European call or put option on a stock. Assume that the stock price at time t is S_t. The payoff at maturity T of a call (put) with exercise price K is $(S_T - K)^+$ (resp. $(K - S_T)^+$). An investor who believes that the stock price is going to rise can buy certain shares of the stock or calls. If one believes that the stock price is going to fall, one can sell short certain shares or buy puts. Here "sell short" means selling shares that one does not own. In either case, if one is correct, one makes money; If one is wrong, one loses money. Speculators often prefer options to stocks, because options often provide larger return rate than stocks, known as *leverage effect*, when the forecast of stock price movement is correct, while limiting the loss to the premium of the option when the forecast is wrong. Speculators take an additional risk in order to obtain leverage.

The second use of options is spreading. A spread trading strategy involves taking a position in two or more options of the same type (i.e. calls or puts). Spreaders want a option position that is less risky than a pure long or short option position. There are two types of spread trading strategy. A *calendar* or *time spread* consists in the simultaneous purchase and sale of options on the same asset with the same strike price but with different maturities. A *strike* or *price spread* consists in the simultaneous purchase and sale of options on the same assets with the same maturity but with different strike prices.

The third use of options is "hedging". In contrast to a "perfect hedge", here by hedging we mean a risk-monitoring strategy that aims at offsetting the gains (losses) on one asset by the losses (gains) on another asset, and thereby reducing the global risk exposure of the resulting position. For example, an investor possesses shares of the stock and believes that the stock price is going to fall. If in the time being he/she could not (or doesn't want to) sell shares, he/she can buy put options to hedge against the possible falling of the stock price. Here, contrary to speculators, hedgers use puts (resp. calls) to hedge against the falling (resp. rising) of the asset price.

All the above uses of options are provided for option buyers. What about option sellers (or *writers*)? They face far greater risk exposures than buyers. In fact, the writer of an option has the possibility of an arbitrarily large loss with a profit limited to the premium of the option. One raises naturally the following question : Why would anyone write options? One reason is that there are always different investors taking opposite views on the asset price movement. The writer of both call and put options can at least make benefit from premium paid by those investors with a wrong forecast. Another reason is that the premium for an option is usually above the minimal value with which one can build a hedging strategy to cover or generate the payoff of the option.

In summary, as an important risk monitoring tool, options provide various risk-return choices for investors and a risk-shifting function among individuals of different beliefs, preferences, and investment objectives. This risk monitoring has made option markets popular for both individuals and financial institutions.

1.3 Some knowledge in probability theory

1.3.1 Random variables, expectation and independency

Let Ω be an abstract set. A collection $\mathcal{F}$ of subests of Ω is called a *σ-algebra*, if it satisfies the following conditions:

(i) $\Omega \in \mathcal{F}$;

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, where $A^c = \Omega \setminus A$ denotes the complement of A;

(iii) $A_j \in \mathcal{F}, j \geq 1, \Rightarrow \bigcup_{j=1}^{\infty} \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a *measurable space*.

Let $\mathcal{C}$ be a collection of subsets of Ω. The smallest σ-algebra containing $\mathcal{C}$ is called the σ-algebra generated by $\mathcal{C}$, and is denoted by $\sigma(\mathcal{C})$. Let Ω be the real line $\mathbf{R}$. The σ-algebra generated by all open sets is called the *Borel σ-algebra*, and is denoted by $\mathcal{B}(\mathbf{R})$. Let f be a real-valued function defined on Ω. We denote by $\sigma(f)$ the σ-algebra generated by $\{f^{-1}(A), A \in \mathcal{B}(\mathbf{R})\}$. f is called *$\mathcal{F}$-measurable*, if for any $A \in \mathcal{B}(\mathbf{R})$, $f^{-1}(A) \in \mathcal{F}$, i.e. $\sigma(f) \subset \mathcal{F}$.

A *measure* on a measurable space $(\Omega, \mathcal{F})$ is a mapping μ from $\mathcal{F}$ to $[0, \infty]$ with $\mu(\emptyset) = 0$ having the countable additivity, i.e. for any countable collection of sets in $\mathcal{F}$ with $A_i \cap A_j = \emptyset, \forall j \neq i$, we have $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$. The triplet $(\Omega, \mathcal{F}, \mu)$ is called a *measure space.* If $\mu(\Omega) = 1$, we call μ a *probability measure*, and $(\Omega, \mathcal{F}, \mu)$ a *probability space.* In probability theory, an element A of $\mathcal{F}$ is called an *event.* If $\mathbf{P}(A) = 1$, we call A a *certain event.* A property that holds except on a set of probability zero is said to hold *almost surely* w.r.t. $\mathbf{P}$, or simply $\mathbf{P}$-a.s.. A real-valued $\mathcal{F}$-measurable function is called a *random variable* (abbreviated as r.v.). Two a.s. equal r.v.'s will not be distinguished.

A finite or countable collection (A_i) is called a *partition* of Ω, if $A_i \cap A_j = \emptyset, \forall j \neq i$ and $\bigcup_i A_i = \Omega$. A *simple* r.v. is a r.v. X of the form

$$X(\omega) = \sum_j a_j I_{A_j}(\omega),$$

where (A_i) is a finite partition of Ω with each $A_i \in \mathcal{F}$, and I_{A_j} is the indicator of A_j, i.e. $I_{A_j}(\omega) = 1$, when $\omega \in A_j$, and $I_{A_j}(\omega) = 0$ otherwise. In this case the *expectation* $\mathbf{E}[X]$ of X w.r.t. $\mathbf{P}$ is defined as

$$\mathbf{E}[X] = \sum_j a_j \mathbf{P}(A_j).$$

For a non-negative r.v. X we can find an increasing sequence (X_n) of simple r.v.'s such that $\lim_{n\to\infty} X_n = X$, a.s.. We define $\mathbf{E}[X] = \lim_{n\to\infty} \mathbf{E}[X_n]$ as the expectation of X w.r.t. $\mathbf{P}$. For a general r.v. X, we define $\mathbf{E}[X] = \mathbf{E}[X^+] - \mathbf{E}[X^-]$, as long as $\mathbf{E}[X^+]$ or $\mathbf{E}[X^-]$ is finite. If $\mathbf{E}[X]$ is finite, X is said to be *integrable.*

Two events A and B are called *independent*, if $\mathbf{P}(AB) = \mathbf{P}(A)\mathbf{P}(B)$. Two collections of events $\mathcal{A}$ and $\mathcal{B}$ are called *independent*, if any A from $\mathcal{A}$ and any B from $\mathcal{B}$ are independent. The events in a collection $(A_t, t \in T)$ are said to be (*mutually*) *independent*, if for any finite subset S of T we have

$$\mathbf{P}\Big(\bigcap_{t\in S} A_t\Big) = \prod_{t\in S} \mathbf{P}(A_t).$$

Let $(\mathcal{C}_t, t \in T)$ be a family of collections of events. If the events in any collection $(A_t, t \in T)$, with each element A_t belonging to $\mathcal{C}_t$, are independent, then we call such a family an *independent family.* A family $(X_t, t \in T)$ of r.v.'s is called an independent family, if $(\sigma(X_t), t \in T)$ is an independet family. A r.v. X is called independent of a σ-algebra $\mathcal{A}$ if $\sigma(X)$ and $\mathcal{A}$ are independent.

An important fact about a finite independent sequence $(X_i, i = 1, \ldots, n)$ of integrable r.v.'s is that

$$\mathbf{E}\Big[\prod_i X_i\Big] = \prod_i \mathbf{E}[X_i]. \tag{1.1}$$

1.3.2 Conditional expectations

Let A and B be two events with $\mathbf{P}(A) > 0$. The probability that B hapens under the condition that A happens is obviously equal to $\mathbf{P}(AB)/\mathbf{P}(A)$. It is called the *conditional probability of B relative to A*, and denoted by $\mathbf{P}(B|A)$.

Let $(B_j)_{1\le j\le m}$ be a finite partition of Ω with $B_j \in \mathcal{F}$ and $\mathbf{P}(B_j) > 0$, $\forall 1 \le j \le m$. Let $\mathcal{G}$ be the σ-algebra generated by (B_j). For an integrable r.v. X, we put

$$\mathbf{E}[X|\mathcal{G}] = \sum_{j=1}^{m} \frac{\mathbf{E}[XI_{B_j}]}{\mathbf{P}(B_j)} I_{B_j},$$

and call $\mathbf{E}[X|\mathcal{G}]$ the *conditional expectation of X relative to $\mathcal{G}$*. If $(A_i)_{1\le i\le n}$ is a finite partitions of Ω with all A_i's in $\mathcal{F}$ and $X = \sum_{i=1}^n a_i I_{A_i}$ is a simple r.v., then it is easy to show that

$$\mathbf{E}[X \mid \mathcal{G}] = \sum_{j=1}^{m} \sum_{i=1}^{n} a_i \mathbf{P}(A_i|B_j) I_{B_j}. \tag{1.2}$$

Now we extend the definition of conditional expectation to the general case. Let μ and ν be two finite measures on $(\Omega, \mathcal{F})$. If for any $A \in \mathcal{F}$ with $\mu(A) = 0$ we have

$\nu(A) = 0$, then ν is said to be *absolutely continuous* w.r.t. μ. In this case, there exists a unique non-negative $\mathcal{F}$-measurable function ξ such that

$$\nu(A) = \int_A \xi d\mu, \quad \forall A \in \mathcal{F}.$$

This result is called the *Radon-Nikodym theorem.* We call ξ the *Radon-Nikodym derivative* of ν w.r.t. μ, and denote it by $\frac{d\nu}{d\mu}|_{\mathcal{F}}$ or simply $\frac{d\nu}{d\mu}$. If μ and ν are mutually absolutely continuous, we say that μ and ν are *equivalent.* In this case $\frac{d\nu}{d\mu}$ can be chosen to be strictly positive and we have $(\frac{d\nu}{d\mu})^{-1} = \frac{d\mu}{d\nu}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ and X an integrable r.v.. Put

$$\nu(A) = \mathbf{E}[XI_A], \quad \forall A \in \mathcal{G}.$$

Then ν is a signed measure (i.e. the difference of two measures) on $(\Omega, \mathcal{G})$, which is absolutely continuous w.r.t. $\mathbf{P}$. By the Radon-Nikodym theorem there exists a unique $\mathcal{G}$-measurable r.v. Y such that

$$\mathbf{E}[YI_A] = \mathbf{E}[XI_A], \quad \forall A \in \mathcal{G}. \tag{1.3}$$

We call Y the *conditional expectation* of X w.r.t. $\mathcal{G}$, and denote it by $\mathbf{E}[X|\mathcal{G}]$.

Apart from the linearity, the conditional expectation has the following properties:

1) $\mathbf{E}[\mathbf{E}[X \mid \mathcal{G}]] = \mathbf{E}[X]$;

2) If X and XY are integrable and Y is $\mathcal{G}$-measurable, then

$$\mathbf{E}[XY|\,\mathcal{G}] = Y\mathbf{E}[X|\,\mathcal{G}]; \tag{1.4}$$

3) If $\mathcal{G}_1$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbf{E}[\mathbf{E}[X \mid \mathcal{G}] \mid \mathcal{G}_1] = \mathbf{E}[X \mid \mathcal{G}_1]; \tag{1.5}$$

4) (*Jensen's inequality*) If f is a convex function on $\mathbf{R}$ and X and $f(X)$ are integrable r.v.'s, then

$$f(\mathbf{E}[X \mid \mathcal{G}]) \leq \mathbf{E}[f(X) \mid \mathcal{G}]. \tag{1.6}$$

The following two theorems will be frequently used in the sequal.

Theorem 1.1 Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, $g(x, y)$ a non-negative Borel function on $\mathbf{R}^2$ and X a $\mathcal{G}$-measurable r.v.. Then for any r.v. Y we have

$$\mathbf{E}[g(X, Y) \mid \mathcal{G}] = \mathbf{E}[g(x, Y) \mid \mathcal{G}]|_{x=X}. \tag{1.7}$$

In particular, if Y is independent of $\mathcal{G}$, we have

$$\mathbf{E}[g(X, Y) \mid \mathcal{G}] = \mathbf{E}[g(x, Y)]|_{x=X}. \tag{1.8}$$

Proof If A and B are two Borel sets and $g(x, y) = I_A(x)I_B(y)$, then from (1.4) we know that (1.7) holds. The genaral case follows from the fact that any non-negative Borel function is a limit of an increasing sequence of simple Borel functions.

Theorem 1.2 (*Bayes rule*) Let $\mathbf{Q}$ be a probability measure equivalent to $\mathbf{P}$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. We put

$$\xi = \frac{d\mathbf{Q}}{d\mathbf{P}}, \quad \eta = \mathbf{E}[\xi|\,\mathcal{G}].$$

If X is a $\mathbf{Q}$-integrable r.v., then we have

$$\mathbf{E_Q}[X|\,\mathcal{G}] = \eta^{-1}\mathbf{E}[X\xi|\,\mathcal{G}]. \tag{1.9}$$

Proof For any $A \in \mathcal{G}$, we have

$$\begin{aligned}\mathbf{E}[X\xi I_A] &= \mathbf{E_Q}[XI_A] = \mathbf{E_Q}[\mathbf{E_Q}[X|\,\mathcal{G}]I_A] = \mathbf{E}[\mathbf{E_Q}[X|\,\mathcal{G}]\xi I_A] \\ &= \mathbf{E}[\mathbf{E_Q}[X|\,\mathcal{G}]\eta I_A].\end{aligned}$$

Since $\mathbf{E_Q}[X|\,\mathcal{G}]\eta$ is $\mathcal{G}$-measurable, we get

$$\mathbf{E}[X\xi|\,\mathcal{G}] = \mathbf{E_Q}[X|\,\mathcal{G}]\eta,$$

which gives (1.9).

1.3.3 Martingales

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $(\mathcal{F}_n, 0 \le n \le N)$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$. We call $(\mathcal{F}_n)$ a *filtration*. For notational convenience, we put $\mathcal{F}_{-1} = \mathcal{F}_0$. A sequence $(X_n, 0 \le n \le N)$ of r.v.'s defined on $(\Omega, \mathcal{F})$ is said to be *adapted* (resp. *predictable*) if each X_n is $\mathcal{F}_n$- (resp. $\mathcal{F}_{n-1}$-) measurable.

Definition 1.3 An adapted sequence $(M_n)_{0\le n\le N}$ is called a *martingale* (resp. *supermartingale, submartingale*), if for each n, M_n is integrable w.r.t. $\mathbf{P}$ and

$$\mathbf{E}[M_{n+1}|\mathcal{F}_n] = M_n (\text{resp.} \le M_n, \;\; \ge M_n), \;\; \forall n \le N-1. \tag{1.10}$$

Let (X_n) be a supermartingale. Put

$$A_n = X_0 + \sum_{j=1}^{n}(X_{n-1} - \mathbf{E}[X_n|\mathcal{F}_{n-1}]); \; M_n = X_n + A_n, \;\; n \ge 1,$$

and let $A_0 = 0, M_0 = X_0$. Then A is a non-decreasing predictable process with $A_0 = 0$ and M is a martingale. We call the expression $X = M - A$ the *Doob's decomposition* of supermartingale X.

The following two propositions will be used in Chapter 2.

Proposition 1.4 Let $(M_n)_{0\le n\le N}$ be a martingale and $(H_n)_{0\le n\le N}$ a predictable sequence such that H_0M_0 and $H_n\Delta M_n, 1 \le n \le N$, are integrable. Put

$$X_0 = H_0M_0, \; X_n = H_0M_0 + \sum_{i=1}^{n} H_i\Delta M_i, \; n \ge 1.$$

Then $(X_n)_{0\le n\le N}$ is a martingale.

Proof Clearly, (X_n) is an adapted sequence and each X_n is integrable. Moreover, since for $n \ge 0$, H_{n+1} is $\mathcal{F}_n$-measurable, we have

$$\begin{aligned}\mathbf{E}[X_{n+1} - X_n|\mathcal{F}_n] &= \mathbf{E}[H_{n+1}(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= H_{n+1}\mathbf{E}[M_{n+1} - M_n|\mathcal{F}_n] = 0,\end{aligned}$$

which means that (X_n) is a martingale.

Proposition 1.5 Let (M_n) be an adapted sequence of integrable random variables. If for any bounded predictable sequence (H_n) we have $\mathbf{E}[\sum_{j=1}^{N} H_j\Delta M_j] = 0$, then (M_n) is a martingale.

Proof Let $1 \le j \le N$. For any $A \in \mathcal{F}_{j-1}$, we put $H_n = 0, n \ne j, H_j = I_A$. Then (H_n) is a bounded predictable sequence and by assumption $\mathbf{E}[I_A(M_j - M_{j-1})] = 0$. That means $\mathbf{E}[M_j|\mathcal{F}_{j-1}] = M_{j-1}$. Thus (M_n) is a martingale.

CHAPTER 2

The Discrete-Time Model

In this chapter we shall introduce the main ideas related to the theory of option pricing and hedging in the discrete-time setting. Firstly, we use the binomial tree model to illustrate the risk-neutral valuation principle. Secondly, we present the general discrete-time model. This model was introduced by Harrison and Pliska (1981) and further discussed in the case of finite states by Taqqu and Willinger (1987). We shall show that the existence of an equivalent probability measure under which the discounted price process of securities is a martingale is equivalent to the absence of arbitrage and that the uniqueness of such a martingale measure is equivalent to the completeness of the market. This unique martingale measure enables one to uniquely price any contingent claim. We follow closely the first chapter of the book by Lamberton and Lapeyre (1996).

2.1 The binomial tree model

It is common knowledge that an asset price moves randomly and one cannot predict future's values of an asset price. However, by a statistical analysis of historical data one can establish some models for asset price movements. There are two types of model: the discrete-time model and continuous-time model. The simplest discrete-time model is the *binomial tree model*, introduced by Cox, Ross, and Rubinstein (1979) as a technique tool for pricing a contingent claim. This model is far from being a realistic one. We shall only use this model to illustrate two important methods for pricing options: *arbitrage pricing* and *risk-neutral valuation*, which are essentially equivalent.

Suppose that there are two assets in the market. One is a riskless security with riskless interest rate r per unit time, another one is a no-dividend-paying stock whose current price (i.e. the price at time zero) is S_0. Assume that at time $n+1$ the stock price S_{n+1} will be either uS_n or dS_n, where $d < u$ are constants. Obviously, the absence of arbitrage opportunity is equivalent to the condition "$d < 1 + r < u$". We are interested in valuing a European contingent claim ξ with maturity N. We assume that ξ depends only on the stock price S_N at time N.

2.1.1 The one-period case

In this subsection we consider the one-period (i.e. $N = 1$) case, the multiperiod case will be studied in the next subsection. In the one-period case, at time 1 the contingent claim takes one of the two values, say, ξ_u or ξ_d, which corresponds respectively to the stock prices uS_0 or dS_0 at time 1. Consider a portfolio consisting of a long position in α_0 shares of the stock and a short position in the contingent claim. The wealth of the portfolio at time 1 is equal to $\alpha_0 S_1 - \xi$. In order to find a value α_0 that makes the portfolio riskless (i.e. the wealth of the portfolio does not depend on the up or down movements of the stock price) we solve the equation

$$\alpha_0 u S_0 - \xi_u = \alpha_0 d S_0 - \xi_d,$$

and obtain $\alpha_0 = \frac{\xi_u - \xi_d}{(u-d)S_0}$. The wealth of the portfolio at time 1 is then $X_1 = \frac{d\xi_u - u\xi_d}{u-d}$. Thus, by the law of one price the wealth X_0 of the portfolio at time zero is equal to $X_1/1+r$. Consequently, the price of the contingent claim ξ at time 0 is given by

$$C_0 = \alpha_0 S_0 - X_0 = \frac{(1+r-d)\xi_u + (u-(1+r))\xi_d}{(1+r)(u-d)}. \tag{2.1}$$

This method of pricing by no arbitrage argument is called *arbitrage pricing.*

A careful reader may notice that the contingent claim pricing formula (2.1) does not involve the probabilities of the stock price moving up or down. This somewhat surprising fact stems from the true meaning of arbitrage pricing. An explication of this fact is that the probabilities of future up or down movements are already embedded into the current price of the stock.

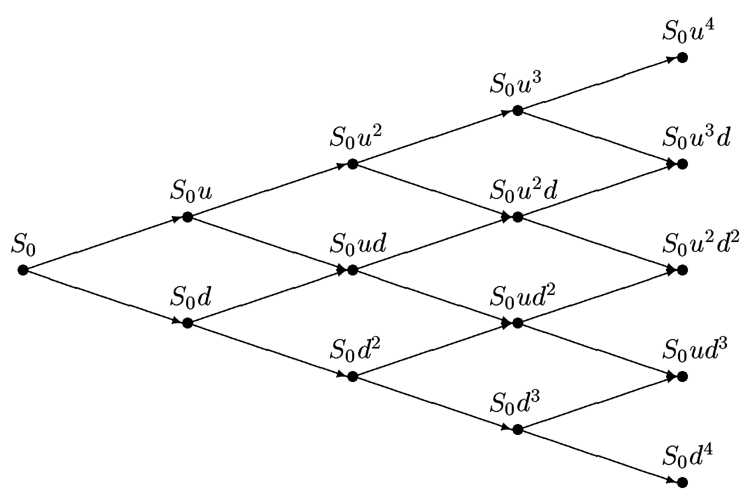
If we put $q = \frac{1+r-d}{u-d}$, then (2.1) can be rewritten as

$$C_0 = (1+r)^{-1}[q\xi_u + (1-q)\xi_d]. \tag{2.2}$$

If we interpret the values q or $1-q$ as the probabilities of the stock price moving up or down, (2.3) then states that today's price of the contingent claim is the expectation of its discounted future value under this new probability measure. Since $qu + (1-q)d = 1+r$, it is readily seen that this new probability measure is the unique measure under which the expected rate of return on the stock, i.e. $\mathbf{E}[S_1]/S_0 - 1$, is just the same as the riskless rate r, and the expectation of the discounted price $(1+r)^{-1}S_1$ is equal to S_0. This new probability measure is called *risk-neutral probability measure.*

2.1.2 The multiperiod case

Now we turn to the multiperiod binomial tree model, which is shown in the following figure:



Our objective is to determine the value at any time n of a European contingent claim ξ with maturity N. Let Ω denote the set of all stock price paths running from time zero up to time N. It represents the uncertainty of the stock price movements. Ω contains 2^N elements. Each element is a possible realization of the movements of the stock price. At time n, we have n nodes. We number these nodes from top to bottom. For each $\omega \in \Omega$ we denote by $\omega(n)$ the serial number of the node passed through by path ω at time n. Put

$$\Omega_{n,j} = \{\omega : \ \omega(n) = j\}.$$

Then

$$\omega, \omega' \in \Omega_{n,j} \Longrightarrow S_n(\omega) = S_n(\omega'). \tag{2.3}$$

If we regard each node as an origin point and consider one-period-movement of the stock price from this node, we come to the situation of one-period binomial model. So by the"backwards induction" and by a repeated application of equation (2.2) we can give the value of the claim at any time $n = 0, 1, \cdots, N-1$. More precisely, let us define a probability measure on Ω by

$$\mathbf{P}^*(\omega) = q^{\sum_{i=0}^{N-1} \alpha_i(\omega)}(1-q)^{N-\sum_{i=0}^{N-1} \alpha_i(\omega)}, \quad \omega \in \Omega, \tag{2.4}$$

where $\alpha_i(\omega) = 1$ or 0, when the stock price goes up or down at the i-th step of path ω. We denote by C_n the value at time n of the contingent claim ξ. From (2.1), (2.2) and (2.3) we see that

$$\omega, \omega' \in \Omega_{n,j} \Longrightarrow C_n(\omega) = C_n(\omega'), \tag{2.5}$$

and

$$C_{n,j} = (1+r)^{-1}[qC_{n+1,j} + (1-q)C_{n+1,j+1}], \tag{2.6}$$

where $C_{n,j} \hat{=} C_n(\omega)$, $\forall \omega \in \Omega_{n,j}$. Since

$$\mathbf{P}^*(\Omega_{n+1,j}|\Omega_{n,j}) = q, \quad \mathbf{P}^*(\Omega_{n+1,j+1}|\Omega_{n,j}) = 1-q,$$

we can rewrite (2.6) as

$$C_{n,j} = (1+r)^{-1}\left[C_{n+1,j}\mathbf{P}^*(\Omega_{n+1,j}|\Omega_{n,j}) + C_{n+1,j+1}\mathbf{P}^*(\Omega_{n+1,j+1}|\Omega_{n,j})\right]. \tag{2.7}$$

If we denote by $\mathcal{F}_n$ the σ-algebra generated by the sequence $(S_j, 0 \le j \le n)$, or, in our case, it is the same generated by the family $\{\Omega_{n,j},\ 1 \le j \le n\}$, then (2.7) just means

$$C_n = (1+r)^{-1}\mathbf{E}^*[C_{n+1}|\mathcal{F}_n]. \tag{2.8}$$

In other words, the sequence of discounted values of the contingent claim $\{(1+r)^{-n}C_n,\ 0 \le n \le N\}$ forms a $\mathbf{P}^*$- martingale. Here $(1+r)^{-n}$ is called the *discount factor* at time n. In particular, from the last statement of the previous section we know that $\mathbf{P}^*$ is the unique probability measure on Ω under which the sequence $\{(1+r)^{-n}S_n, 0 \le n \le N\}$ of the discounted stock prices forms a martingale. We call a probability measure with such property a *risk-neutral probability measure* or *martingale measure.*

From (2.8) we get a formula for pricing the contingent claim ξ:

$$C_n = (1+r)^{-(N-n)}\mathbf{E}^*[\xi|\mathcal{F}_n]. \tag{2.9}$$

This formula is an example of an important general principle in contingent claim pricing, known as *risk-neutral valuation principle*, which states that any security dependent on a stock price can be valued on the assumption that the world is risk-neutral.

Assume now $\xi = f(S_N)$ with f being a positive function. We are going to deduce an explicit expression for C_n. Put $T_n = S_n/S_{n-1}$, for $n = 1, \cdots, N$. It is easy to verify that the random variables $T_1, \cdots, T_n$ are independent, identically distributed (i.i.d.) and their distribution is: $\mathbf{P}^*(T_1 = u) = q = 1 - \mathbf{P}^*(T_1 = d)$. In particular, for each $i \geq n+1$, T_i is independent of $\mathcal{F}_n$. Consequently, since $S_N = S_n \prod_{i=n+1}^N T_i$, from (2.9) we can apply Theorem 1.1 to get

$$\begin{aligned}
C_n &= (1+n)^{-(N-n)}\mathbf{E}^*\Big[f(S_n \prod_{i=n+1}^{N} T_i)\,\Big|\,\mathcal{F}_n\Big] \\
&= (1+n)^{-(N-n)}\mathbf{E}^*\Big[f(x \prod_{i=n+1}^{N} T_i)\Big]\Big|_{x=S_n} \\
&= (1+r)^{-(N-n)} \sum_{j=0}^{N-n} \binom{N-n}{j} q^j (1-q)^{N-n-j} f(S_n u^j d^{N-n-j}).
\end{aligned}$$

2.2 Basic concepts of the discrete-time model

Now we turn to the general discrete-time model. Let N be the time horizon. Uncertainty up to time N is represented by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω represents the set of all possible states. Let $\mathcal{F}_n$ be a sub-σ-algebra of $\mathcal{F}$, which represents the information available at time n. Then $\{\mathcal{F}_n, 0 \leq n \leq N\}$ constitutes a filtration, i.e. an increasing sequence of sub-σ- algebras of $\mathcal{F}$. For notational convenience, we put $\mathcal{F}_{-1} = \mathcal{F}_0$.

Assume that the market consists in $(d+1)$ assets, whose prices at time n constitute an adapted $\mathbf{R}^{d+1}$-valued non-negative random vector $S_n = (S_n^0, \cdots, S_n^d)$. The asset indexed by 0 is a riskless asset whose price at time n is $S_n^0 = S_0^0(1+r)^n$, where $r > 0$ is the riskless interest rate per unit time. Other assets are risky. In the following we assume $S_0^0 = 1$ and define $\beta_n = (1+r)^n$ and denote by γ_n the discount factor $(1+r)^{-n}$ at time n.

A *trading strategy* is a predictable $\mathbf{R}^{d+1}$-valued stochastic sequence

$$\phi = \{(\phi_n^0, \cdots, \phi_n^d),\ 0 \leq n \leq N\},$$

where ϕ_n^i denotes the number of shares of asset i held in the portfolio at time n. The assumption means that we can only utilize the information available at time $(n-1)$ to make a decision on the positions in the portfolio at time n. If $\phi_n^0 < 0$, we have borrowed the amount $|\phi_n^0|$ in the riskless asset. If $i \geq 1$ and $\phi_n^i < 0$, we say that we are *short* a number ϕ_n^i of asset i. Short-selling and borrowing are allowed.

For $a, b \in \mathbf{R}^d$, let $a \cdot b$ denote the scalar product of a and b. The *wealth of the portfolio* $\phi_n = (\phi_n^0, \cdots, \phi_n^d)$ at time n is

$$V_n(\phi) = \phi_n \cdot S_n = \sum_{i=0}^{d} \phi_n^i S_n^i\,. \tag{2.10}$$

The discounted wealth is

$$\widetilde{V}_n(\phi) = \gamma_n V_n(\phi) = \phi_n \cdot \widetilde{S}_n, \tag{2.11}$$

where $\widetilde{S}_n = (1, \gamma_n S_n^1, \cdots, \gamma_n S_n^d)$ is the vector of discounted prices.

A strategy (ϕ_n) is called *self-financing* if

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n, \ \forall 0 \leq n \leq N-1. \tag{2.12}$$

It means that at time n, once the price vector S_n is quoted, the investor readjusts his/her positions from ϕ_n to ϕ_{n+1} without bringing or withdrawing any wealth. It is easy to prove that (2.12) is equivalent to

$$V_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta S_i, \quad \forall 1 \leq n \leq N, \tag{2.13}$$

or

$$\widetilde{V}_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta \widetilde{S}_i, \ \forall 1 \leq n \leq N, \tag{2.14}$$

where $\Delta S_i = S_i - S_{i-1}$ and $\Delta \widetilde{S}_i = \widetilde{S}_i - \widetilde{S}_{i-1}$.

In contrast to a self-financing strategy, a ***strategy with consumption*** is a strategy (ϕ_n) with the property that there is a predictable non-negative sequence (c_n) null at $n = 0$, such that

$$\phi_{n+1} \cdot S_n = \phi_n \cdot S_n - c_{n+1}, \quad 0 \leq n \leq N-1. \tag{2.15}$$

Here c_n represents the wealth withdrawn for consumption at time n. The wealth at time n of a strategy with consumption (ϕ_n) is still defined by $V_n(\phi) = \phi_n \cdot S_n$. It is easy to see that (2.15) is equivalent to

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j - \sum_{j=1}^{n} c_j, \quad 1 \leq n \leq N. \tag{2.16}$$

A strategy (ϕ_n) is said to be *admissible* if its wealth process $V(\phi)$ is non-negative. A strategy (ϕ_n) is said to be *tame* (or *c-admissible*) if its discounted wealth process is bounded from below by some real constant c. The above two notions are standard in the literature. However, it seems that these definitions are too restrictive. For example, a strategy with a short sale of assets is not necessarily a tame strategy. So we propose another notion. A strategy (ϕ_n) is said to be *allowable* if there exists a positive constant c such that $V(\phi)_n \geq -c \sum_{i=0}^{d} S_n^i, \ 0 \leq n \leq N$.

An *arbitrage strategy* is an allowable self-financing strategy with zero initial wealth and non-zero final wealth. In option pricing theory we always assume that the market excludes any arbitrage opportunity.

Lemma 2.1 Let Γ denote the convex cone of non-negative non-zero random variables (ξ is zero r.v., if $\mathbf{P}(\xi = 0) = 1$). For any (vector-valued) predictable process $(\phi_n^1, \cdots, \phi_n^d)_{0 \leq n \leq N}$, we define

$$\widetilde{G}_n(\phi) = \sum_{j=1}^{n} (\phi_j^1 \Delta \widetilde{S}_j^1 + \cdots + \phi_j^d \Delta \widetilde{S}_j^d).$$

Then there exists a predictable process (ϕ_n^0) such that $(\phi_n^0, \phi_n^1, \cdots, \phi_n^d)$ constitutes a self-financing strategy with initial wealth zero and that its discounted wealth process is just $(\widetilde{G}_n(\phi))$. If the market has no arbitrage, then $\widetilde{G}_N(\phi) \notin \Gamma$.

Proof We put

$$\phi_n^0 = \sum_{j=1}^{n-1}(\phi_j^1 \Delta \widetilde{S}_j^1 + \cdots + \phi_j^d \Delta \widetilde{S}_j^d) - (\phi_n^1 \widetilde{S}_{n-1}^1 + \cdots + \phi_n^d \widetilde{S}_{n-1}^d). \tag{2.17}$$

Then (ϕ_n^0) is a predictable process. By (2.11) and (2.14) it is readily verified that the strategy $\phi = (\phi^0, \cdots, \phi^d)$ is a self-financing strategy with initial wealth zero and its discounted wealth process is just $(\widetilde{G}_n(\phi))$.

Now we are going to prove the second statement. Suppose that $\widetilde{G}_N(\phi) \in \Gamma$. Put

$$m = \sup\{k : \mathbf{P}(\widetilde{G}_k(\phi) < 0) > 0\}.$$

Here, by convention, $\sup \emptyset = 0$. We denote by A the event $\{\widetilde{G}_m(\phi) < 0\}$ and define a new process ψ by

$$\psi_j(\omega) = \begin{cases} 0 & \text{if } j \le m \\ I_A(\omega)\phi_j(\omega) & \text{if } j > m \end{cases}$$

Then $\psi = (\psi_1, \cdots, \psi_N)$ is a predictable process and

$$\widetilde{G}_j(\psi) = \begin{cases} 0 & \text{if } j \le m \\ I_A\big(\widetilde{G}_j(\phi) - \widetilde{G}_m(\phi)\big) & \text{if } j > m \end{cases}$$

Thus, $\widetilde{G}_j(\psi) \ge 0$ for all $j \in \{0, \cdots, N\}$ and $\widetilde{G}_N(\psi) > 0$ on A. This contradicts the assumption of no arbitrage, because $(\widetilde{G}_n(\psi))$ is the discounted wealth process of an admissible self-financing strategy with initial wealth zero.

2.3 Martingale characterization for no-arbitrage

Characterizing stochastic processes which can be transformed into martingales by means of an equivalent change of measure is of particular interest in financial economics. Harrison and Kreps (1979), Harrison and Pliska (1981), and Kreps (1981) studied the relationship between the question of the existence of equivalent martingale measures for the securities price process and the economic concept of "no-arbitrage". Under appropriate assumptions on the price process, such as integrability, they obtained some fundamental results. The most important fact in the option pricing theory, as was pointed out in Harrison and Kreps (1979), is that the absence of arbitrage follows from the existence of an equivalent martingale measure for the (discounted) price process of securities. Fortunately, the proof of this fact is quite easy. The converse fact that the absence of arbitrage implies the existence of an equivalent measure is however rather difficult. In the discrete-time case, the proof of this fact for an arbitrary probability space was given by Dalang-Morton-Willinger (1990).

We start with the assumption that Ω is a finite set and $\mathcal{F}$ is the family of all subsets of Ω and $\mathbf{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. Moreover, we assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_N = \mathcal{F}$. Under these assumptions every real-valued random variable takes only a finite number of values, and is thus bounded.

The following theorem gives a complete characterization of no-arbitrage.

Theorem 2.2 Under the above assumptions, there exists no arbitrage strategy if and only if there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ (in the present case, this means that $\mathbf{P}^*(\{\omega\}) > 0$ for all $\omega \in \Omega$) such that the $\mathbf{R}^d$-valued process of discounted prices $(\widetilde{S}_n)_{0 \le n \le N}$ of assets is a $\mathbf{P}^*$-martingale.

Proof Sufficiency. Assume that there exists a probability $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that $(\widetilde{S}_n)$ is a martingale. By Proposition 1.4 and (2.14), for any allowable self-financing strategy (ϕ_n), the discounted wealth process $\widetilde{V}_n(\phi)$ is a $\mathbf{P}^*$-martingale. In particular, if $V_0(\phi) = 0$, then $\mathbf{E}[\widetilde{V}_N(\phi)] = 0$. Since $V_N(\phi) \geq 0$ and $\mathbf{P}(\{\omega\}) > 0, \forall \omega \in \Omega$, we must have $V_N(\phi) = 0$. That means there exists no arbitrage strategy.

Necessity. Assume there exists no arbitrage strategy. Let $\mathcal{V}$ be the set of random variables $\widetilde{G}_N(\phi)$ with $\phi = (\phi_n^1, \cdots, \phi_n^d)$ being ($\mathbf{R}^d$-valued) predictable process. By Lemma 2.1, $\mathcal{V} \cap \Gamma = \emptyset$. In particular, $\mathcal{V}$ does not intersect the convex compact set $K = \{X \in \Gamma : \sum_\omega X(\omega) = 1\}$. Put $C = K - \mathcal{V} = \{x - y : x \in K, y \in \mathcal{V}\}$. Since $\Omega = \{\omega_1, \cdots, \omega_m\}$ is a finite set, we can regard a random variable defined on Ω as a vector in $\mathbf{R}^m$. Thus C is a closed convex subset of $\mathbf{R}^m$ which doesn't contain the origin. By the convex sets separation theorem (see Dudley (1989), p.152 or Lamberton-Lapeyer (1996), p.178) there exists a linear functional f defined on $\mathbf{R}^m$ such that $f(x) \geq \alpha, \forall x \in C$ with an $\alpha > 0$. Since $C = K - \mathcal{V}$ and $\mathcal{V}$ is a subspace of $\mathbf{R}^m$ we must have $f(x) = 0, \forall x \in \mathcal{V}$ and $f(x) \geq \alpha > 0, \forall x \in K$. It means that there exists $\big(\lambda(\omega)\big)_{\omega \in \Omega}$ such that (1)$\forall X \in K, \sum_\omega \lambda(\omega) X(\omega) > 0$ and (2) for any predictable ϕ, $\sum_\omega \lambda(\omega) \widetilde{G}_N(\phi) = 0$. From (1) we see that $\lambda(\omega) > 0$ for all $\omega \in \Omega$. Put

$$\mathbf{P}^*(\{\omega\}) = \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')}.$$

Then $\mathbf{P}^*$ is equivalent to $\mathbf{P}$ and by (2), for any predictable ϕ,

$$\mathbf{E}^* \sum_{j=1}^N \phi_j \Delta \tilde{S}_j = 0.$$

Therefore, according to Proposition 1.5, $(\widetilde{S}_n)_{0 \leq n \leq N}$ is a $\mathbf{P}^*$-martingale.

Now we turn to an arbitrary probability space case. In this case we have still a martingale characterization for no-arbitrage. The result, due to Dalang et al. (1990), is usually referred to as the *fundamental theorem of asset pricing*.

Theorem 2.3 There exists no arbitrage strategy if and only if there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that the $\mathbf{R}^d$-valued process of discounted prices $(\widetilde{S}_n)_{0 \leq n \leq N}$ of assets is a $\mathbf{P}^*$-martingale. In this case $\mathbf{P}^*$ may be chosen such that the Radon-Nikodym derivative $d\mathbf{P}^*/d\mathbf{P}$ is bounded.

The proof of the "if" part is quite easy, but the proof of the "only if" part is complicated. So we omit the proof.

Note that the integrability of the price process is not assumed in Theorem 2.3. Some special cases of Theorem 2.3 were derived previously e.g. by Harrison and Kreps (1979), Taqqu and Wilinger (1987). The original proof of Theorem 2.3 is mainly based on a measurable selection theorem. Alternative (elementary) proofs are due to Schachermayer (1992), Kabanov and Kramkov (1994) and Rogers (1994). For an extension of Theorem 2.6 to the infinite-horizon case see Scharchermayer (1994).

2.4 Complete markets and option pricing in finite state case

A (*European*) *contingent claim* is an $\mathcal{F}$-measurable non-negative random variable. Let $\mathbf{P}^*$ be an equivalent martingale measure. A contingent claim ξ is said to be $\mathbf{P}^*$-*attainable* (or *replicatable*), if the discounted value $\gamma_T \xi$ is $\mathbf{P}^*$-integrable and there exists an admissible self-financing strategy such that its discounted wealth process is a $\mathbf{P}^*$-martingale and its terminal wealth at time N is ξ. In this case we say that such a strategy is a $\mathbf{P}^*$-*hedging strategy* for ξ.

In this section we only consider the case of finite state. In this case, a market is said to be *complete* if any contingent claim is attainable. The following theorem characterizes the completeness of a market. We refer to Lamberton and Lapeyre (1996) for its proof.

Theorem 2.4 Assume Ω is a finite set. A market with no-arbitrage is complete if and only if there exists a unique probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which the $\mathbf{R}^d$-valued process of discounted prices $(\widetilde{S}_n)_{0\leq n\leq N}$ of assets is a $\mathbf{P}^*$-martingale.

For an arbitrary probability space case a characterization theorem for the completeness was obtained by Willinger and Taqqu (1988). In this case one needs to imposes a condition on the structure of the filtration $(\mathcal{F}_n)$.

In the following we assume that the market has no arbitrage and is complete. We denote by $\mathbf{P}^*$ the unique martingale measure.

Let ξ be a European contingent claim with maturity N. Assume $\gamma_T\xi$ is $\mathbf{P}^*$-integrable. Let $\underset{\sim}{\phi}$ be a heading strategy for ξ. By Proposition 1.4 the discounted wealth process $(\widetilde{V}_n(\phi))$ is a $\mathbf{P}^*$-martingale. Consequently,

$$V_n(\phi) = \beta_n \mathbf{E}^*[\gamma_N \xi | \mathcal{F}_n], \quad n = 0, \cdots, N, \tag{2.18}$$

where γ_N is the discount factor at time N. This formula gives the value of the contingent claim ξ at any time $n \leq N$. If (ψ_n) is a strategy with consumption, then the discounted wealth process $(\widetilde{V}_n(\psi))$ is a $\mathbf{P}^*$-supermartingale.

Now we turn to the pricing of an *American option* with maturity N, which is defined as an $(\mathcal{F}_n)$-adapted non-negative sequence (Z_n). This means that the option pays Z_n if it is exercised at time n. We will price the American option by a "backward induction" argument. The value U_N at time N of the option is obviously Z_N. If a writer of an American option wants to cover his/her positions at times $N-1$ and N, he/she must earn at time $N-1$ the maximum between Z_{N-1} and the amount necessary at time $N-1$ to replicate Z_N at time N. By (2.18), the latter amount is $\gamma_{N-1}^{-1}\mathbf{E}^*[\widetilde{Z}_N|\mathcal{F}_{N-1}]$. So from the writer's account the value U_{N-1} at time $N-1$ of the option should be defined as

$$U_{N-1} = \max\Big(\mathrm{Z_{N-1}}, \beta_{\mathrm{N-1}}\mathbf{E}^*[\widetilde{\mathrm{Z}}_{\mathrm{N}}|\mathcal{F}_{\mathrm{N-1}}]\Big).$$

By induction, for $n = 0, \cdots, N-1$, we define the value at time n of the American option by

$$U_n = \max\Big(\mathrm{Z_n}, \beta_{\mathrm{n}}\mathbf{E}^*[\gamma_{\mathrm{n+1}}\mathrm{U_{n+1}}|\mathcal{F}_{\mathrm{n}}]\Big). \tag{2.19}$$

It is easy to prove the following theorem.

Theorem 2.5 The sequence $(\widetilde{U}_n)_{0\leq n\leq N}$ is a $\mathbf{P}^*$-supermartingale. It is the smallest $\mathbf{P}^*$-supermartingale that dominates the sequence $(\widetilde{Z}_n)_{0\leq n\leq N}$.

Remark 2.6 By Doob's decomposition of $\widetilde{U}$ it is easy to see that there is a strategy with consumption (ϕ_n) such that $V_n(\phi) = U_n$ for all $0 \leq n \leq N$. Obviously, this strategy *super-hedges* the American option in the sense that $V_n(\phi) \geq Z_n$ for all $0 \leq n \leq N$. On the other hand, if a strategy with consumption (ϕ_n) super-hedges the American option (Z_n), then we can prove that $V_n(\phi) \geq U_n$ for all $0 \leq n \leq N$. This means that the value sequence (U_n) of the American option defined by (2.19) is the minimal wealth sequence within all wealth sequences of the super-hedging strategies.

CHAPTER 3

Brownian Motion and Itô Calculus

In the previous chapter we have studied discrete-time models for financial markets. However, the price changes in the market are actually so frequent that a discrete-time model can barely follow the moves. Over the past three decades, the continuous-time model has proven to be a convenient and productive tool in finance. In fact, the continuous-time approach often leads to closed form solutions or analytical expressions, suitable for numerical computations and allows the use of stochastic calculus, which sometimes facilitates the derivation of more precise theoretical solutions than can otherwise be derived from its discrete-time counterpart.

In this chapter we will give a short presentation of Itô calculus. We will review some fundamental results on continuous-time martingales and summarize the construction of the stochastic integrals with respect to Brownian motion; present some useful tools for Itô calculus, such as Itô's formula, the martingale representation theorem, Girsanov's theorem and the Feynman-Kac formula; and introduce Itô stochastic differential equations. Finally, we make a short presentation of continuous semimartingales. The proofs of all results are omitted. Readers can find the omitted proofs in Karatzas and Shreve (1988) and Revuz and Yor (1991).

3.1 Brownian motion and martingales

In the continuous-time case, we encounter two types of time horizons: $[0,T]$ or $[0,\infty)$, where $T>0$ is a constant. We will work on a given complete probability space $(\Omega,\mathcal{F},\mathbf{P})$, as well as a *filtration* $(\mathcal{F}_t)$ satisfying the *usual conditions.* Here a filtration is an increasing family of σ-algebra included in $\mathcal{F}$, and by the usual conditions we mean that each $\mathcal{F}_t$ contains the null sets of $\mathcal{F}$ and $(\mathcal{F}_t)$ is right-continuous (i.e. $\cap_{s>t}\mathcal{F}_s=\mathcal{F}_t$). A complete probability space $(\Omega,\mathcal{F},\mathbf{P})$ together with a usual fltration $(\mathcal{F}_t)$ will be called a *filtered probability space* and denoted by $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbf{P})$.

Definition 3.1 A (*standard*) *Brownian motion* $(B_t)_{t\geq 0}$ on a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$ is a real valued continuous process, with independent and stationary increments, such that $B_0=0$ and for $s>t, B_s-B_t$ is normally distributed with mean zero and variance $(s-t)$.

If $(B_t^i), 1\leq i\leq d$ are independent Brownian motions, we call the $\mathbf{R}^d$-valued process $(B_t)=(B_t^1,\cdots,B_t^d)$ a d-dimensional Brownian motion. Let $\mathcal{F}_t^B$ denote the σ-algebra generated by the union of $\sigma(B_s, s\leq t)$ and the null sets of $\mathcal{F}$. We call $(\mathcal{F}_t^B)_{t\geq 0}$ the *natural filtration* of the Brownian motion (B_t). It is well-known that $(\mathcal{F}_t^B)_{t\geq 0}$ satisfies the usual condition.

Let $(\mathcal{F}_t)$ be a usual filtration on a probability space $(\Omega,\mathcal{F},\mathbf{P})$. We shall also need a definition of Brownian motion w.r.t. $(\mathcal{F}_t)$.

Definition 3.2 An $\mathbf{R}^d$-valued continuous process (X_t) with $X_0=0$ is an $(\mathcal{F}_t)$-*Brownian motion* or *Wiener process*, if for any $t\geq 0, X_t$ is $\mathcal{F}_t$-measurable, and for

$s > t, X_s - X_t$ is independent of the σ-algebra $\mathcal{F}_t$ and is normally distributed in $\mathbf{R}^d$ with mean zero and covariance matrix $(s-t)I$.

In Chapter 1 we have defined the notions of martingale, supermartingale or submartingale for discrete-time processes. For continuous time processes the same notions can be defined in a similar fashion. We will sometimes need the notion of a *local martingale.* In order to give its definition we need the notion of a *stopping time.* An $(\mathcal{F}_t)$-stopping time is a non-negative random variable τ with the property that for any $t \geq 0$, the set $[\tau \leq t]$ belongs to $\mathcal{F}_t$. A real-valued right-continuous process (M_t) is called a local martingale w.r.t. the filtration $(\mathcal{F}_t)$, if there exists an increasing sequence (τ_n) of stopping times tending to ∞ such that for each $n \geq 1$ the stopped process X^{τ_n} is a martingale, where $X_t^{\tau_n} = X_{t\wedge\tau_n}$. Note that almost all paths of a local martingale have finite left-hand limits on $(0,\infty)$ and any non-negative local martingale is a supermartingale.

When (M_t) is a right-continuous adapted process defined only on a finite interval $[0,T]$, it is called a local martingale if there exists an increasing sequence (τ_n) of stopping times tending to T such that $\mathbf{P}(\tau_n = T) \to 1$ and for each $n \geq 1$ the stopped process X^{τ_n} is a martingale.

Now we present subsequently *Doob's optional sampling theorem*, the *Doob inequality* and the *Doob-Meyer decomposition*, which are fundamental in the martingale theory.

Theorem 3.3 If (M_t) is a right-continuous martingale (resp. supermartingale) w.r.t. the filtration $(\mathcal{F}_t)$ and if τ_1 and τ_2 are two bounded stopping times such that $\tau_1 \leq \tau_2$, then M_{τ_2} is integrable and

$$\mathbf{E}[M_{\tau_2}|\mathcal{F}_{\tau_1}] = M_{\tau_1} \quad (\text{resp. } \leq M_{\tau_1}), \ \mathbf{P}\text{-a.s..}$$

Theorem 3.4 Let $T > 0$ be a real number. If $(M_t)_{0\leq t\leq T}$ is a right-continuous martingale, then

$$\mathbf{E}\big[\sup_{0\leq t\leq T} |M_t|^2\big] \leq 4\mathbf{E}[|M_T|^2].$$

Theorem 3.5 A right-continuous supermartingale X can be expressed in the form $X = M - A$, where M is a martingale and A is an adapted non-decreasing process null at 0. If X is continuous, then such a decomposition is unique and M and A are also continuous.

The following theorem is called *Lévy's martingale characterization* for Brownian motion.

Theorem 3.6 Let (X_t) be an $\mathbf{R}^d$-valued $(\mathcal{F}_t)$-adapted continuous process. Then X is an $(\mathcal{F}_t)$-Brownian motion, if and only if for any $1 \leq i,j \leq d, (X_t^i)$ and $(X_t^i X_t^j - \delta_{ij}t)$ are (local) martingale.

3.2 Stochastic integration and martingale representation

Now we turn to the construction of stochastic integrals . Let $T > 0$ be a real number and $(B_t)_{0\leq t\leq T}$ an $(\mathcal{F}_t)$-Brownian motion. We want to define (indefinite) integrals of the type $\int_0^t \theta(s)dB_s$ for a certain class of measurable processes θ. Here by measurability we mean the joint measurability w.r.t. the product σ-algebra $\mathcal{B}([0,T]) \times \mathcal{F}_T$.

We start with a *simple process* $(\theta(t))$. Here we mean that there is a partition of $[0,T]$: $0 = t_0 < t_1 <, \cdots, < t_N = T$, such that for all $j \geq 0$,

$$\theta(t) = \xi_j, \quad t \in (t_j, t_{j+1}],$$

where ξ_j is an $\mathcal{F}_{t_j}$-measurable bounded random variable. For such a process it is reasonable to define, for any $t_k < t \leq t_{k+1}$,

$$\int_0^t \theta(s)dB_s = \sum_{0\leq j\leq k-1} \xi_j(B_{t_{j+1}} - B_{t_j}) + \xi_k(B_t - B_{t_k}).$$

Obviously, for any $t \in [0,T]$ the above expression can be rewritten as

$$\int_0^t \theta(s)dB_s = \sum_{0\leq j\leq N} \xi_j(B_{t_{j+1}\wedge t} - B_{t_j\wedge t}).$$

One can check that $(\int_0^t \theta(s)dB_s)$ is a continuous martingale w.r.t. $(\mathcal{F}_t)$, and

$$\mathbf{E}\Big[\Big(\int_0^t \theta(s)dB_s\Big)^2\Big] = \mathbf{E}\Big[\int_0^t \theta(s)^2 ds\Big]. \tag{3.1}$$

Consequently, by the Doob inequality (Theorem 3.4) we get

$$\mathbf{E}\Big[\sup_{t\leq T}\Big|\int_0^t \theta(s)dB_s\Big|^2\Big] \leq 4\mathbf{E}\Big[\int_0^t \theta(s)^2 ds\Big]. \tag{3.2}$$

Now we extend the above integral to a lager class of measurable adapted processes. Let $\mathcal{L}$ denote the set of all measurable adapted processes defined on $[0,T]$. Put

$$\mathcal{H}^2 = \Big\{\theta : \theta \in \mathcal{L},\ \mathbf{E}\Big[\int_0^T \theta(s)^2 ds\Big] < \infty\Big\}.$$

An important fact is that if θ is in $\mathcal{H}^2$, then there exists a sequence (θ^n) of simple process such that

$$\lim_{n\to\infty} \mathbf{E}\Big[\int_0^T |\theta(s) - \theta^n(s)|^2 ds\Big] = 0.$$

For a proof see Karatzas and Shreve (1988), p.134. By using this fact and (3.2) we can find a subsequence of (θ^n), denoted again by (θ^n), such that $\int_0^t \theta^n(s)dB_s$ converges a.s. to a continuous process, which is denoted by $\int_0^t \theta(s)dB_s$ or simply $(\theta.B)_t$. We call it the *stochastic integral* of θ w.r.t. Brownian motion (B_t). One can prove that $(\int_0^t \theta(s)dB_s)$ is a continuous martingale w.r.t. $(\mathcal{F}_t)$, and (3.1) and (3.2) are still valid.

Let (B_t) be a d-dimensional $(\mathcal{F}_t)$-Brownian motion and $H = (H^1, \cdots, H^d)$ be an $\mathbf{R}^d$-valued process in $(\mathcal{H}^2)^d$. We can define the stochastic integral of H w.r.t. B, denoted again by $H.B$, as

$$\int_0^t H_s dB_s = \sum_{j=1}^d \int_0^t H_s^j dB_s^j.$$

Now we put

$$\mathcal{L}^2 = \Big\{\theta : \theta \in \mathcal{L},\ \int_0^T \theta(s)^2 ds < \infty\Big\},$$

$$\mathcal{L}^1 = \Big\{\theta : \theta \in \mathcal{L},\ \int_0^T |\theta(s)| ds < \infty\Big\}.$$

If $\theta \in (\mathcal{L}^2)^d$ we can still define the stochastic integral of θ w.r.t. B. In this case, $(\int_0^t \theta(s)dB_s)$ is a local martingale. A process of the form

$$X_t = X_0 + \int_0^t \theta(s)dB_s + \int_0^t \phi(s)ds \tag{3.3}$$

with X_0 being $\mathcal{F}_0$-measurable and $\theta \in (\mathcal{L}^2)^d, \phi \in \mathcal{L}^1$ is called an *Itô process.* Since a continuous local martingale with finite variation must equal its initial value, the decomposition of an Itô process into a local martingale and a process of finite variation is unique.

Finally, the stochastic integral can be defined on the infinite interval $[0, \infty)$. In fact, if θ is a measurable adapted process such that for any $t > 0$, $\int_0^t |\theta(s)|^2 < \infty$, then $\theta.B$ is a local martingale on $[0, \infty)$.

The following is the *martingale representation theorem* for a Brownian motion.

Theorem 3.7 Let (B_t) be a d-dimensional Brownian motion and $(\mathcal{F}_t^B)$ its natural filtration. Then (B_t) has the *martingale representation property* in the sense that for any local martingale w.r.t. $(\mathcal{F}_t^B)$, there exists some θ in $(\mathcal{L}^2)^d$ such that

$$M_t = M_0 + \int_0^t \theta(s)dB_s, \quad t \geq 0. \tag{3.4}$$

Moreover, such a θ is unique in the sense that if θ' is another process satisfying (3.4), then $\int_0^t |\theta(s) - \theta'(s)|^2 ds = 0, \forall t > 0$. In particular, any local $(\mathcal{F}_t^B)$-martingale is continuous.

3.3 Itô's formula and Girsanov's theorem

Let (X_t) be an Itô process given by (3.3) and (H_t) a measurable and adapted process such that $H\theta \in (\mathcal{L}^2)^d$ and $H\phi \in \mathcal{L}^1$. Then we can define the stochastic integral of H w.r.t. Itô process X as

$$\int_0^t H_s dX_s = \int_0^t H_s \theta(s)dB_s + \int_0^t H_s \phi(s)ds.$$

The following theorem provides the change of variables formula for Itô processes, called *Itô formula*, which is a powerful tool in Itô calculus.

Theorem 3.8 Let $B = (B^1, \cdots, B^d)$ be a d-dimensional $(\mathcal{F}_t)$- Brownian motion and $X = (X^1, \cdots, X^m)$ be an $\mathbf{R}^m$-valued Itô processes with

$$X_t^i = X_0^i + \sum_{j=1}^d \int_0^t \theta_j^i(s)dB_s^j + \int_0^t \phi^i(s)ds, \quad 1 \leq i \leq m. \tag{3.5}$$

If $f = f(t,x)$ is a function on $\mathbf{R}_+ \times \mathbf{R}^m$ such that it is twice differentiable w.r.t. x and once differentiable w.r.t. t, with continuous partial derivatives in (t,x), then we have

$$\begin{aligned} f(t, X_t) = f(t, X_0) \quad &+ \quad \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)dX_s^i \\ &+ \quad \frac{1}{2}\sum_{i,k=1}^m \int_0^t \frac{\partial^2 f}{\partial x_i x_k}(s, X_s)d\langle X^i, X^k\rangle_s, \end{aligned}$$

where, by definition

$$\langle X^i, X^k \rangle_t = \sum_{j=1}^{d} \int_0^t \theta_j^i(s)\theta_j^k(s)ds \tag{3.6}$$

is the so-called *quadratic covariation* process of X^i and X^k.

As a particular case of Itô's formula, we obtain the *integration by parts formula.*

Theorem 3.9 Let X and Y be two Itô processes. Then

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t X_s dY_s + \langle X, Y \rangle_t. \tag{3.7}$$

The following theorem is an simple application of Itô's formula.

Theorem 3.10 Let X be an Itô process. Put

$$\mathcal{E}(X)_t = \exp\Big\{X_t - X_0 - \frac{1}{2}\langle X, X \rangle_t\Big\}. \tag{3.8}$$

Then $\mathcal{E}(X)$ is the unique solution to the following stochastic integration equation:

$$Z_t = 1 + \int_0^t Z_s dX_s.$$

Expression (3.8) is called *Doléans-Dade's exponential formula.* In particular, if X is a local martingale, then $\mathcal{E}(X)$ is also a local martingale.

The following *Novikov Theorem* gives a useful sufficient condition for $\mathcal{E}(X)$ to be a martingale.

Theorem 3.11 Let $(B_t)_{0\le t\le T}$ be a d-dimensional Brownian motion. If $\theta \in (\mathcal{L}^2)^d$ and satisfies *Novikov's condition*

$$\mathbf{E}\Big[\exp\Big(\frac{1}{2}\int_0^T |\theta(s)|^2 ds\Big)\Big] < \infty, \tag{3.9}$$

then $\mathcal{E}(\theta.B)$ is a martingale (or equivalently, $\mathbf{E}[\mathcal{E}(\theta.B)_T] = 1$).

The following is called *Girsanov's Theorem*, which describes the structure of a Brownian motion under an equivalent change of probability measure.

Theorem 3.12 Let $(B_t)_{0\le t\le T}$ be a d-dimensional $(\mathcal{F}_t)$-Brownian motion. If $\theta \in (\mathcal{L}^2)^d$ and $\mathbf{E}[\mathcal{E}(\theta.B)_T] = 1$, then $B_t^* = B_t - \int_0^t \theta(s)ds$ is a d-dimensional $(\mathcal{F}_t)$-Brownian motion under a new probability measure $\mathbf{P}^*$ with $\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}(\theta.B)_T$.

As an application of Girsanov's theorem we obtain the following formula:

$$\mathbf{E}[f(B^*.)g(B_T^*)] = \mathbf{E}[\mathcal{E}(\theta.B)f(B.)g(B_T)], \tag{3.10}$$

where f is a Borel function on $C([0,T], \mathbf{R}^d)$ and g is a Borel function on $\mathbf{R}$, because

$$\mathbf{E}[f(B^*.)g(B_T^*)] = \mathbf{E}^*\Big[\Big(\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|_{\mathcal{F}_T}\Big)^{-1} f(B^*.)g(B_T^*)\Big] = \mathbf{E}^*[\mathcal{E}(-\theta.B^*)_T f(B^*.)g(B_T^*)],$$

and (B_t^*) is a Brownian motion under $\mathbf{P}^*$.

The following theorem is due to Fujisaki-Kallianpur-Kunita (1972). A proof of this result can also be found in Yan (1980b).

Theorem 3.13 Under the assumption of Theorem 3.12, if $(\mathcal{F}_t)$ is the natural filtration of (B_t), then (B_t^*) has the martingale representation property w.r.t. $(\mathcal{F}_t)$ under $\mathbf{P}^*$.

3.4 Itô SDEs and the Feynman-Kac formula

Let $(B_t)_{t\geq 0}$ be a d-dimensional $(\mathcal{F}_t)$-Brownian motion. Let $b : \mathbf{R}_+ \times \mathbf{R}^m \to \mathbf{R}^m$ and $\sigma : \mathbf{R}_+ \times \mathbf{R}^m \to M^{m,d}$ be Borel measurable maps, where $M^{m,d}$ is the set of all real matrices with m rows and d columns. An $\mathbf{R}^m$-valued continuous $(\mathcal{F}_t)$-adapted process X is said to be a *solution* of the following *Itô stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = \xi, \tag{3.11}$$

with $\xi = (\xi^1, \cdots, \xi^m)$ being $\mathcal{F}_0$-measurable, if X satisfies

$$X_t^i = \xi^i + \int_0^t b^i(s, X_s)ds + \sum_{j=1}^d \int_0^t \sigma_j^i(s, X_s)dB_s^j, \ 1 \leq i \leq m, \ t \geq 0. \tag{3.12}$$

Such a solution is sometimes called a *strong solution*. There is also what is known as a *weak solution*, which we don't discuss here, because in financial mathematics we have no need for it.

In the sequel we denote

$$|x|^2 = \sum_{i=1}^m x_i^2, \quad |\gamma|^2 = \mathrm{tr}(\gamma\gamma^{\mathrm{T}}) = \sum_{\mathrm{j}=1}^{\mathrm{d}} \sum_{\mathrm{i}=1}^{\mathrm{m}} (\gamma^{\mathrm{ij}})^2$$

for $x \in \mathbf{R}^m$ and $\gamma \in M^{m,d}$.

Theorem 3.14 If b and σ are *Lipschtiz* in x:

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y| \tag{3.13}$$

and satisfy the *linear growth condition* in x:

$$|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|), \tag{3.14}$$

where K is a constant, then (3.11) has a unique solution X. Moreover, if $\mathbf{E}[|\xi|^2] < \infty$, then for all $0 \leq T < \infty$ we have $\mathbf{E}\sup_{0\leq t\leq T} |X_t|^2] < \infty$.

Remark 3.15 If b and σ are only *locally Lipschitz* in the sense that for each positive constant L there is a constant K such that (3.13) is satisfied for x and y with $|x| \leq L, |y| \leq L$, then (3.11) still has a unique solution.

The unique solution to (3.11) is a continuous *strong Markov process*, which is usually called an *Itô diffusion*. We denote by $a(t, x)$ the $(d \times d)$ matrix $\sigma(t, x)\sigma^T(t, x)$. We call a the *diffusion matrix* and b the *drift vector* of the diffusion. For every $t \geq 0$, we associate with the diffusion (X_t) a second-order differential operator

$$(\mathcal{A}_t f)(x) = \frac{1}{2} \sum_{i,k=1}^d a_{i,k}(t, x) \frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^d b_i(t, x) \frac{\partial f(x)}{\partial x_i}, \ f \in C^2(\mathbf{R}^d). \tag{3.15}$$

If b and σ do not depend on t, the equation (3.11) and its solution are said to be *time-homogeneous*. In this case, the operator $\mathcal{A}_t, t \geq 0$, is reduced to

$$(\mathcal{A}f)(x) = \frac{1}{2} \sum_{i,k=1}^d a_{i,k}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i}, \ f \in C^2(\mathbf{R}^d). \tag{3.16}$$

$\mathcal{A}$ is called the *generator* of the diffusion (X_t).

The following theorem provides a generalized version of the *Feynman-Kac formula*, which provides a probabilistic representation for the solution of the *Cauchy problem*

$$-\frac{\partial u}{\partial t} + ku = \mathcal{A}_t u + g, \quad (t,x) \in [0,T) \times \mathbf{R}^d \tag{3.17}$$

subject to the terminal condition

$$u(T,x) = f(x), \quad x \in \mathbf{R}^d. \tag{3.18}$$

Here $f : \mathbf{R}^d \to \mathbf{R}, k : \mathbf{R}^d \to \mathbf{R}_+$, and $g : [0,T] \times \mathbf{R}^d$ are continuous functions, and f and g satisfy the *polynomial growth condition* in x (see below (3.19)) or are non-negative.

Theorem 3.16 Let u be a continuous, real valued function on $[0,T] \times \mathbf{R}^d$, of class $C^{1,2}$ on $[0,T) \times \mathbf{R}^d$ satisfying (3.17) and (3.18). Assume that u satisfies the polynomial growth condition in x:

$$\sup_{0 \le t \le T} |u(t,x)| \le M(1 + |x|^{2\mu}), \quad x \in \mathbf{R}^d, \tag{3.19}$$

for some constant $M > 0$, $\mu \ge 1$. Then u admits the representation

$$\begin{aligned} u(t,x) \;&= \mathbf{E}^{t,x}\Big[f(X_T)\exp\Big\{-\textstyle\int_t^T k(\theta, X_\theta)d\theta\Big\} \\ &+ \textstyle\int_t^T g(s, X_s)\exp\Big\{-\textstyle\int_t^s k(\theta, X_\theta)d\theta\Big\}ds\Big], \end{aligned} \tag{3.20}$$

where $\{\mathbf{P}^{t,x}, t \ge 0, x \in \mathbf{R}^d\}$ is the family of probability measures associated with the Markov process (X_t). In particular, such a solution to (3.17) and (3.18) is unique.

In particular, if k does not depend on t, then

$$\begin{aligned} u(t,x) \;&= \mathbf{E}^{0,x}\Big[f(X_t)\exp\Big\{-\textstyle\int_0^t k(X_\theta)d\theta\Big\} \\ &+ \textstyle\int_0^t g(t-s, X_s)\exp\Big\{-\textstyle\int_0^s k(X_\theta)d\theta\Big\}ds\Big] \end{aligned} \tag{3.20$'$}$$

is the unique solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + ku = \mathcal{A}_t u + g, \quad (t,x) \in [0,T) \times \mathbf{R}^d \tag{3.17$'$}$$

subject to the initial condition

$$u(0,x) = f(x), \quad x \in \mathbf{R}^d. \tag{3.18$'$}$$

For a proof we refer the reader to Karatzas and Shreve (1988), pp. 366-368 and 268–270.

For a one-dimensional SDE (i.e. $m = d = 1$), the following result due to Yamada and Watanabe (1971) weakens considerably the conditions on the existence and uniqueness of the solution to (3.11).

Theorem 3.17 Assume $m = d = 1$. In order for (3.11) to have a unique solution it suffices that b is continuous and Lipschitz in x and σ is continuous with the property

$$|\sigma(t,x) - \sigma(t,y)| \le \rho(|x-y|),$$

for all x and y and t, where $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ is a strictly increasing function with $\rho(0) = 0$ and for any $\epsilon > 0$,

$$\int_{(0,\epsilon)} \rho^{-2}(x)dx = \infty.$$

For example, $\rho(x) = \sqrt{x}$ satisfies this condition. We shall use this result later in the study of the term structure of interest rates.

3.5 Semimartingales

Definition 3.18 A right-continuous adapted process X is called a *semimartingale*, if X can be expressed as $X = X_0 + M + A$, where M is a local martingale with $M_0 = 0$ and A is a process with finite variation.

It turns out that the semimartingales constitute the largest class of integrators, w.r.t. which stochastic integrals of *predictable* processes can be reasonably defined. Moreover, the semimartingale property and the stochastic integrals are invariant under an equivalent change of probability.

In the following we only consider continuous semimartingales. Since a continuous local martingale with finite variation must equal its initial value, the decomposition of a continuous semimartingale into a local martingale and a process of finite variation is unique. In this case, we call the expression $X = X_0 + M + A$ the *canonical decomposition* of X and M the *martingale part* of X.

If M is a square-integrable continuous martingale, then M^2 is a submartingale. So by the Doob-Meyer decomposition (Theorem 3.5), there exists a unique increasing continuous process C such that $M^2 - C$ is a martingale. Consequently, if M is a continuous local martingale, then there exists a unique increasing continuous process C with $C_0 = 0$ such that $M^2 - C$ is a local martingale. We denote C by $\langle M, M\rangle$ and call it the *quadratic variation* process of M. Through polarization, one can define the *quadratic covariation* process $\langle M, N\rangle$ of two continuous local martingales M and N. Note that if $\sqrt{\langle M, M\rangle_\infty}$ is integrable, then M is a uniformly integrable martingale.

Let X and Y be two continuous semimartingales with the canonical decompositions $X = X_0 + M + A$ and $Y = Y_0 + N + B$. Then we put $\langle X, Y\rangle = \langle M, N\rangle$ and call it the *quadratic covariation* process of X and Y. This definition is consistent with that for Itô processes, see Theorem 3.8.

Now we turn to the definition of stochastic integral for continuous semimartingales. A process H is said to be *progressively measurable* or simply *progressive*, if $\forall t \geq 0$, restricted on $\Omega \times [0, t]$, H is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable. Let M be a continuous local martingale. If H is a progressive process such that $\int_0^t H_s^2 d\langle M, M\rangle_s < \infty$, $\forall t \geq 0$, then we can define a unique continuous local martingale, denoted by $H.M$, such that $(H.M)_0 = H_0 M_0$ and for any continuous local martingale N we have $\langle H.M, N\rangle = H.\langle M, N\rangle$. In this case, we say that H is integrable w.r.t. M and call $H.M$ the *stochastic integral* of H w.r.t. M. For a continuous semimartingale X with the canonical decomposition $X = X_0 + M + A$, the definition of the stochastic integral of a progressive process H w.r.t. X is defined by $H.X = H_0 X_0 + H.M + H.A$, provided the two integrals in the right side make sense. With this definition of stochastic integral the Itô's formula is still valid .

CHAPTER 4

The Black-Scholes Model

In the early 1970's Black and Scholes (1973) made a breakthrough in option pricing theory by deriving explicit formulas for pricing European vanilla options. Since then there has been a rapid progress in theoretical studies and empirical researches on option pricing as well as its practical applications. This chapter provides the main ideas and results of option pricing in the Black-Scholes setting. First, we derive the Black-Scholes differential equation and valuation formulas for European options. Second, we present practical uses of the Black-Scholes formulas. Third, we introduce a martingale approach to the option pricing. Fourth, we make a brief discussion on the problem of pricing American options.

4.1 The Black-Scholes PDE and valuation formulas

Consider a financial market which consists of two instruments: a risky asset, called simply asset, and a bank account. Assume that the asset pays no dividends and its price process satisfies the Itô SDE

$$dS_t = S_t(\mu dt + \sigma dB_t), \tag{4.1}$$

where $S_0 > 0, \mu$ and σ are constant, (B_t) is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. Such a process (S_t) is called a *geometric Brownian motion.* It is also called a *log-normal* process because by Theorem 3.10,

$$S_t = S_0 \exp\Big\{(\mu - \frac{\sigma^2}{2})t + \sigma B_t\Big\}, \tag{4.2}$$

so that $\log(S_t)$ is normally distributed. We call μ the (*instantaneous*) *expected rate of return* and σ the *volatility* of S. Here one should beware that the *continuously compounded rate of return* $\log\frac{S_t}{S_0}$ is different from the expected rate of return. The value process (β_t) of the bank account is assumed to satisfy

$$d\beta_t = r\beta_t dt, \tag{4.3}$$

where r is the constant interest rate. In the sequel, we always assume $\beta_0 = 1$ so that $\beta_t = e^{rt}$.

We assume that the market is *frictionless.* It means that there are no transaction costs, no bid/ask spread, no restrictions on trade such as margin requirements or sort sale restrictions, there are no taxes, and borrowing and lending are at the same interest rate. In addition, we assume that trading in assets takes place continually in time. A *trading strategy* is a pair of $\mathcal{F}_t$-adapted processes $\{a, b\}$ such that $a \in \mathcal{L}^2$ and $b \in \mathcal{L}^1$, where $a(t)$ denotes the number of units of the asset held at time t, and $b(t)$

the amount of the money in the bank account at time t. The *wealth* V_t at time t of a portfolio $\{a(t), b(t)\}$ is defined as

$$V_t = a(t)S_t + b(t)\beta_t.$$

A trading strategy $\{a, b\}$ is said to be *self-financing* if the change of its wealth is only due to the changes in the assets prices weighted by the portfolio, meaning that for all t,

$$dV_t = a(t)dS_t + b(t)d\beta_t. \tag{4.4}$$

The definition of admissible or tame strategy is similar to the discrete time case. A trading strategy is said to be *allowable*, if there exists a positive constant c such that $V_t \geq -(e^{rt} + S_t)$, for all $t \in [0, T]$, where V_t is the wealth at time t of the strategy.

Consider a European contingent claim of the form $\xi = f(S_T)$ with maturity T, where $f : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function. We want to determine its price Y_t at any time $t \leq T$. We anticipate that $Y_t = F(t, S_t), t < T$ with an unknown F being a $\mathrm{C}^{1,2}$-function on $[0, T) \times (0, \infty)$. By Itô's formula,

$$dY_t = \Big[F_t(t,S_t)+F_x(t,S_t)\mu S_t+\frac{1}{2}F_{xx}(t,S_t)\sigma^2 S_t^2\Big]dt+F_x(t,S_t)\sigma S_t dB_t, \quad t < T. \tag{4.5}$$

On the other hand, assume there is a self-financing trading strategy $\{a, b\}$ with

$$a(t)S_t + b(t)\beta_t = Y_t = F(t, S_t). \tag{4.6}$$

From (4.1), (4.3) and (4.4) we obtain

$$dY_t = [a(t)\mu S_t + b(t)\beta_t r]dt + a(t)\sigma S_t dB_t. \tag{4.7}$$

Identifying coefficients in dB_t and dt of (4.5) and (4.7) leads to

$$a(t) = F_x(t, S_t), \quad b(t) = (\beta_t r)^{-1}\Big[F_t(t, S_t) + \frac{1}{2}F_{xx}(t, S_t)\sigma^2 S_t^2\Big]. \tag{4.8}$$

On the other hand, by (4.6) we have

$$b(t) = \beta_t^{-1}[F(t, S_t) - F_x(t, S_t)S_t]. \tag{4.8$'$}$$

Equations (4.8) and (4.8)$'$ imply

$$F_t(t, S_t) + rS_t F_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 F_{xx}(t, S_t) - rF(t, S_t) = 0. \tag{4.9}$$

In order for (4.9) to hold and $Y_T = F(T, S_T)$, it suffices that F satisfies the equation

$$F_t(t, x) + rxF_x(t, x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t, x) - rF(t, x) = 0 \tag{4.10}$$

for $(t, x) \in [0, T) \times (0, \infty)$ with the final condition

$$F(T, x) = f(x), \quad x \in (0, \infty). \tag{4.11}$$

Equation (4.10) is called the *Black-Scholes partial differential equation.*

Now we consider a European call option $\xi = (S_T - K)^+$, whose price at time t is denoted by $Y_t = C(t, S_t)$. In order to obtain an explicit expression for $C(t,x)$ we solve the Black-Scholes equation

$$C_t(t,x) + rxC_x(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) - rC(t,x) = 0 \tag{4.12}$$

subject to the final condition

$$C(T,x) = (x-K)^+, \quad x \in (0,\infty). \tag{4.13}$$

This final condition is not enough for determining a unique solution of (4.12). From the financial meaning of the option prices, it is readily seen that C should satisfy the following boundary conditions at $x = 0$ and $x = \infty$

$$C(t,0) = 0, \quad C(t,x) \sim x \quad \text{as x} \to \infty. \tag{4.14}$$

For solving equation (4.12), we make the following substitutions:

$$x = Ke^y, \; t = T - 2\tau/\sigma^2, \; C(t,x) = Ke^{\alpha\tau+\beta x}U(\tau,y),$$

where $\alpha = -\frac{1}{2}(k_1 - 1)$, $\beta = -\frac{1}{4}(k_1+1)^2$ with $k_1 = 2r/\sigma^2$. Then the problem is reduced to solve the heat equation

$$U_\tau = U_{yy}, \quad (\tau, y) \in (0, \frac{1}{2}\sigma^2 T] \times \mathbf{R}$$

subject to the initial condition

$$U(0,y) = \max(\mathrm{e}^{\frac{1}{2}(\mathrm{k}_1+1)\mathrm{y}} - \mathrm{e}^{\frac{1}{2}(\mathrm{k}_1-1)\mathrm{y}}, 0). \tag{4.15}$$

Conditions (4.14) allow us to express the unique solution of this heat equation by

$$U(\tau,y) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} U(0,z) e^{-(y-z)^2/4\tau} dz. \tag{4.16}$$

From (4.16) we get the Black-Scholes formula for a European call option:

$$C(t,x) = xN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{4.17}$$

where $N(z)$ is the cumulative standard normal distribution function and

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\log(x/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{4.18}$$

By using the put-call parity formula we can get the corresponding Black-Scholes formula for a European put option:

$$P(t,x) = Ke^{-r(T-t)}N(-d_2) - xN(-d_1).$$

One important feature of the Black-Scholes model is that the expected rate of return on the asset does not enter the Black-Scholes equation and Black-Scholes formulas. A careful reader may further notice that if we replace the constant μ in (4.1) by an adapted process $(\mu(t))$, the same argument as above still leads to the validity of (4.9), because two terms involving μ_t in (4.5) and (4.7) will remain to be equal. Consequently, the Black-Scholes equation and Black-Scholes formulas are applicable to this case. This remarkable fact was first observed by Merton (1973). An economical explanation for this somewhat surprising fact is similar to that for the binomial tree model. As in the latter case, this fact leads to a risk-neutral valuation (see Section 4.4 below).

4.2 A generalized Black-Scholes model

In this section we generalize Black-Scholes model to the case where the bank account earns a time-dependent interest rate $r(t)$ and the underlying asset pays dividends and has time-dependent expected rate of return, volatility and dividend yield. We denote them by $\mu(t), \sigma(t)$ and $q(t)$ respectively. Then the asset price process satisfies

$$dS_t = S_t[\mu(t)dt + \sigma(t)dB_t], \tag{4.1)*$$

and the value process (β_t) of the bank account satisfies

$$d\beta_t = r(t)\beta_t dt. \tag{4.2)*$$

In the present case, a trading strategy $\{a(t), b(t)\}$ is said to be *self-financing* if its wealth process $V_t = a(t)S_t + b(t)\beta_t$ satisfies:

$$dV_t = a(t)dS_t + b(t)d\beta_t + q(t)a(t)S_t dt. \tag{4.4)*$$

The last term in the right side of equation (4.4) means that the gain from dividends is reinvested in the market.

We assume that $r(t), \mu(t), \sigma(t), q(t)$ are deterministic functions of t. Consider again a European contingent claim $\xi = f(S_T)$. We anticipate that its price Y_t at time t can be expressed as $F(t, S_t)$ with F being a $C^{1.2}$-function on $[0, T) \times (0, \infty)$. By a similar derivation as that for equation (4.10) we obtain the following PDE:

$$-r(t)F(t,x) + F_t(t,x) + (r(t) - q(t))xF_x(t,x) + \frac{1}{2}\sigma^2(t)x^2F_{xx}(t,x) = 0. \tag{4.12)*$$

By making the following substitutions of variables:

$$y = xe^{-\int_t^T (q(s)-r(s))ds}; \ \tau = \int_t^T \sigma^2(s)ds; \ U(\tau, y) = F(t,x)e^{\int_t^T r(s)ds},$$

equation $(4.12)^*$ becomes

$$U_t = \frac{1}{2}y^2 U_{yy}, \tag{4.12)'$$

which is a standard Black-Scholes equation.

If we consider a European call option $\xi = (S_T - K)^+$, then its price at time t is equal to $C(t, S_t)$, where $C(t, x)$ is given by a generalized Black-Scholes formula:

$$C(t,x) = \widetilde{x}N(\widetilde{d}_1) - Ke^{-(T-t)\widetilde{r}}N(\widetilde{d}_2), \tag{4.17)'$$

where

$$\widetilde{x} = xe^{-\int_t^T q(s)ds}, \quad \widetilde{r} = \frac{1}{T-t}\int_t^T r(s)ds,$$

and $\widetilde{d}_1$ and $\widetilde{d}_2$ have the same expressions as (4.18), the only difference is that x, r and σ^2 therein are replaced by $\widetilde{x}, \widetilde{r}$ and $\frac{1}{T-t}\int_t^T \sigma^2(s)ds$, respectively. We refer the reader to Wilmott-Dewynne-Howison (1994) for more details on the derivation of this formula.

4.3 Practical uses of the Black-Scholes formulas

4.3.1 Historical and implied volatilities

Note that the only unknown parameter in Black-Scholes formulas is the volatility, which is difficult to measure. One might use historic data of the asset prices to calculate the standard deviation of asset returns as an estimate of the volatility, known as *historical volatility*. In doing so one needs the data over a long time period. This violates the assumption of constant volatility. However, forecasting the volatility is a critical factor in trading options. What can we do? Fortunately, the market "knows" implicitly the volatility through the quoted option prices. That means, admitting that the Black-Scholes model is correct and the option prices are fair, we can take the quoted prices of options on the same asset with different maturities and/or strike prices to deduce, from the Black-Scholes formula, estimates of the volatility. By taking some kind of weighted average over these estimates we obtain an estimate of the volatility, known as *implied volatility*. The latter can be considered as a forecast of the future volatility. Empirical studies have shown that implied volatilities are better suited than historical volatilities for predicting an asset's future volatility. Once one knows approximately the future volatility, one can use the Black-Scholes formula once again to detect under- and over-priced options and to build investment strategies accordingly. It is one of the primary uses of the Black-Scholes formula by practitioners, including arbitragers and brokerage houses. Here we should mention that a statistical method of forecasting volatility, called GARCH method, has also received attention in the financial community. GARCH stands for "generalized autoregressive conditional heterocedasticity".

4.3.2 Delta hedging and option's price sensitivity

Another primary use of the Black-Scholes formula is that it provides useful measures of the sensitivity of an option's price to various parameter changes. These sensitivity measures prove to be very useful and effective tools for monitoring an option's position risk exposure. By definition, the *delta* of an option measures the change in the option price for a unit change in the underlying asset's price. The sensitivity of delta to changes in the value of the underlying asset is called the *gamma*. The sensitivity of the option price to changes in the time to maturity (resp. in the volatility, in the interest rate) is called the *theta* (resp. *Vega, rho*).

From the Black-Scholes formula we see that the price of a call option depends on S_t, K, σ, r, and $T-t$, the time to maturity of the option. Since we have

$$xN'(d_1) = Ke^{-r(T-t)}N'(d_2),$$

it is easy to prove that

$$\begin{aligned}
\Delta = \frac{\partial C}{\partial x} &= N(d_1) > 0, \\
\Gamma = \frac{\partial^2 C}{\partial x^2} &= \frac{1}{x\sigma\sqrt{T-t}}N'(d_1) > 0, \\
V = \frac{\partial C}{\partial \sigma} &= x\sqrt{r}N'(d_1) > 0, \\
\rho = \frac{\partial C}{\partial r} &= T - te^{-r(T-t)}KN(d_2) > 0,
\end{aligned}$$

$$\theta = \frac{\partial C}{\partial t} \quad = \quad -\frac{x\sigma}{2\sqrt{T-t}}N(d_1) - Kre^{-r(T-t)}N(d_2) < 0.$$

From (4.8) we obtain immediately a hedging strategy $\{a(t), b(t)\}$ for the call option, where $a(t)$ is given by the delta $\frac{\partial C}{\partial x}(t, S_t)$ at time of the option. So this hedging is also called the *delta hedging.* In practice, in using this delta hedging strategy, there are two things one should keep in mind. First, due to transaction costs, one should rebalance the portfolio only when the position's delta has moved noticeably from its target level. To this point, the option's gamma help us to know how frequently one should rebalance the portfolio. The higher the gamma, the more frequent rebalancing of the portfolio. Second, one should recalculate the (implied) volatility as often as possible.

The theta of a call option is always negative. This suggests that a long position in a call will loss its time value with the passage of time. This loss can only be avoided by setting a theta-neutral position consisting of short and long holdings in options that have the same theta. If one believes that the volatility is not constant, one should also take the Vega into consideration.

Another sensitivity measure of an option's price is the so-called *elasticity* or *lambda*, denoted by λ. It also refers to the *leverage* of the option position. It measures the percentage change in the option price for 1 percentage change in the underling asset's price. In the Black-Scholes setting, we have

$$\lambda = \frac{\partial C}{\partial x}\frac{x}{C} = \frac{xN(d_1)}{C}.$$

From (4.17) we see that $\lambda > 1$ always holds. This phenomenon is called *leverage effect.* For a put option, $\lambda = -\frac{\partial P}{\partial x}\frac{x}{P}$, where P stands for the price of the put. It is always negative, but not necessary less than -1. It means that a put option doess not necessarily have a leverage effect.

4.4 Martingale method in contingent claim pricing

In this section we assume that the filtration $(\mathcal{F}_t)$ is the natural filtration of the Brownian motion (B_t). We introduce a martingale method for the contingent claim pricing and hedging in a generalized Black-Scholes model. For the clarity of the presentation we only consider the case where the risky asset pays dividends at a constant rate q and other parameters of the model are all constants. The price process of the risky asset is still assumed to obey equation (4.1), but the wealth process V_t of a self-financing strategy $\{a, b\}$ satisfies $(4.4)^*$ with $q(t) = q$. If we put

$$\mu_q = \mu + q, \quad X_t = e^{qt}S_t,$$

then (4.1) is reduced to

$$dX_t = X_t(\mu_q dt + \sigma dB_t), \tag{4.1$'$}$$

and the self-financing condition $(4.4)^*$ becomes

$$dV_t = a(t)e^{-qt}dX_t + b(t)d\beta_t. \tag{4.4$'$}$$

We denote by $\widetilde{X}_t$ the discounted value of X_t, i.e. $\widetilde{X}_t = e^{-rt}X_t$.

In the present case where the risky asset pays dividends the definition of allowable strategy should be modified. A strategy is said to be *allowable*, if there exists a positive constant c such that $V_t \geq -c(e^{rt} + X_t)$, for all $t \in [0, T]$, where V_t is the wealth at time t of the strategy. The definition of arbitrage strategy is similar to the discrete-time case.

The starting point of martingale methods for option pricing is the following observation.

Lemma 4.1 A trading strategy $\{a, b\}$ is self-financing if and only if its discounted wealth process $(\widetilde{V}_t)$ satisfies

$$d\widetilde{V}_t = a(t)e^{-qt}d\widetilde{X}_t. \tag{4.19}$$

Proof Assume $\{a, b\}$ is self-financing. Since $\widetilde{V}_t = e^{-rt}V_t$, by (3.7), (4.4)′ and (4.3) we have

$$\begin{aligned} d\widetilde{V}_t &= -V_t r e^{-rt}dt + e^{-rt}dV_t \\ &= -[a(t)e^{-qt}X_t + b(t)e^{rt}]re^{-rt}dt + e^{-rt}[a(t)e{-}qtdX_t + b(t)e^{rt}rdt] \\ &= a(t)e^{-qt}[X_t d(e^{-rt}) + e^{-rt}dX_t] \\ &= a(t)e^{-qt}d\widetilde{X}_t. \end{aligned}$$

Similarily, we can prove "if" part.

We will prove that there exists a unique probabilty measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that the process $\widetilde{X}_t$ is a $\mathbf{P}^*$-martingale. In fact, we can rewrite (4.1)′ as

$$d\widetilde{X}_t = \widetilde{X}_t[(\mu_q - r)dt + \sigma dB_t].$$

Consequently, if we put $\frac{d\mathbf{P}^*}{d\mathbf{P}}|_{\mathcal{F}_T} = \mathcal{E}(-\theta.B)_T$ with $\theta(t) = \theta = (\mu_q - r)/\sigma$, then by the Girsanov's theorem (Theorem 3.12) $B_t^* = B_t + \theta t$ is a $\mathbf{P}^*$-Brownian motion and

$$d\widetilde{X}_t = \widetilde{X}_t \sigma dB_t^*. \tag{4.20}$$

Thus $(\widetilde{X}_t)$ is a $\mathbf{P}^*$-martingale. It is easy to see that such a probability measure is unique. By Lemma 4.1, the discounted wealth process of a self-financing strategy is a local martingale under $\mathbf{P}^*$. Thus, for an allowable self-financing strategy, its discounted wealth process is a $\mathbf{P}^*$-supermartingale, because it can be expressed as the difference of a non-negative local $\mathbf{P}^*$-martingale and a $\mathbf{P}^*$-martingale. From this it is easy to prove that the market has no arbitrage.

We call the probability measure $\mathbf{P}^*$ the *equivalent martingale measure* for the market. By contrast, $\mathbf{P}$ is called the *objective* or *physical* probability measure. On the other hand, (4.1) can be rewritten as

$$dS_t = S_t[(r - q)dt + \sigma dB_t^*]. \tag{4.21}$$

It means that under this measure $\mathbf{P}^*$ the expected rate of return of the risky asset is equal to $r - q$, i.e. the expected rate of rturn of the risky asset plus the dividend rate is equal to the interest rate of the bank account. For this reason the equivalent martingale measure $\mathbf{P}^*$ is also called a *risk-neutral probability measure.*

Note that (4.1) can be expressed as

$$dS_t = S_t[(r - q + \sigma\eta)dt + \sigma dB_t],$$

where $\eta = \frac{\mu_q - r}{\sigma}$. We call η the *market price of risk* or *risk premium* of the risky asset. It represents the excess rate of return above the risk-free rate of return per unit of extra risk. To further explain the economic meaning of the market price of risk, we add a new risky asset in the market. We assume that this asset pays no dividends and its price process (W_t) is the following Itô process

$$dW_t = W_t[\alpha_t dt + \beta_t dB_t].$$

Thus the discounted price process $(\widetilde{W}_t)$ satisfies

$$d\widetilde{W}_t = \widetilde{W}_t[(\alpha_t - r)dt + \beta_t dB_t].$$

So in order for the new market to have no arbitrage, $(\widetilde{W}_t)$ must be a $\mathbf{P}^*$-martingale. Consequently, we must have ${\beta_t}^{-1}(\alpha_t - r) = \eta$. This means that in an arbitrage-free market the assets having the same sources of uncertainty must have the same market price of risk.

The following theorem is the main result of the risk-neutral valuation.

Theorem 4.2 Let ξ be a European contingent claim which is integrable under $\mathbf{P}^*$. Then there exists an admissible self-financing strategy $\{a, b\}$ replicating ξ such that its wealth process (V_t) is given by

$$V_t = \mathbf{E}^*\left[e^{-r(T-t)}\xi | \mathcal{F}_t\right], \tag{4.22}$$

or equivalently, $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale. Moreover, such an admissible self-financing strategy $\{a, b\}$ replicating ξ is essentially unique. More precisely, we have

$$a(t) = (\sigma^2 \widetilde{S}_t^2)^{-1} \frac{d\langle \widetilde{V}, \widetilde{S} \rangle_t}{dt}. \tag{4.23}$$

In particular, if $V_t = F(t, S_t)$ with $F \in C^{1,2}([0,T) \times \mathbf{R}_+$, then $a(t) = F_x(t, S_t)$. It implies that we are in a delta hedging situation.

Proof We define V_t by (4.22). Then $(\widetilde{V}_t)$ is a $\mathbf{P}^*$- martingale. Since $(\mathcal{F}_t)$ is also the natural filtration of (B_t^*), by the martingale representation theorem (Theorem 3.7) there exists an $H \in \mathcal{L}^2$ such that

$$\widetilde{V}_t = V_0 + \int_0^t H_s dB_s^*, \quad t \in [0,T]. \tag{4.24}$$

Put

$$a(t) = H_t/(\sigma \widetilde{S}_t), \quad b(t) = \widetilde{V}_t - a(t)\widetilde{S}_t. \tag{4.25}$$

Then $\{a, b\}$ is a hedging strategy for ξ and (V_t) is its wealth process. By (4.21) and (4.25) we have

$$a(t)e^{-qt} d\widetilde{X}_t = a(t)e^{-qt}\widetilde{X}_t \sigma dB_t^* = H_t dB_t^* = d\widetilde{V}_t.$$

Thus by Lemma 4.1 the strategy $\{a, b\}$ is self-financing and admissible.

Now let $\{a, b\}$ be an admissible self-financing strategy replicating ξ. Then by (4.19) and (4.20) we have

$$d\widetilde{V}_t = a(t)e^{-qt}\sigma \widetilde{X}_t dB_t^* = a(t)\sigma \widetilde{S}_t dB_t^*,$$

from which we get immediately (4.23).

Now assume that $V_t = F(t, S_t)$, where $F \in C^{1,2}([0,T) \times \mathbf{R}^+$. We claim that $a(t) = F_x(t, S_t)$. In fact, by Itô's formula,

$$\begin{aligned} d\widetilde{V}_t &= d\Big(e^{-rt}F(t,S_t)\Big) = e^{-rt}F_x(t,S_t)dS_t + \text{“}dt\text{” term} \\ &= F_x(t,S_t)d\widetilde{S}_t + \text{“}dt\text{” term} \\ &= F_x(t,S_t)e^{-qt}d\widetilde{X}_t + \text{“}dt\text{” term} \\ &= F_x(t,S_t)e^{-qt}d\widetilde{X}_t \\ &= F_x(t,S_t)\widetilde{S}_t\sigma dB_t^*. \end{aligned}$$

Here the equality before the last is due to the fact that $(\widetilde{V}_t)$ and $\widetilde{X}_t$ are $\mathbf{P}^*$-martingales, which implies that the "dt" term must vanish. Thus the claim follows from (4.23).

Remark It is natural to define V_t as the *"fair " price* at time t of the contingent claim ξ, because with this price there does not exist any arbitrage opportunity for both the seller and buyer of the contingent claim. This method of pricing contingent claims is called *pricing by arbitrage* or *arbitrage pricing*. Equation (4.22) is called the *risk-neutral valuation* formula.

Corollary 4.3 If $\xi = f(S_T)$, then $V_t = F(t, S_t)$, where

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{\infty} f\Big(e^{-q(T-t)}xe^{(r-\sigma^2/2)(T-t)+\sigma y\sqrt{T-t}}\Big)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy. \qquad (4.26)$$

Proof We express S_T as

$$S_T = e^{-qT}X_T = e^{-qT}X_t(X_T X_t^{-1}) = e^{-q(T-t)}S_t \exp\{(r-\sigma^2/2)(T-t)+\sigma(B_T^*-B_t^*)\}.$$

Since S_t is measurable w.r.t. $\mathcal{F}_t$ and $B_T^* - B_t^*$ is independent of $\mathcal{F}_t$, by (4.20) and Theorem 1.1 we have

$$V_t = \mathbf{E}^*\Big[e^{-r(T-t)}f(e^{-q(T-t)}x\exp\{(r-\sigma^2/2)(T-t)+\sigma(B_T^*-B_t^*)\}\Big]\Big|_{x=S_t},$$

from which we get $V_t = F(t, S_t)$.

4.5 Pricing American contingent claims

Now we turn to the problem of pricing American contingent claims in the Black-Scholes framework. We assume that the asset pays no-dividends. In the continuous-time case, an *American contingent claim* is naturally defined as an adapted non-negative process $(h_t)_{0\le t\le T}$. For simplicity, we only consider an American contingent claim of the form $h_t = g(S_t)$. For a call, we have $g(x) = (x-K)^+$, and for a put, $g(x) = (K-x)^+$.

Similar to the discrete-time case, for pricing American contingent claims we should introduce the *strategy with consumption*. By such a strategy we mean a trading strategy $\phi = \{a, b\}$ with the property that

$$a(t)S_t + b(t)\beta_t = a(0)S_0 + b(0) + \int_0^t a(s)dS_s + \int_0^t b(s)d\beta_s - C_t$$

for all $t \in [0,T]$, where (C_t) is an adapted, continuous, non-decreasing process null at $t = 0$. C_t represents the cumulative consumption up to time t. We denote by

$V_t(\phi)$ the *wealth* at time t of ϕ, namely, $V_t(\phi) = a(t)S_t + b(t)\beta_t$. A strategy with consumption $\phi = \{a, b\}$ is said to *super-hedge* the American contingent claim (h_t), if for all $t \in [0, T], V_t(\phi) \geq h_t$, a.s.. So a super-hedging strategy is automatically admissible.

Let $\mathcal{T}_{t,T}$ be the set of all stopping times taken values in $[t, T]$. Put

$$\Phi(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbf{E}^* \left[e^{-r(\tau - t)} g\big(x \exp\{(r - (\sigma^2/2))(\tau - t) + \sigma(B_\tau - B_t)\}\big)\right]$$

where we assume $g(x) \leq A + Bx$ so that $\Phi(t, x)$ is well defined. It is easy to prove that the process $e^{-rt}\Phi(t, S_t)$ is a supermartingale that dominates the process $g(S_t)$ for all $t \in [0, T]$.

The following theorem is the main result concerning the pricing American contingent claims. For a proof we refer the reader to Karatzas and Shreve (1988), p.376–378.

Theorem 4.4 There is a trading strategy with consumption ϕ such that ϕ super-hedges $(g(S_t))$ and $V_t(\phi) = \Phi(t, S_t)$, $\forall t \in [0, T]$. Moreover, for any trading strategy with consumption ψ super hedging $(g(S_t))$, we have $V_t(\psi) \geq \Phi(t, S_t)$ for all $t \in [0, T]$.

We call $\Phi(t, S_t)$ the *selling price* at time t of the American contingent claim. The following theorem shows that the American call option and the European call option have the same price at any time t, if the underlying asset pay no dividend.

Theorem 4.5 In the call option case, i.e. $g(x) = (x - K)^+$, we have $\Phi(t, x) = C(t, x)$, where C is defined by (4.17).

Proof We only consider the case $t = 0$, other cases being similar. Since $\widetilde{S}_t$ is martingale under $\mathbf{P}^*$, for any stopping time taking values in $[0, T]$, we have

$$\widetilde{S}_\tau - e^{-rT}K = \mathbf{E}^*[e^{-rT}(S_T - K)|\mathcal{F}_\tau] \leq \mathbf{E}^*[e^{-rT}(S_T - K)^+|\mathcal{F}_\tau],$$

from which we get

$$(\widetilde{S}_\tau - e^{-r\tau}K)^+ \leq (\widetilde{S}_\tau - e^{-rT}K)^+ \leq \mathbf{E}^*[e^{-rT}(S_T - K)^+|\mathcal{F}_\tau].$$

By taking expectations we obtain

$$\mathbf{E}^*[(\widetilde{S}_\tau - e^{-r\tau}K)^+] \leq \mathbf{E}^*[e^{-rT}(S_T - K)^+].$$

This implies the result.

We will further study the problem of pricing American put options in Chapter 6.

CHAPTER 5

Exotic Options

Any option which is not vanilla is called an *exotic option.* Exotic options are widely used today by banks, corporations and institutional investors, in their management of risk. The simplest type of exotic options are the *binary* or *digital* options. The payoff at time t of a binary option is given by $B\mathcal{H}(S_t - K)$, where $\mathcal{H}(\cdot)$ is the Heaviside function, S_t is the price at time t of the underlying asset and B is a constant. Other types of exotic options are the *barrier options, Asian options* and *lookback options*, which will be defined shortly. They are all *path-dependent* in the sense that their payoff at exercise or expiry depends on history of the underlying asset. Note that any American-style option is path-dependent but not necessarily exotic.

In this chapter we will study the problem of pricing path-dependent exotic options. These are barrier options, Asian options, and lookback options. For simplicity, we continue to work in the Black-Scholes setting and consider only European-style options. For some options we can get explicit expressions for their prices. For others we can only deduce PDEs which govern their prices. We shall use at ease the notations of Chapter 4 concerning the Black-Scholes model.

5.1 Barrier options

Barrier Options are options that are either worthless ("out") or established ("in"), when the price of the underlying asset crosses a particular level ("barrier"). Common examples of single-barrier options are "down-and-out", "down-and-in", "up-and-out" and "up-and-in" options for calls or puts. A *double-knock binary option* or *up-and-down out binary option* is a simplest double-barrier option. It is characterized by two barriers, L (lower barrier) and U (upper barrier): the option knocks out if either barrier is touched. Otherwise, the option gives at maturity the usual binary payoff. Barrier options have become increasingly popular over the last few years. Since their price is less expensive than the standard options, barrier options may be an appropriate hedge in a number of situations. For instance, a down-and-in put with a low barrier offers an inexpensive protection against a downward move of the underlying asset. Pricing barrier options in the Black-Scholes setting is not difficult, as will be seen below. Closed-form solutions for all types of single-barrier options have been offered by Goldman *et al.* (1979) a long time ago (see also Rubinstein (1991). Two different analytic expressions for double-barrier options have been worked out recently by Geman and Yor (1996) and Hui (1996).

5.1.1 Single-Barrier options

We only consider a down-and-out option with strike price K, maturity T, and an *out-barrier* $X < K$, the other cases can be treated in a similar way. The payoff at expiry of this option is the same as that for call option (i.e. $(S_T - K)^+$), provided that S_t never falls below X before T. Therefore, if, instead of $C(t, S_t)$, we denote by $\widetilde{C}(t, S_t)$ the price at time t of the barrier option and make the same change of variables as in Section 4.2 of Chapter 4, the present problem reduces to the solution of the following heat equation

$$U_\tau = U_{yy}, \quad (\tau, y) \in (0, \frac{1}{2}\sigma^2 T] \times \mathbf{R}$$

with

$$U(0, y) = \max(\mathrm{e}^{\frac{1}{2}(\mathrm{k}_1+1)\mathrm{y}} - \mathrm{e}^{\frac{1}{2}(\mathrm{k}_1-1)\mathrm{y}}, 0), \quad \mathrm{y} \geq \log(\mathrm{X}/\mathrm{K}),$$
$$U(t, y) \sim e^{(1-\alpha)y - \beta\tau} \quad \text{as} \quad \mathrm{y} \to \infty,$$

and

$$U(t, \log(X/K)) = 0,$$

where $k_1 = \frac{2r}{\sigma^2}$. In dealing with this last boundary condition by the so-called *method of images* we can obtain

$$\widetilde{C}(t, x) = C(t, x) - \left(\frac{x}{X}\right)^{-(k_1-1)} C(t, X^2/x). \tag{5.1}$$

See Wilmott-Dewynne-Howison (1993) or Rubinstein (1992) for more details about the derivation.

5.1.2 Double-barrier options

In this subsection we follow closely Hui (1996). We consider a double-knock option with two barriers L, U, maturity T, and the binary payoff $g(S_T)$. For a call or put, the strike price K satisfies $L < K < U$. We assume that the asset pays a dividend at rate D_0. If the price at time t of the double-knock option is given by $C(t, S_t)$, then the corresponding PDE remains as $(4.12)'$. The only change is the boundary conditions:

$$C(t, L) = C(t, U) = 0, \quad t < T; \quad C(T, x) = g(x), \quad L < x < U.$$

By making a similar change of variables as in Section 4.3 of Chapter 4 (replacing K by L), the pricing problem is reduced to the solution of the following heat equation

$$U_\tau = U_{yy}, \quad (\tau, y) \in (0, \frac{1}{2}\sigma^2 T] \times (0, \log(U/L)$$

subject to the boundary conditions

$$U(\tau, 0) = 0, \quad U\left(\tau, \log\frac{U}{L}\right) = 0, \quad \tau > 0$$

and the initial condition

$$U(0, y) = \frac{g(Le^y)e^{-\alpha y}}{U}, \quad 0 < y < \log(U/L).$$

Now we assume the function g is continuous and satisfies the linear growth condition. Then since the boundary conditions are homogeneous, we can use the method of separation of variables to solve this equation. The solution is:

$$U(\tau, y) = \sum_{n=1}^{\infty} b_n(\tau) \sin\big(\frac{n\pi y}{W}\big), \tag{5.2}$$

where $W = \log(U/L)$, $b_n(\tau) = b_n \exp\{-(\frac{n\pi y}{W})\}$, and (b_n) is the sequence of the Fourier coefficients of $U(0, y)$, namely,

$$b_n = \frac{2}{LW} \int_0^W g(Le^y) e^{-\alpha y} \sin\big(\frac{n\pi y}{W}\big) dy.$$

Thus we can get an expression for $C(t, x)$. If $g(x) = R$ is a constant, then

$$b_n = \frac{2\pi n R}{LW^2} \left[\frac{1 - (-1)^n W^{-\alpha}}{\alpha^2 + (n\pi/W)^2} \right].$$

In this case, the n-th term in the series (5.2) is of the order $n^{-1} e^{-n^2(T-t)}$.

We refer the reader to Geman and Yor (1996) for a probabilistic approach to this problem. Their result is expressed in terms of the Laplace transform w.r.t. the maturity T.

5.2 Asian options

By an Asian option we mean an option whose payoff depends on a suitably defined average of the asset price. There are two kinds of average: geometric and arithmetic. For each kind of average there are two types of options: the *average strike* and *average rate* options. They are similar to a European vanilla option, the only difference is that in the payoff, for the former, the strike price is replaced by the average, and for the latter, the asset price is replaced by the average. The sampling for the average can be discrete or continuous. In this section we only consider the average rate options with continuous sampling.

There are two types of average rate call options whose payoffs are equal to

$$\xi_1 = \left(\exp\left\{ \frac{1}{T} \int_0^T \log(S_u) du \right\} - K \right)^+$$

and

$$\xi_2 = \left(\frac{1}{T} \int_0^T S_u du - K \right)^+$$

respectively. The first one uses the geometric average, while the second one uses the arithmetic average. We denote by $C_t^{(i)}$ the price at time t of $\xi_i, i = 1, 2$.

We consider the geometric average case first. We denote by $\mathbf{P}^*$ the equivalent martingale measure for $\widetilde{S}$. Then we have

$$C_t^{(1)} = e^{-r(T-t)} \mathbf{E}^*[\xi_1 | \mathcal{F}_t]. \tag{5.3}$$

Put

$$I_t = \int_0^t \log(S_u)du. \tag{5.4}$$

Then

$$\begin{aligned}\xi_1 &= \Big(\exp\Big\{\frac{1}{T}I_t + \frac{1}{T}\int_t^T \log(S_u S_t^{-1})du + \frac{T-t}{T}\log S_t\Big\} - K\Big)^+, \\ &= (X_t Y_t - K)^+,\end{aligned}$$

where

$$X_t = e^{I_t/T} S_t^{(T-t)/T}, \quad Y_t = \exp\Big\{\frac{1}{T}\int_t^T \log S_u S_t^{-1} du\Big\}. \tag{5.5}$$

Note that

$$S_t = S_0 \exp\Big\{(r - \frac{\sigma^2}{2})t + \sigma B_t^*\Big\},$$

we have

$$Y_t = \exp\Big\{\frac{1}{T}\int_t^T \Big[(r - \frac{\sigma^2}{2})(u-t) + \sigma(B_u^* - B_t^*)\Big]du\Big\} = e^{r^*(T-t)+Z_t}$$

with

$$r^* = (r - \frac{\sigma^2}{2})\frac{T-t}{2T}, \quad Z_t = \frac{1}{T}\int_t^T \sigma(B_u^* - B_t^*)du. \tag{5.6}$$

Since Z_t is independent of $\mathcal{F}_t$ and X_t is $\mathcal{F}_t$-measurable, by Theorem 1.1 we have $C_t^{(1)} = e^{-r(T-t)}F(t, X_t)$, where

$$F(t,x) = \mathbf{E}^*[(xe^{r^*(T-t)+Z_t} - K)^+].$$

Note that Z_t is a Gaussian random variable with mean zero and variance $\sigma^{*2}(T-t)$, with

$$\sigma^{*2} = \frac{\sigma^2(T-t)^2}{3T}. \tag{5.7}$$

We have

$$\begin{aligned}F(t,x) &= e^{xr^*(T-t)}\textstyle\int_{-\infty}^{\infty}\Big(e^{\sigma^*\sqrt{T-t}y} - Ke^{-r^*(T-t)}\Big)^+ \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= xe^{(r^* + \frac{\sigma^{*2}}{2})(T-t)}N(d_1^*) - KN(d_2^*),\end{aligned} \tag{5.8}$$

where

$$d_1^* = \frac{\log(x/K) + (r^* + \sigma^{*2})(T-t)}{\sigma^*\sqrt{T-t}}, \quad d_2^* = \frac{\log(x/K) + r^*(T-t)}{\sigma^*\sqrt{T-t}}. \tag{5.9}$$

Now we turn to the pricing of an arithmetic average rate call option. In this case we have

$$C_t^{(2)} = \mathbf{E}^*\Big[e^{-r(T-t)}\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)^+ \Big| \mathcal{F}_t\Big]. \tag{5.10}$$

Following Rogers and Shi (1995) we put

$$M_t = \mathbf{E}^*\Big[\Big(\int_0^T S_u du - TK\Big)^+ \Big| \mathcal{F}_t\Big]. \tag{5.11}$$

Since $\int_t^T S_t^{-1} S_u du$ is independent of $\mathcal{F}_t$, we have

$$M_t = S_t \mathbf{E}^* \Big[\Big(\int_t^T S_t^{-1} S_u du - S_t^{-1} \big(TK - \int_0^t S_u du \big) \Big)^+ \Big| \mathcal{F}_t \Big] = S_t f(t, Y_t),$$

where

$$f(t,x) = \mathbf{E}^* \Big[\Big(\int_t^T S_t^{-1} S_u du - x \Big)^+ \Big] \tag{5.12}$$

and $Y_t = S_t^{-1}\big(TK - \int_0^t S_u du\big)$. For the case $x \le 0$, we have

$$f(t,x) = \mathbf{E}^* \Big[\Big(\int_t^T S_t^{-1} S_u du - x \Big) \Big] = \int_t^T e^{r(u-t)} du - x = r^{-1}(e^{r(T-t)} - 1) - x.$$

It remains to treat with the case $x \ge 0$. Since

$$dS_t^{-1} = -S_t^{-2} dS_t + S_t^{-3} d\langle S, S \rangle_t = S_t^{-1}[(\sigma^2 - r)dt - \sigma dB_t^*],$$

we have

$$dY_t = \Big(TK - \int_0^t S_u du \Big) dS_t^{-1} - dt = Y_t[(\sigma^2 - \sigma dB_t^*] - dt.$$

Consequently,

$$d\langle Y, Y \rangle_t = Y_t^2 \sigma^2 dt$$

and

$$d\langle S, f(\cdot, Y.) \rangle_t = f_x(t, Y_t) d\langle S, Y \rangle_t = -f_x(t, Y_t) S_t Y_t \sigma^2 dt.$$

In the following we use the notation $A \sim B$ to mean the fact that $A - B$ is a local martingale under $\mathbf{P}^*$. Therefore, by Itô's formula we obtain

$$\begin{aligned} dM_t &= S_t df(t, Y_t) + f(t, Y_t) dS_t + d\langle S, f(\cdot, Y.) \rangle \\ &= S_t \Big[f_t(t, Y_t) dt + f_x(t, Y_t) dY_t + \frac{1}{2} f_{xx}(t, Y_t) d\langle Y, Y \rangle_t \Big] \\ &\quad + f(t, Y_t) dS_t + d\langle S, f(\cdot, Y.) \rangle_t \\ &\sim S_t \Big(f_t - (1 + rY_t) f_x + \frac{1}{2} \sigma^2 Y_t^2 f_{xx} + rf \Big)(t, Y_t) dt. \end{aligned}$$

Since (M_t) is a martingale under $\mathbf{P}^*$, the above term on the right hand side must vanish. This leads to the following PDE:

$$f_t - (1 + rx) f_x + \frac{\sigma^2 x^2}{2} f_{xx} + rf = 0, \quad x \ge 0. \tag{5.13}$$

The first boundary condition is (by (5.12))

$$f(T, x) = 0.$$

The second one is

$$f(t, 0) = r^{-1}(e^{r(T-t)} - 1),$$

because

$$\mathbf{E}^*[S_t^{-1} S_u] = \mathbf{E}^* \Big[\exp \Big\{ \sigma(B_u^* - B_t^*) - (\frac{\sigma^2}{2} - r)(u - t) \Big\} \Big] = e^{r(u-t)}.$$

In addition, since

$$f_x(t,x) = -\mathbf{P}^*\Big(\int_t^T S_t^{-1} S_u du \geq x\Big),$$

we have the third boundary condition:

$$\lim_{x\to\infty} f_x(t,x) = 0.$$

Unfortunately, the analytic expression for the solution of (5.13) has not been found yet. However, a lower bound for the option price was provided in Rogers and Shi (1995).

5.3 Lookback options

By a lookback option we mean an option whose payoff depends on the maximum or minimum realised asset price over the option's life. Like Asian options, there are two types of options: the *lookback strike* and *lookback rate*, in both call and put varieties. They are similar to a European vanilla option, the only difference is that in the payoff, for the former, the strike price is replaced by the sampled maximum or minimum, and for the latter, the asset price is replaced by the sampled maximum or minimum. The sampling of the underlying asset price can be discrete or continuous. In this section we only consider the continuous sampling.

5.3.1 Lookback strike options

The payoff of a lookback strike call (resp. put) is defined by

$$\xi = S_T - \min_{0\leq s\leq T} S_s, \quad (\text{resp. } \eta = \max_{0\leq s\leq T} S_s - S_T).$$

We denote the prices at time t of a call and a put by C_t and P_t respectively. By Theorem 4.2, we have

$$C_t = e^{-r(T-t)}\mathbf{E}^*[\xi \mid \mathcal{F}_t], \quad P_t = e^{-r(T-t)}\mathbf{E}^*[\eta \mid \mathcal{F}_t]. \tag{5.14}$$

We are going to deduce an explicit expression for P_t from the following well known result due to Shepp (1966)

$$\mathbf{P}\Big(\max_{s\leq t}(\sigma B_s + \lambda s) \leq x\Big) = N\Big(\frac{x-\lambda t}{\sigma\sqrt{t}}\Big) - e^{2\lambda x/\sigma^2} N\Big(\frac{-x-\lambda t}{\sigma\sqrt{t}}\Big), \tag{5.15}$$

where $x \geq 0$ and $N(z)$ is the cumulative standard normal distribution. We let $\lambda = r - \frac{1}{2}\sigma^2$ and set

$$M_t = \max_{0\leq s\leq t} S_s, \quad L_t = \max_{t\leq s\leq T} S_s. \tag{5.16}$$

Then M_t is $\mathcal{F}_t$-measurable. Since

$$S_t^{-1} L_t = \exp\{\max_{t\leq s\leq T}(\sigma(B^* s - B_t^*) + \lambda(s-t)\},$$

$S_t^{-1}L_t$ is independent of $\mathcal{F}_t$. By using these notations we have

$$\begin{aligned} P_t &= e^{-r(T-t)}\mathbf{E}^*[M_T - S_T|\mathcal{F}_t] = e^{-r(T-t)} S_t \mathbf{E}^*[\max(S_t^{-1}M_t, S_t^{-1}L_t)|\mathcal{F}_t] - S_t \\ &= e^{-r(T-t)} S_t \mathbf{E}^*[\max(y, S_t^{-1}L_t)]|_{y=S_t^{-1}M_t} - S_t \ . \end{aligned}$$

We denote $\mathbf{P}\Big(\max_{s\le t}(\sigma B_s+\lambda s)\le x\Big)$ by $F_t(x)$, then we have

$$\begin{aligned}
\mathbf{E}^*\big[\max(y,S_t^{-1}L_t)\big] &= \mathbf{E}^*\big[\exp\{\max\big(\log y,\max_{t\le s\le T}[\sigma(B_s^*-B_t^*)+\lambda(s-t)]\big)\}\big] \\
&= \mathbf{E}\big[\exp\{\max(\log y,\max_{0\le s\le T-t}\big(\sigma B_s+\lambda s)\big)\}\big] \\
&= yF_{T-t}(\log y)+\int_{\log y}^{\infty} e^x F'_{T-t}(x)dx.
\end{aligned}$$

After computations we get

$$\begin{aligned}
P_t = \; & S_t(-1+N(d_3)(1+\sigma^2/2r)) \\
& +M_t e^{-r(T-t)}\Big(N(d_1)-\sigma^2/2r(S_t^{-1}M_t)^{(2r/\sigma^2)-1}N(d_2)\Big),
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
d_1 &= \frac{\log(M_t/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{-\log(M_t/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_3 &= \frac{-\log(M_t/S_t)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}$$

Similarly, from the result

$$\mathbf{P}\Big(\min_{s\le t}(\sigma B_s+\lambda s)\le -x\Big)=N\Big(\frac{-x-\lambda t}{\sigma\sqrt{t}}\Big)+e^{-2\lambda x/\sigma^2}N\Big(\frac{-x+\lambda t}{\sigma\sqrt{t}}\Big)$$

we obtain an explicit expression for C_t.

The explicit expressions for P_t and C_t were derived by Goldman-Sosin-Gatto (1979).

5.3.2 Lookback rate options

We only consider the sampled maximum and call option case. In this caes, the payoff of a lookback rate call option is defined by

$$\xi=(\max_{0\le s\le T}S_s-K)^+.$$

We denote by C_t its price at time t. By Theorem 4.2, we have

$$C_t=e^{-r(T-t)}\mathbf{E}^*[\xi\mid\mathcal{F}_t]. \tag{5.18}$$

Using the notations of the previous subsection and letting $K_t=\max(M_t,K)$ we have

$$\begin{aligned}
\mathbf{E}^*[\xi\mid\mathcal{F}_t] &= \mathbf{E}^*[\max(M_T,K)-K\mid\mathcal{F}_t] \\
&= \mathbf{E}^*[\max(K_t,L_t)-K_t\mid\mathcal{F}_t]+K_t-K \\
&= \mathbf{E}^*[(L_t-K_t)^+\mid\mathcal{F}_t]+K_t-K \\
&= S_t\mathbf{E}^*[(S_t^{-1}L_t-S_t^{-1}K_t)^+\mid\mathcal{F}_t]+K_t-K \\
&= S_t\mathbf{E}^*[(S_t^{-1}L_t-y)^+]\mid_{y=S_t^{-1}K_t}+K_t-K \\
&= S_t\mathbf{E}^*\Big[\exp\Big\{\Big(\max_{t\le s\le T}(\sigma(B_s^*-B_t^*)+\lambda(s-t))-y\Big)^+\Big\}\Big]+K_t-K \\
&= S_t\mathbf{E}[\exp\{(\max_{0\le s\le T-t}(\sigma B_s+\lambda s)-y)^+\}]+K_t-K \\
&= -y(1-F_{T-t}(\log y))+\int_{\log y}^{\infty} e^x F'_{T-t}(x)dx.
\end{aligned}$$

From the same computations as in the previous subsection we get

$$\begin{aligned} C_t &= S_t N(d_3)(1+\sigma^2/2r)) \\ &+ K_t e^{-r(T-t)}\Big(N(d_1) - \sigma^2/2r(S_t^{-1}K_t)^{(2r/\sigma^2)-1}N(d_2)\Big) - e^{-r(T-t)}K, \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} d_1 &= \frac{\log(K_t/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{-\log(K_t/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_3 &= \frac{-\log(K_t/S_t) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

CHAPTER 6

The Itô Process and Diffusion Models

In this chapter we will introduce a general framework for a financial market. In the first section we present some basic concepts and fundamental results on martingale methods in the pricing and hedging of European contingent claims under the Itô process setting. In the second section we show that within the diffusion process framework the pricing and hedging of European contingent claims can be reached through a PDE approach. In the third section we address the problem of pricing American contingent claims in the diffusion model. Finally, we make a brief discussion on stochstic volatility models.

6.1 The Itô process model

6.1.1 The numeraire and self-financing strategies

We fix a finite time-horizon T. Let $B = (B^1, \cdots, B^d)$ be a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $(\mathcal{F}_t)$ the natural filtration of (B_t) and by $\mathcal{L}$ the set of all measurable $(\mathcal{F}_t)$-adapted processes. We adopt the same notations $\mathcal{L}^1$ and $\mathcal{L}^2$ as defined in Chapter 3. For an Itô process X with the canonical decomposition

$$X_t = X_0 + \int_0^t a(t)dB_t + \int_0^t b(t)dt,$$

we put

$$\mathcal{L}^2(X) = \Big\{\theta \in \mathcal{L} : \theta a \in (\mathcal{L}^2)^d,\, \theta b \in \mathcal{L}^1\Big\}. \tag{6.1}$$

We consider a financial market which consists of $m+1$ assets. The price process (S_t^i) of each asset i is assumed to be a strictly positive Itô process. Since its logarithm is also an Itô process, we can represent (S_t^i) as

$$dS_t^i = S_t^i\Big[\sigma^i(t)dB_t + \mu^i(t)dt\Big],\ S_0^i = p_i,\quad 0 \le i \le m. \tag{6.2}$$

We call $\mu = (\mu^0, \cdots, \mu^m)$ the *vector of expected rate of return* and σ the *volatility matrix.*

We specify arbitrarily one of the assets, say, asset 0, as the numeraire asset. We set $\gamma_t \hat{=} (S_t^0)^{-1}$ and call γ_t the *deflator* at time t. By Itô's formula we have

$$d\gamma_t = -\gamma_t\Big[\sigma^0(t)dB_t + (\mu^0(t) - |\sigma^0(t)|^2)dt\Big].$$

We set $S_t = (S_t^1, \cdots, S_t^m)$ and $\widetilde{S}_t = (\widetilde{S}_t^1, \cdots, \widetilde{S}_t^m)$, where $\widetilde{S}_t^i = \gamma_t S_t^i$. Then we have

$$d\widetilde{S}_t^i = \widetilde{S}_t^i\Big[a^i(t)dB_t + b^i(t)dt\Big],\quad 1 \le i \le m, \tag{6.3}$$

where

$$a^i(t) = \sigma^i(t) - \sigma^0(t); \quad b^i(t) = \mu^i(t) - \mu^0(t) + |\sigma^0(t)|^2 - \sigma^i(t)\cdot\sigma^0(t).$$

In particular, if asset 0 is a bank account with interest rate process $(r(t))$, then

$$a^i(t) = \sigma^i(t), \quad b^i(t) = \mu^i(t) - r(t).$$

A *trading strategy* is a pair $\phi = \{\theta^0, \theta\}$ of $\mathcal{F}_t$-adapted processes, where

$$\theta(t) = (\theta^1(t), \cdots, \theta^m(t)), \quad \theta^i \in \mathcal{L}^2(S^i), \ \forall 0 \le i \le m.$$

$\theta^i(t)$ represents the numbers of units of asset i held at time t. The wealth V_t at time t of a trading stratey $\phi = \{\theta^0, \theta\}$ is

$$V_t = \theta^0(t)S^0(t) + \theta(t)\cdot S(t). \tag{6.4}$$

Its deflated wealth at time t is $\widetilde{V}_t = V_t\gamma_t$. A trading strategy $\phi = \{\theta^0, \theta\}$ is said to be *self-financing*, if

$$V_t = V_0 + \int_0^t \theta^0(u)dS_u^0 + \int_0^t \theta(u)dS_u. \tag{6.5}$$

Put $S_t^{m+1} = \sum_{i=0}^m S_t^i$. As in the discrete-time case, a strategy $\phi = \{\theta^0, \theta\}$ is said to be *allowable*, if there exists a positive constant c such that the wealth process (V_t) is bounded from below by $-cS_t^{m+1}$. The definition of admissible or tame strategy is similar to the discrete-time case.

Similar to the case of Black-Scholes model (see Lemma 4.1), we have the following characterization of the self-financing strategy.

Lemma 6.1 A trading strategy $\phi = \{\theta^0, \theta\}$ is self-financing if and only if

$$d\widetilde{V}_t = \theta(t)d\widetilde{S}_t. \tag{6.6}$$

Proof Assume that $\phi = \{\theta^0, \theta\}$ is a self-financing strategy. We rewrite dS_t and $d\gamma_t$ as

$$dS_t = \sigma_S(t)dB_t + \mu_S(t)dt,$$
$$d\gamma_t = \sigma_\gamma(t)dB_t + \mu_\gamma(t)dt.$$

Applying Itô's formula to the product $V_t\gamma_t$ we obtain (noting that $d(S_t^0\gamma_t) = 0$)

$$\begin{aligned} d\widetilde{V}_t &= V_t d\gamma_t + \gamma_t dV_t + d\langle V, \gamma\rangle_t \\ &= (\theta(t)\cdot S_t)d\gamma_t + \gamma_t\theta(t)dS_t + [\theta(t)^T\sigma_S(t)]\sigma_\gamma(t)dt + \theta^0(t)d(S_t^0\gamma_t) \\ &= \theta(t)d[S_t d\gamma_t + \gamma_t dS_t + d\langle S, \gamma\rangle_t] = \theta(t)d\widetilde{S}_t. \end{aligned}$$

Similarly, we can prove the "if" part.

6.1.2 Equivalent martingale measures and no-arbitrage

Assume that asset j is taken as the numeraire asset. Let $\mathbf{Q}$ be a probability measure equivalent to the "objective" probability measure $\mathbf{P}$. If the deflated price process is a (vector-valued) $\mathbf{Q}$-martingale, we call $\mathbf{Q}$ an *equivalent martingale measure* for the market. We denote by $\mathcal{M}^j$ the set of all equivalent martingale measures. Here the superscript j indicates that asset j is taken as the numeraire. We will see below in Theorem 6.9 that for any pair (i,j) of indices from $\{0,\cdots,m\}$ there exists a bijection from $\mathcal{M}^i$ onto $\mathcal{M}^j$.

A market is said to have *arbitrage* opportunity if there exists an allowable self-financing strategy such that its initial wealth V_0 is zero but its terminal wealth V_T is non-negative and $\mathbf{P}\big(V_T>0\big)>0$.

In the following we specify asset 0 as the numeraire. By Lemma 6.1, the deflated wealth process of any self-financing strategy is a local $\mathbf{Q}$-martingale for each $\mathbf{Q}\in\mathcal{M}^0$. As a consequence, for any $\mathbf{Q}\in\mathcal{M}^0$, the wealth process of any allowable self-financing strategy is a $\mathbf{Q}$-supermartingale.

If $\mathcal{M}^0\neq\emptyset$, we say that the market is a *fair* market. The following theorem shows that the fairness of a market implies no-arbitrage.

Theorem 6.2 A fair market has no arbitrage.

Proof Let $\mathbf{P}^*\in\mathcal{M}^0$. Let $\phi=\{\theta^0,\theta\}$ be an allowable self-financing strategy with initial wealth zero. As mentioned above, its deflated wealth process $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-supermartingale. Therefore, we must have $\mathbf{E}^*[\widetilde{V}_T]\le 0$. So there is no arbitrage.

Remark For a general semimartingale model of financial market, Frittelli and Lakner (1994) showed that the fairness of a market is equivalent to "no-free-lunch", slightly weaker than no-arbitrage. By using a theorem of Kusuoka (1993) and Delbaen (1992) Yan (1997) gives a new characterization of fairness of a market. In view of the economic meaning of no-free-lunch, an equivalent martingale measure is also called a *risk-neutral probability measure.*

One raises naturally a question: what conditions should we impose on coefficients a and b of the diffusion $(\widetilde{S}_t)$ such that the market is fair? The following theorem gives a partial answer to this question.

Theorem 6.3 If the market is fair, the linear equation

$$a(t)\psi(t)=b(t),\quad dt\times d\mathbf{P}-\text{a.e., a.s. on }[0,T]\times\Omega \tag{6.7}$$

has a solution $\psi\in(\mathcal{L}^2)^d$. Conversely, if

$$\mathbf{E}\left[\exp\left\{\frac{1}{2}\int_0^T|a^i(t)|^2dt\right\}\right]<\infty,\quad 1\le i\le m, \tag{6.8a}$$

and equation (6.7) has a solution $\psi\in(\mathcal{L}^2)^d$ satisfying

$$\mathbf{E}\left[\exp\left\{\frac{1}{2}\int_0^T|\psi(t)|^2dt\right\}\right]<\infty, \tag{6.8b}$$

then the probability measure $\mathbf{Q}$ with Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}=\mathcal{E}(-\psi.B)_T$ is an equivalent martingale measure.

Proof Let $\mathbf{Q}\in\mathcal{M}^0$. Put

$$M_t=\mathbf{E}\big(\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathcal{F}_t\big).$$

Then (M_t) is a $\mathbf{P}$-martingale. By the martingale representation theorem for Brownian motion there exists $\phi\in(\mathcal{L}^2)^d$ such that $dM_t=\phi(t)dB_t$. Set $\psi(t)=-\phi(t)/M_t$. Then $M=\mathcal{E}(-\psi.B)$ and by Girsanov's theorem $B_t^*=B_t+\int_0^t\psi(s)ds$ is a Brownian motion under $\mathbf{Q}$. Moreover, by Theorem 3.13 (B_t^*) has also the martingale representation property w.r.t. $(\mathcal{F}_t)$ under $\mathbf{Q}$. Thus there exists some $\sigma^* \in (\mathcal{L}^2)^{m\times d}$ such that

$$d\widetilde{S}_t = \sigma^*(t)dB_t^* = \sigma^*(t)(dB_t + \psi(t)dt).$$

According to the uniqueness of the representation of Itô process $(\widetilde{S}_t)$ and the invariance of the stochastic integral under a change of probability, from (6.3) we know that $\sigma^*(t) = \widetilde{S}_t a(t),\ dt\times d\mathbf{P}$-a.e., a.s., and consequently, $a(t)\psi(t) = b(t),\ dt\times d\mathbf{P}$-a.e., a.s.. So $(\psi(t))$ is a solution of equation (6.7).

Now assume that a satisfies $(6.8a)$, ψ is a solution of (6.7) and verifies $(6.8b)$. By the Novikov theorem (Theorem 3.11) $\mathcal{E}(-\psi.B)$ is a $\mathbf{P}$-martingale. So we can define a probability measure $\mathbf{Q}$ such that $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\psi.B)_T$. In order to prove that $(\widetilde{S}_t)$ is a $\mathbf{Q}$-martingale, it suffices to prove that the product $\mathcal{E}(-\psi.B)\widetilde{S}$ is a $\mathbf{P}$-martingale. By (6.3) and (3.8) we have

$$\widetilde{S}_t^i = \widetilde{S}_0^i \exp\Big\{\int_0^t [a^i(s)dB_s + b^i(s)ds] - \frac{1}{2}\int_0^t |a^i(s)|^2 ds\Big\}.$$

Thus from (6.7) we know that

$$\mathcal{E}(-\psi.B)_t\widetilde{S}_t^i = \widetilde{S}_0^i \exp\Big\{\int_0^t (a^i(s)-\psi(s))dB_s - \frac{1}{2}\int_0^t |a^i(s)-\psi(s)|^2 ds\Big\}.$$

Once again by the Novikov Theorem $\mathcal{E}(-\psi.B)\widetilde{S}^i$ is a $\mathbf{P}$-martingale.

This theorem leads us to pose the following definition.

Definition 6.4 If a satisfies $(6.8a)$ and equation (6.7) has a solution ψ which satisfies $(6.8b)$, the market is called *standard*.

According to Theorem 6.3 and 6.2 a standard market is fair.

Remark Assume that asset 0 is a bank account with the interest rate $r(t)$ and equation (6.7) has a solution $\eta \in (\mathcal{L}^2)^d$. We call such a η a *market price of risk* (process). Note that in this case equation (6.7) is reduced to

$$\sigma(t)\eta(t) = \mu(t) - r(t),\ \ dt\times d\mathbf{P}-\text{a.e., a.s. on } [0,T]\times\Omega. \qquad (6.7)'$$

So the economic meaning of a market price of risk is that η provides a proportional relationship between mean rates of change of prices $\mu - r$ and the amounts σ of "risk" in asset price changes. If $\mathbf{Q}$ is an eqivalent martingale measure and $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\eta.B)_T$, then η is just a market price of risk.

The following theorem provides a sufficient condition for the existence of a unique equivalent martingale measure.

Theorem 6.5 Assume that $m \geq d$, a satisfies $(6.8a)$ and $a^T(t)a(t)$ are non-degenerated for a.e. t, where $a^T(t)$ stands for the transpose of $a(t)$. Put $\psi(t) = (a^T(t)a(t))^{-1}a^T(t)b(t)$. If ψ satisfies (6.7) and $(6.8b)$, then there exists a unique equivalent martingale measure $\mathbf{P}^*$ for the market. Moreover, we have

$$\mathbf{E}\Big[\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|\mathcal{F}_t\Big] = \exp\Big\{-\int_0^t \psi(s)dB_s - \frac{1}{2}\int_0^t |\psi(s)|^2 ds\Big\},\ 0\leq t\leq T.$$

Proof Under the assumptions of the theorem, the market is standard, so by Theorem 6.3 there exists an equivalent martingale measure. To prove the uniqueness, let $\mathbf{Q}$ be an equivalent martingale measure. There exists a $\theta \in (\mathcal{L}^2)^d$ such that $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(\theta.B)_T$. By Theorem 6.3, we have $a(t)\theta(t) = b(t)$. Consequently, applying $(a^T(t)a(t))^{-1}a^T(t)$ to both sides of this equation we get $\theta(t) = \psi(t)$. The uiqueness is thus proved.

Remark If $m = d$, then ψ satisfies (6.7) automatically.

By a *European contingent claim* we mean a non-negative $\mathcal{F}_T$-measurable random variable.

Definition 6.6 Let $\mathbf{Q} \in \mathcal{M}^0$. A European contingent claim ξ is said to be $\mathbf{Q}$-*replicatable* (or *attainable*) if there exists an admissible self-financing strategy such that its terminal wealth is equal to ξ and its deflated wealth process is a $\mathbf{Q}$-martingale for some $\mathbf{Q} \in \mathcal{M}^0$. Such a trading strategy is called a $\mathbf{Q}$-*hedging strategy* for ξ.

Now assume that there exists a unique equivalent martingale measure $\mathbf{P}^*$ for the market. According to a theorem due to Jacod and Yor (1977), the uniqueness of equivalent martingale measure implies the martingale representation property under the equivalent measure. From this result we can prove that the market is *complete* in the sense that any European contingent claim ξ with the deflated value $\gamma_T\xi$ being $\mathbf{P}^*$-integrable is $\mathbf{P}^*$-replicatable. In our case we can prove this result directly, as will be shown below.

Theorem 6.7 If the conditions in Therorem 6.5 are satisfied, then the market is complete.

Proof Let $\mathbf{P}^*$ be the unique equivalent martingale measure. By Theorem 6.5 we have $\frac{d\mathbf{P}^*}{d\mathbf{P}} = \mathcal{E}(-\psi.B)_T$, where

$$\psi(t) = (a^T(t)a(t))^{-1}a^T(t)b(t).$$

We put $B_t^* = B_t + \int_0^t \psi(s)ds$. Then (B_t^*) is a $\mathbf{P}^*$-Brownian motion. Let ξ be a non-negative contingent claim such that $\mathbf{E}^*[\gamma_T\xi] < \infty$. We put

$$V_t = \gamma_t^{-1}\mathbf{E}^*[\gamma_T\xi \mid \mathcal{F}_t].$$

Then the deflated value process $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale. Since by Theorem 3.13 (B_t^*) also has the martingale representation property w.r.t. $(\mathcal{F}_t)$ under $\mathbf{P}^*$, there exist two processes H and K in $(\mathcal{L}^2)^{m\times d}$ such that

$$d\widetilde{V}_t = H_t dB_t^* = H_t(dB_t + \psi(t)dt),$$

$$d\widetilde{S}_t = K_t dB_t^* = K_t(dB_t + \psi(t)dt).$$

From (6.3) we know that $K_t = \widetilde{S}_t a(t)$ and $a(t)\psi(t) = b(t)$. Thus we have

$$\widetilde{S}_t^{-1}(a^T(t)a(t))^{-1}a^T(t)d\widetilde{S}_t = dB_t + \psi(t)dt.$$

Set

$$\theta(t) = H_t\widetilde{S}_t^{-1}(a^T(t)a(t))^{-1}a^T(t), \quad \theta^0(t) = {S_t^0}^{-1}\Big[V_t - V_0 - \int_0^t \theta(u)dS_u\Big].$$

Then $\phi = \{\theta^0, \theta\}$ is a hedging strategy for ξ, whose wealth process is (V_t). By definition, the market is complete.

Remark If $m = d$ and the market is standard, then the market is complete if and only if $a(t,\omega)$ is non-singular, for $(t,\omega) \in [0,T] \times \Omega$, a.e., a.s.. See Karatzas (1997).

6.1.3 Pricing and hedging of European contingent claims

In this subsection we study the problem of the pricing and hedging of European contingent claims. We assume that the market is fair.

Let ξ be a contingent claim. One raises naturally a question: what is a "fair" price process of ξ? Assume that $\gamma_T \xi$ is $\mathbf{P}^*$-integrable for some $\mathbf{P}^* \in \mathcal{M}^0$. We put

$$V_t = \gamma_t^{-1} \mathbf{E}^*[\gamma_T \xi \mid \mathcal{F}_t]. \tag{6.9}$$

If we consider (V_t) as the price process of an asset, then the market augmented with this asset is still fair, because the deflated price process of this asset is a $\mathbf{P}^*$-martingale. So it seems that (V_t) can be considered as a candidate for a"fair" price process of ξ. However this definition of "fair" price depends on the choice of the equivalent measure. We will show that for replicatable contingent claims this definition is reasonable.

Theorem 6.8 Let $\mathbf{P}^*, \mathbf{Q} \in \mathcal{M}^0$ and ξ be a $\mathbf{P}^*$- and $\mathbf{Q}$-replicatable contingent claim. Let (V_t) (resp. (U_t) be the wealth process of a $\mathbf{P}^*$- (resp. $\mathbf{Q}$-)hedging strategy for ξ. Then (V_t) and (U_t) are the same. Moreover, V_t is given by (6.9).

Proof Put $\widetilde{V}_t = \gamma_t V_t$, $\widetilde{U}_t = \gamma_t U_t$. Then $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale and a $\mathbf{Q}$-supermartingale and $(\widetilde{U}_t)$ is a $\mathbf{Q}$-martingale and a $\mathbf{P}^*$-supermartingale. Note that $U_T = V_T = \xi$ we have

$$\mathbf{E}^*[\widetilde{V}_T \mid \mathcal{F}_t] = \widetilde{V}_t \geq \mathbf{E}_{\mathbf{Q}}[\widetilde{V}_T \mid \mathcal{F}_t] = \widetilde{U}_t.$$

Thus we have $V_t \geq U_t$, a.s.. Similarily, we have $U_t \geq V_t$, a.s.. Hence $V = U$. The last assertion of the theorem is obvious.

Remark According to Theorem 6.8, for a $\mathbf{P}^*$-replicatable contingent claim ξ it is natural to define its *"fair" price* at time t by (6.9). We call this method of pricing the *risk-neutral valuation* (or *pricing by arbitrage*, or *arbitrage pricing*).

The following theorem shows that the risk-neutral valuation is invariant under the change of numeraire.

Theorem 6.9 Let $j \in \{0, 1, \ldots, m\}$. For a $\mathbf{P}^* \in \mathcal{M}^0$ we define a probability measure $\mathbf{Q}$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}^*} = \frac{S_0^0}{S_0^j} (S_T^0)^{-1} S_T^j. \tag{6.10}$$

We denoted it by $h_j(\mathbf{P}^*)$. Then hj is a bijection from $\mathcal{M}^0$ onto $\mathcal{M}^j$. Moreover, if $\mathbf{P}^* \in \mathcal{M}^0$ and ξ is a $\mathbf{P}^*$-replicatable contingent claim, then ξ is a $h_j(\mathbf{P}^*)$-replicatable contingent claim, and its "fair" price process is invariant under the change of numeraire.

Proof We let $\gamma_t' = (S_t^j)^{-1}$ and put

$$\widehat{S}_t^i = \gamma_t' S_t^i, \quad 0 \leq i \leq m.$$

Let $\mathbf{P}^* \in \mathcal{M}^0$. We define a probability measure $\mathbf{Q}$ by (6.10). Since $(S_t^0)^{-1} S_t^j$ is a $\mathbf{P}^*$-martingale, we must have

$$M_t := \mathbf{E}^*\left[\frac{d\mathbf{Q}}{d\mathbf{P}^*} \,\middle|\, \mathcal{F}_t\right] = \frac{S_0^0}{S_0^j} (S_t^0)^{-1} S_t^j. \tag{6.11}$$

From the fact that

$$M_t \widehat{S}_t^i = M_t \gamma_t' S_t^i = \frac{S_0^j}{S_0^0} \widetilde{S}_t^i$$

we know that $\mathbf{Q}$ is an equivalent martingale measure for the market with asset j as the numeraire asset, i.e. $\mathbf{Q} \in \mathcal{M}^j$. Now assume that ξ is a $\mathbf{P}^*$-replicatable contingent claim. We have

$$\mathbf{E}_{\mathbf{Q}}[(\gamma'_T)^{-1}\xi] = \mathbf{E}^*[M_T(\gamma'_T)^{-1}\xi] = \frac{S_0^0}{S_0^j}\mathbf{E}^*[\gamma_T\xi] = (S_0^j)^{-1}V_0.$$

This implies that a $\mathbf{P}^*$-hedging strategy for ξ is also a $\mathbf{Q}$-hedging strategy for ξ. So ξ is a $\mathbf{Q}$-replicatable contingent claim. Moreover, by the Bayes rule we have

$$\begin{aligned}(\gamma'_t)^{-1}\mathbf{E}_{\mathbf{Q}}[\gamma'_T\xi|\mathcal{F}_t] &= (\gamma'_t)^{-1}M_t^{-1}\mathbf{E}^*[M_T\gamma'_T\xi|\mathcal{F}_t] \\ &= \gamma_t^{-1}\mathbf{E}^*[\gamma_T\xi|\mathcal{F}_t].\end{aligned}$$

This proves that the "fair" price process of ξ is invariant under the change of numeraire.

Now assume that the conditions in Theorem 6.5 are satisfied. So there exists a unique equivalent martingale measure $\mathbf{P}^*$ for the market and by Theorem 6.7 the market is complete. Let ξ be a European contingent claim such that $\gamma_T\xi$ is $\mathbf{P}^*$-integrable. From the proof of Theorem 6.7, there exists actually a $\mathbf{P}^*$-hedging strategy for ξ. So in this case, the "fair" price process of ξ is given by (6.9).

In general, if a contingent claim ξ is not replicatable, we can not define its "fair" price process. In this case we need new kinds of trading strategies. Similar to the discrete-time case, a *strategy with consumption* is a trading strategy $\phi = \phi = \{\theta^0, \theta\}$ with the property that for all $t \in [0, T]$,

$$\theta^0(t)S_t^0 + \theta(t)\cdot S_t = \theta^0(0)S_0^0 + \theta(0)\cdot S_0 + \int_0^t \theta^0(u)dS_u^0 + \int_0^t \theta(u)dS_u - C_t$$

where (C_t) is an adapted , continuous, non-decreasing process null at $t = 0$. C_t represents the cumulative consumption up to time t. By contrast, a *strategy with reinvestment* is a trading strategy $\phi = \{\theta^0, \theta\}$ with the property that for all $t \in [0, T]$,

$$\theta^0(t)S_t^0 + \theta(t)\cdot S_t = \theta^0(0)S_0^0 + \theta(0)\cdot S_0 + \int_0^t \theta^0(u)dS_u^0 + \int_0^t \theta(u)dS_u + R_t$$

where (R_t) is an adapted, continuous, non-decreasing process null at $t = 0$. R_t represents the cumulative reinvestment up to time t. For a strategy ϕ of these two kinds we denote by $V_t(\phi)$ the wealth at time t of ϕ, namely, $V_t(\phi) = \theta^0(t)S_t^0 + \theta(t)\cdot S_t$. Similar to the discrete-time case, the deflated wealth process of a strategy with consumption is a $\mathbf{Q}$-supermartingale for any equivalent martingale measure $\mathbf{Q}$. By contrast, the deflated wealth process of a strategy with reinvestment is a local $\mathbf{Q}$-submartingale for any equivalent martingale measure $\mathbf{Q}$. We denote by $\mathcal{G}_c$ the set of all admissible strategies with consumption and by $\mathcal{G}_r$ the set of all admissible strategies with reinvestment.

The following definition seems to be reasonable.

Definition 6.10 Let ξ be a European contingent claim such that ξ is $\mathbf{Q}$-integrable for some $\mathbf{Q} \in \mathcal{M}^0$. We put

$$V_t^s = \text{essinf}\{V_t(\phi) : \phi \in \mathcal{G}_c, V_T(\phi) \geq \xi\},$$

$$V_t^b = \text{esssup}\{V_t(\phi) : \phi \in \mathcal{G}_r, V_T(\phi) \leq \xi\}.$$

We call V_t^s and V_t^b the *seller's price* (or *upper-price*) and *buyer's price* (or *lower-price*) at time t of ξ, respectively.

Note that there exists a version of V^s (resp. V^b) such that V^s (resp. V^b) is a $\mathbf{Q}$-supermartingale (resp. local $\mathbf{Q}$-submartingale).

We refer the reader to Karatzas (1997) or Musiela and Rutkowski (1997) for an account of this subject. Note that our definition of buyer's price is a little different from that given in the above two books.

6.2 PDE approach to contingent claim pricing

In this and the next section we assume that the market consistes of $m+1$ assets, one of which is a bank account. We denote by S_t^0 the value process of the bank account and $S_t = (S_t^1, \cdots, S_t^m)$ the price processes of the other assets. We take (S_t^0) as the numeraire. Assume that the interest rate process is of the form $r(t, S_t)$ where $r : \mathbf{R}_+ \times \mathbf{R}^m \to \mathbf{R}_+$ is Borel measurable and (S_t) is a diffusion process. Moreover, we assume that there exists a unique equivalent martingale measure $\mathbf{P}^*$ for $(\widetilde{S}_t)$. Then under $\mathbf{P}^*$, (S_t) can be expressed as:

$$dS_t^i = S_t^i\Big[\sigma^i(t, S_t)dB_t^* + r(t, S_t)dt\Big],\ S_0^i = p_i, \quad 1 \le i \le m, \tag{6.12}$$

where $\sigma : \mathbf{R}_+ \times \mathbf{R}^m \to M^{m,d}$ is Borel measurable and (B_t^*) is a d-dimensional Brownian motion under $\mathbf{P}^*$. If $r(t,x)x$ and the matrix $(x^i\sigma_j^i)$ are Lipschtiz in x and satisfy the linear growth condition in x, then according to Theorem 3.14, (6.12) has a unique solution.

We shall show how the problem of European contingent claim pricing in this model is related to a parabolic PDE. Let a European contingent claim ξ be of the form $g(S_T)$ with a non-negative Borel function on $\mathbf{R}_+^d$. Assume that $\mathbf{E}^*[\gamma_T g(S_T)] < \infty$. By (6.9), the price at time t of an contingent claim ξ is given by

$$V_t = \mathbf{E}^*\left[e^{-\int_t^T r(s,S_s)ds} g(S_T) \,\Big|\, \mathcal{F}_t\right]. \tag{6.13}$$

If V_t can be expressed as $V_t = F(t, S_t)$, then by Markovian property of the diffusion (S_t), intuitively we should have

$$F(t,x) = \mathbf{E}^*[e^{-\int_t^T r(s,S_s)ds} g(S_T)|S_t = x].$$

By using the notation in (3.20) we can rewrite this expression as

$$F(t,x) = \mathbf{E}^{*t,x}[e^{-\int_t^T r(s,S_s)ds} g(S_T)]. \tag{6.14}$$

Consequently, under some purely technical conditions, $F(t,x)$ solves the following parabolic PDE

$$-\frac{\partial u}{\partial t} + ru = \mathcal{A}_t u, \quad (t,x) \in [0,T) \times \mathbf{R}_+^d \tag{6.15}$$

subject to the terminal condition $u(T,x) = g(x)$, where $(\mathcal{A}_t f)(x)$ is defined by (3.15) with $a(t,x) = x\sigma(t,x)(x\sigma(t,x))^T$ and $b_i(t,x) = r(t,x)x_i$. The Black-Scholes differential equation (4.10) is a particular case of (6.15).

6.3 Pricing American contingent claims

Now we address the problem of pricing American contingent claims in the diffusion model setting. We will use the same notations adopted in the previous section. Recall that an American contingent claim is defined as an adapted non-negative process $(h_t)_{0\leq t\leq T}$. For simplicity, we only consider an American contingent claim of the form $h_t = g(t, S_t)$. If $m = 1$, for an American call , we have $g(t, x) = (x - K)^+$, and for an American put, $g(t, x) = (K - x)^+$.

Let $\mathcal{T}_{t,T}$ be the set of all stopping times taking values in $[t, T]$. Put

$$\Phi(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbf{E}^{*t,x}\left[e^{-\int_t^\tau r(s,S_s)ds} g(\tau, S_\tau)\right], \tag{6.16}$$

where function g is assumed to be good enough such that $\Phi(t, x)$ is well defined. It is not difficult to prove that the process $\gamma_t\Phi(t, S_t)$ is the supermartingale that dominates the process $\gamma_t g(t, S_t)$ for all $t \in [0, T]$.

The following two theorems are main results concerning pricing American contingent claims. For a proof we refer the reader to Karatzas (1988).

Theorem 6.11 There exists a trading strategy with consumption ϕ such that ϕ super-hedges $(g(t, S_t))$ and its wealth process $V_t(\phi)$ is given by $V_t(\phi) = \Phi(t, S_t)$, $\forall t \in [0, T]$. Moreover, for any trading strategy with consumption ψ which super-hedges $(g(t, S_t))$, we have $V_t(\psi) \geq g(t, S_t)$ for all $t \in [0, T]$.

We call $\Phi(0, S_0)$ the *upper price* or *selling price* at time 0 of the American contingent claim.

Theorem 6.12 Under some technical conditions, $\Phi(t, x)$ solves the following system of partial differential inequalities:

$$\frac{\partial u}{\partial t} + \mathcal{A}_t u - ru \leq 0,\ \ u \geq g \ \text{ in } [0,\mathrm{T}] \times \mathbf{R}^{\mathrm{m}}, \tag{6.17}$$

$$(\frac{\partial u}{\partial t} + \mathcal{A}_t u - ru)(g - u) = 0, \text{ in } [0,\mathrm{T}] \times \mathbf{R}^{\mathrm{m}}, \tag{6.18}$$

$$u(T,x) = g(T,x) \ \text{ in } \mathbf{R}^{\mathrm{m}}. \tag{6.19}$$

Now we turn to the optimal exercise problem on American contingent claims. Let $\tau \in \mathcal{T}_{0,T}$. If one exercises the American contingent claim at the stopping time τ, the initial value of the payoff $g(\tau, S_\tau)$ is given by

$$V_0^\tau = \mathbf{E}^*\left[e^{-\int_0^\tau r(s,S_s)ds} g(\tau, S_\tau)\right].$$

Put

$$\tau^* = \inf\left\{t \in [0,T] : \Phi(t, S_t) = g(t, S_t)\right\}. \tag{6.20}$$

Then τ^* is a stopping time which maximizes V_0^τ within $\mathcal{T}_{0,T}$. So it is reasonable to consider τ^* as the optimal exercise time for the American contingent claim. For an American put option in the Black-Scholes setting, we set

$$s_f(t) = \sup\left\{x \in \mathbf{R}_+ :\ \Phi(t,x) = (K - x)^+\right\}. \tag{6.21}$$

The function $s_f(t)$ is called the *critical price* (or *optimal stopping boundary*), which is not known a priori. It turns out that

$$\tau^* = \inf\Big\{t \in [0,T] : S_t = s_f(t)\Big\}, \tag{6.22}$$

and Φ satisfies the following *free boundary* condition

$$\Phi(t, s_f(t)) = (K - s_f(t))^+, \ \ \frac{\partial \Phi}{\partial x}(t, s_f(t)) = -1. \tag{6.23}$$

So the problem of pricing and optimal exercising American put options is reduced to solving a free boundary problem for a PDE. We refer the reader to Wilmott-Dewynne-Howison (1993) for a detailed treatment of this problem.

6.4 Stochastic volatility models

The simplist diffusion model for the price process (S_t) of a risky asset is

$$dS_t = S_t[\mu(t, S_t)dt + \sigma(t, S_t)dB(t)]. \tag{6.24}$$

We call $\mu(t, S_t)$ and $\sigma(t, S_t)$ the *stochastic* expected rate of return and volatility of the asset price. Let $\sigma_t = \sigma(t, S_t)$. If $\sigma(t, x)$ is a $C^{1,2}$-function then by Itô's formula σ_t itself is a diffusion process. But this diffusion process is driven by the same Brownian motion (B_t). More generally, we should model the price process (S_t) of a risky asset by the following equation:

$$dS_t = S_t[\mu(t, S_t)dt + \sigma_t dB_t], \tag{6.25}$$

with the volatility (σ_t) itself modeled by a diffusion process:

$$d\sigma_t = a(t, \sigma_t)dt + b(t, \sigma_t)dW_t. \tag{6.26}$$

Here B and W are two different one-dimensional Wiener processes defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, with the quadratic covariation process $\langle B, W\rangle_t = \rho t$, where ρ is a constant with $|\rho| \le 1$. Such a model is called a *stochastic volatility model.* Many authors have proposed various models of stochastic volatility, such as

$$\begin{aligned} d\sigma_t &= \kappa(\nu - \sigma_t)dt + \theta dW_t, \\ d\sigma_t &= \kappa\sigma_t(\nu - \sigma_t)dt + \theta\sigma_t dW_t. \end{aligned}$$

Unfortunately, since the volatility is not a tradable asset, markets with stochastic volatility models are incomplete. Consequently, many contingent claims can not be priced by arbitrage. However, for a general volatility model specified by (6.25)–(6.26), if two Wiener processes B and W are independent, the price process V_t of a European option written on the risky asset can be shown to be expressed as $F(t, S_t, \sigma_t)$ with $F(t, x, y)$ being a $C^{1,2,2}$-function on $[0, T) \times (0, \infty) \times \mathbf{R}$ satisfying the following PDE:

$$-rF + F_t + rxF_x + (a + \lambda b)y^2 F_y + \frac{1}{2}x^2y^2F_{xx} + \frac{1}{2}b^2F_{yy} = 0 \tag{6.27}$$

(cf. Hull and White (1987)), where $\lambda = \lambda(t, y)$ and $\lambda(t, \sigma_t)$ represents the market price of the volatility risk, which needs to be exogenously specified. In some cases, a closed-form expression for the option's price is available.

CHAPTER 7

Term Structure Models for Interest Rates

In the Black-Scholes model, it was assumed that the interest rates are a constant or a deterministic function. For short-dated options on stock-like assets, it is an acceptable approximation. However, for interest rate derivatives, it is an unreasonable assumption. Therefore, we must address the problem of random interest rates. There are different approaches to model the term structure of interest rates. They can be divided into two types: short-rate models and forward rate models. These two approaches are pioneered by Vasicek (1977) and Heath-Jarrow-Morton (1987, 1992), respectively. Recently, Flesaker and Hughston (1996) introduced a new approach to model the term structure of the interest rates. In this chapter we will present these three approaches. Some best-known models for the term structure of interest rates are presented and the valuations of some interest rate devivatives are briefly discussed. We omit the discussion about hedging, for which we refer the reader to Duffie (1996), p. 140–141. Very recently, Rogers (1997) proposed the potential approach to the term structure of interest rates and foreign exchange rates. This genaral approach will not be presented here because it is somewhat beyond our scope.

7.1 The bond market

Throughout the sequel, we fix a time horizon $[0,T]$ and consider a d-dimensional Brownian motion B on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $(\mathcal{F}_t)$ the natural filtration of B.

We consider a financial market, called *bond market*, which consists of a bank account and discount bonds with all possible maturities. By a *discount bond* (or *zero-coupon bond*) we mean a financial security which pays no dividends and is sold at a price lower than the face value paid at maturity. In the following we call a discount bond maturing at time s an *s-bond*, denote its price at time t by $P(t,s)$ and assume that $P(s,s)$ is equal to 1 (i.e. one unit of bank account).

The *yield-to-maturity* (or simply, *yield*) at time $t < s$ of an s-bond is defined as

$$Y(t,s) = -\frac{\log(P(t,s))}{s-t}. \tag{7.1}$$

It is a measure of future values of interest rates at current time t. The difference in yields at different maturities reflects market beliefs about future interest rates. A *yield curve* at time t is the graph of $Y(t,s)$ against maturities s. The dependence of the yield curve on the time to maturity, $s-t$, is called the *term structure of interest rates*. The *short rate* $r(t)$ at time t is defined as $\lim_{s>t, s\to t} Y(t,s)$, when the limit exists a.s.. In the sequel, we assume that $r(t)$ exists for all $t \in [0,T]$ and admits a measurable version. Moreover we assume $\int_0^T r(t)dt < \infty$.

If $P(t,s)$ is differentiable w.r.t. s, then another measure of future values of interest rates is the *forward rates* $f(t,s)$, for $t \le s$, which is defined by

$$f(t,s) = -\frac{\partial \log P(t,s)}{\partial s} = -\frac{(\partial P(t,s))/\partial s}{P(t,s)}. \tag{7.2}$$

Given the forward rates $f(t,s)$, we can recover the bond prices $P(t,s)$ by

$$P(t,s) = \exp\Big\{ -\int_t^s f(t,u)du \Big\}. \tag{7.3}$$

An *interest rate derivative* is a financial contract whose payoffs are contingent on future interest rates or bond prices. In order to be able to price an interest rate derivative, we need to model the dynamic behavior of interest rates and/or bond prices over the derivative's life. The basic principle is to assure the absence of arbitrage in the bond market. If $P(t,s)_{t\le s}$, for $s \le T$, are known deterministic smooth functions, then under no-arbitrage condition, $P(t,s)$ must have the form

$$P(t,s) = \exp\{-\int_t^s r(u)du\},$$

where $r(t)$ is the short rate at time t. It means that in this case the bond prices are completely determined by the short rates. However, in the uncertain world, this is no longer true. In fact, assume we are given a short rate process $(r(t))$, which is a measurable $(\mathcal{F}_t)$-adapted non-negative process. If $\mathbf{P}^*$ is a probability measure equivalent to $\mathbf{P}^*$ and we put

$$P(t,s) = \mathbf{E}^*\Big[e^{-\int_t^s r(u)du}\Big| \mathcal{F}_t\Big], \quad t \le s \le T, \tag{7.4}$$

then $P(t,s)_{t\le s}$, $s \le T$ defines bond prices and $\mathbf{P}^*$ is an equivalent martingale measure for this bond market. So different equivalent probabilitiy measures lead to different models for bond prices. We will see below (in Section 7.2) that selecting an equivalent probability measure consists in specifying the market price of risk.

7.2 Short rate models

7.2.1 One-factor models

We assume that the short rate process $(r(t))$ is modeled, under the objective probability measure $\mathbf{P}$, by a diffusion process

$$dr(t) = \mu_0(t, r(t))dt + \sigma(t, r(t))dB_t, \quad t \le T, \tag{7.5}$$

where (B_t) is a one-dimensional Brownian motion. Since the only state-variable in equation (7.5) is the short rate, we call such kind of model a *one-factor model.* In order to model a bond market related to this short rate process, first we select appropriately a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ as an equivalent martingale measure for the bond market, and then according to the risk-neutral valuation formula (7.4) to model bond price processes. For simplicity, we only consider those equivalent probability measures $\mathbf{P}^*$ whose Radon-Nikodym derivatives w.r.t. $\mathbf{P}$ have the form

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\Big\{ -\int_0^T \lambda(u, r(u))dB_u - \frac{1}{2}\int_0^T \lambda^2(u, r(u))du \Big\}, \tag{7.6}$$

where $\lambda(t,x)$ is a Borel function on $[0,T]\times\mathbf{R}$. Consequently, selecting such a probability measure consists in specifying a function λ. The latter can be estimated by using the market deta, because $\lambda(t,r(t))_{0\le t\le s}$ is the market price of risk for the s-bond (see Chapter 6 for the meaning of the market price of risk). Once we know function λ, the short rate process $(r(t))$ modeled by (7.5), can be remodeled, in the "risk-neutral" world, as

$$dr(t)=\mu(t,r(t))dt+\sigma(t,r(t))dB_t^*,\quad t\le T, \tag{7.7}$$

where $\mu(t,x)=\mu_0(t,x)-\sigma(t,x)\lambda(t,x)$, (B_t^*) is a one-dimensional $\mathbf{P}^*$-Brownian motion, and

$$B_t^*=B_t+\int_0^t\lambda(u,r(u))du.$$

By Feynman-Kac formula we know that, under some regularity conditions, the s-bond price process can be expressed as $P(t,s)=F(t,r(t);s)$, where $F(t,x;s)$ is a $C^{1,2}$-function $F(t,x;s)$ on $[0,T]\times\mathbf{R}$, for any fixed $s\in(0,T]$, and is the unique solution of the PDE

$$F_t(t,x;s)+\mu(t,x)F_x(t,x;s)+\frac{1}{2}\sigma(t,x)F_{xx}(t,x;s)-rF(t,x;s)=0, \tag{7.8}$$

with the terminal condition $F(s,x;s)=1$.

As examples of one-factor models we present now two best-known models: the Vasicek and CIR models. In the Vasicek model, it is assumed that the short rate process $r(t)$ in the risk-neutral world (i.e under the equivalent martingale measure $\mathbf{P}^*$) satisfies a stochastic differential equation (SDE) of the form

$$dr(t)=a(b-r(t))dt+\sigma dB_t^*, \tag{7.9}$$

where a,b,σ are positive constants, (B_t^*) is a Brownian motion under $\mathbf{P}^*$. Such a process is called an *Ornstein-Uhlenbeck process.* It seems that the short rate behaves like a stock price. But one important difference between the short rates and stock prices is that short rates appear over time to be pulled back to some long-run average level, a phenomenon known as *mean reversion.* In fact, if the market price of risk is a constant λ then from (7.9) we know that the short rate is pulled to a level $b+\frac{\lambda\sigma}{a}$ at rate a, because $B_t^*=B_t+\lambda t$. It is easy to verify that the unique solution to (7.9) is given by

$$r(t)=r(0)e^{-at}+b(1-e^{-at})+\sigma e^{-at}\int_0^t e^{as}dB_s^*. \tag{7.10}$$

Since $r(t)$ is normally distributed, we have $\mathbf{P}(r(t)<0)>0$, which is obviously unreasonable. Nevertheless, this simple model has an advantage that it provides an explicit expression for the s-bond's prices as follows:

$$P(t,s)=e^{A(t,s)-B(t,s)r(t)}, \tag{7.11}$$

where

$$B(t,s)=\frac{1-e^{-a(s-t)}}{a} \tag{7.12}$$

and

$$A(t,s)=\frac{(B(t,s)-st)(a^2b-\sigma^2/2)}{a^2}-\frac{\sigma^2B(t,s)^2}{4a} \tag{7.13}$$

These formulas can be derived either from solving equation (7.8) or calculating the conditional expectation in (7.4) by using (7.10).

As mentioned above, a drawback of the Vasicek model is that the short rate can become negative. In order to overcome this problem, Cox-Ingersoll-Ross (1985) has proposed to model the behavior of the short rate, in the risk-neutral world, by the following SDE:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dB_t^*, \tag{7.14}$$

where a, b, σ are positive constants. According to Theorem 3.17, equation (7.14) has a unique solution, which must be non-negative. By solving equation (7.8) we obtain the same expression (7.11) for the s-bond prices, where

$$B(t,s) = \frac{2(e^{\gamma(s-t)} - 1)}{(\gamma + a)(e^{\gamma(s-t)} - 1) + 2\gamma} \tag{7.15}$$

and

$$A(t,s) = \frac{2ab}{\sigma^2}\log\left[\frac{2\gamma e^{(a+\gamma)(s-t)/2}}{(\gamma + a)(e^{\gamma(s-t)} - 1) + 2\gamma}\right] \tag{7.16}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$.

In both models, $B(t,s)$ and $A(t,s)$ are deterministic functions of s and t, and the bond prices have the form (7.11). The yield curve $Y(t,s)$ at time t is a linear function of the short rate $r(t)$:

$$Y(t,s) = \frac{1}{s-t}\big[B(t,s)r(t) - A(t,s)\big].$$

Therefore, both models are said to possess an *affine term structure.* For such a model, possible shapes of the yield curve are upward-sloping, downward-sloping, and slightly humped. We refer the reader to Duffie (1992) for a detailed disscussion on such models.

In practice, practitioners use the historical data of the short rate to estimate the parameters b, a, and σ, then calculate values for a set of traded bonds and options based on estimated parameters and compare them with market values, and finally adjust values of parameters. This procedure should be repeated until the model fits well the historical data. However, it is very difficult to adjust values of the parameters so that the bond prices fit today's observed bond prices. In order to overcome this shortcome Hull and White (1990) have extended these models to the case of time-dependent coefficients as follows:

$$dr(t) = (\Phi(t) - a(t)r(t))dt + \sigma(t)dB_t^*,$$

$$dr(t) = (\Phi(t) - a(t)r(t))dt + \sigma(t)\sqrt{r(t)}dB_t^*.$$

These extended modeles still possess an affine term structure.

7.2.2 Multi-factor models

The one-factor short rate models presented above provide explicit expressions for bond prices. However these models do not fit the real interest rate movement well. A more realistic short rate model should include some other economic variables, such as the long-term interest rate (or long rate), the yields on a fixed number of bonds, the sort rate volatility, etc. The sources of uncertainty are represented by a multi-dimensional Brownian motion. Such a model is called a *multi-factor model*.

The first of multi-factor models, proposed by Brennan and Schwartz (1979) is a two-dimensional diffusion model, in which the state variables are the short rate and long rate. The latter is represented by the reciprocal of the price of a consol. A *consol* is a special kind of coupon-bearing bond with no final maturity date. However, a recent result of Dybvig *et al.* (1996) shows that the long rate is non-decreasing. So it can never be modeled by a diffusion. A three-factor model has been recently proposed by Chen (1996). In this model, in addition to the short rate, two other factors are the short-term mean rate and the short rate volatility.

In the last few years, there have been many papers dealing with the so-called *higher-dimensional squared-Gauss-Markov prosses model*, which is described as

$$dX_t = (a(t) + C_t X_t)dt + \sigma_t dB_t^*,$$

$$r(t) = \frac{1}{2}|X_t|^2,$$

where (B_t^*) is a d-dimensional Brownian motion under the equivalent martingale measure $\mathbf{P}^*$, and σ, C are $\mathbf{R}^d \times \mathbf{R}^d$-valued functions on $\mathbf{R}^+$ and a is an $\mathbf{R}^d$-valued function. This model has the advantage that it leads to an explicit formula for bond prices. We refer the reader to Rogers (1995) for references on this model.

7.3 The HJM model

Heath, Jarrow, and Morton have proposed in 1987 another way to model the term structure (see Heath-Jarrow-Morton (1992)). The HJM model represents the term structure in terms of the forward interest rates. In this way, the model fits automatically today's yield curve. A discrete-time analog of the HJM model has been proposed by Ho and Lee (1986). Given a stochastic model f of forward interest rates, we will assume that $r(t) = \lim_{s>t, s\to t} f(t,s)$ defines the short rate at time t. For each fixed maturity s the HJM model for forward interest rates is described, in the risk-neutural world, by an Itô process:

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s)du + \int_0^t \sigma(u,s)dB_u^*, \quad t \le s, \tag{7.17}$$

where (B_t^*) is a d-dimensional Brownian motion under the equivalent martingale measure $\mathbf{P}^*, \{\mu(t,s) : 0 \le t \le s\}$ and $\{\sigma(t,s) : 0 \le t \le s\}$ are measurable adapted processes valued in $\mathbf{R}$ and $\mathbf{R}^d$ respectively such that (7.17) is well defined as an Itô process, and the initial forward curve, $f(0,s)$, is deterministic and satisfies the condition that $\int_0^T f(0,u)du < \infty$.

Assume that $(r(t))$ is the interest rate process. Put

$$W_t = \mathbf{E}^*\Big[e^{-\int_0^s r(u)du}\Big|\,\mathcal{F}_t\Big] = e^{-\int_0^t r(u)du}P(t,s).$$

Since $(W_t)_{0\le t\le s}$ is a strictly positive martingale, by the martingale representation theorem for Brownian motion, there exists a $\mathbf{R}^d$-valued adapted process (H_t) such that

$$W_t = W_0 \exp\Big\{\int_0^t H(u,s)dB_u^* - \frac{1}{2}\int_0^t |H(u,s)|^2 du\Big\},$$

namely,

$$d_t P(t,s) = P(t,s)\Big[r(t)dt + H(t,s)dB_t^*\Big]. \tag{7.18}$$

On the other hand, by (7.4), we have

$$W_t = e^{-\int_0^t r(u)du - \int_0^s f(t,u)du}.$$

Thus, under some technical conditions ensuring the applicability of a stochastic Fubini theorem, by comparing the martingale parts of the two expressions of $\log W_t$ we obtain

$$H(t,s) = \int_t^s \sigma(t,u)du. \tag{7.19}$$

As a consequence, $\mu(t,s)$ must be expressed as

$$\mu(t,s) = \sigma(t,s) \cdot \int_t^s \sigma(t,u)du. \tag{7.20}$$

For more details of the proof we refer the reader to Duffie (1996), p.151–153. From (7.17) and (7.20) we obtain

$$r(t) = f(0,t) + \int_0^t \sigma(v,t) \cdot \int_v^t \sigma(v,u)du\,dv + \int_0^t \sigma(v,t)dB_v^*.$$

In particular, when $\sigma(t,s)$ is a constant then we obtain the continuous-time limit of the Ho-Lee model:

$$dr(t) = \Phi(t)dt + \sigma dB_t^*,$$

where

$$\Phi(t) = \sigma^2 t.$$

7.4 The Flesaker-Hughston model

Recently, Flesaker and Hughston (1996) have proposed a new approach to the term stucture modelling of interest rates. The key point of this approach stems from the following observation on (7.4): Let the s-bond price process $P(t,s)$ be defined by (7.4). Set

$$\eta_t = \frac{d\mathbf{P}}{d\mathbf{P}^*}\bigg|_{\mathcal{F}_t}, \quad 0 \le t \le T,$$

then by the Bayes rule,

$$P(t,s) = A_t^{-1}\mathbf{E}[A_s \mid \mathcal{F}_t], \quad s \ge t, \tag{7.21}$$

where

$$A_s = \eta_s \exp\Big\{-\int_0^s r(\tau)d\tau\Big\}. \tag{7.22}$$

Since (η_t) is a $\mathbf{P}$-martingale, (A_t) is a $\mathbf{P}$-supermartingale. Note that the expression (7.22) is nothing but the product decomposition of the supermartingale (A_t). Now assume that (A_t) is a strictly positive $\mathbf{P}$-surmatingale and the bond price $P(t,s)$ is modeled by (7.21). If the product decomposition of the supermartingale A is of the form (7.22) with η being a $\mathbf{P}$-martingale and r being a nonnegative process, then the corresponding short rate process must be r, and the probability measure $\mathbf{P}^*$ with density process η^{-1} is an equivalent martingale measure for price processes of bonds with different maturities. As an example (due to Flesaker and Hughston (1996)), let

$$A_t = f(t) + g(t)M_t, \quad t \in [0,T], \tag{7.23}$$

where $f, g : [0,T] \to \mathbf{R}_+$ are strictly positive decreasing functions with $f(0)+g(0) = 1$, and (M_t) is a strictly positive martingale defined on a filtered probability space $\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, with $M_0 = 1$. Then it follows immediately from (7.21) that

$$P(t,s) = \frac{f(s)+g(s)M_t}{f(t)+g(t)M_t}, \quad t \in [0,s]. \tag{7.24}$$

This model fits easily the initial curve: it suffices to choose f and g such that

$$P(0,s) = f(s) + g(s), \quad s \in [0,T]. \tag{7.25}$$

In order to get an explicit expression for the short rate we assume that $(\mathcal{F}_t)$ is the natural filtration of a Brownian motion (B_t). Since (M_t) is a strictly positive martingale, it must be of the form $M_t = \mathcal{E}(\sigma.B)_t$, where σ is an adapted measurable process. Let $A_t = \eta_t C_t$ be the product decomposition of the supermartingale A, where η is a strictly positive local martingale and C is a strictly positive decreasing process with $\eta_0 = C_0 = 1$. η must be of the form $\eta_t = \mathcal{E}(\gamma.B)_t$. Thus by Itô's formula, we have

$$\eta_t dC_t + C_t \eta_t \gamma_t dB_t = dA_t = f'(t)dt + M_t g'(t)dt + g(t)M_t\sigma_t dB_t. \tag{7.26}$$

By comparing the "dB_t" terms and the remaining terms on both sides of (7.26) we find that

$$\gamma_t = \frac{\sigma_t g(t) M_t}{f(t)+g(t)M_t},$$
$$dC_t = C_t \frac{f'(t)+g'(t)M_t}{f(t)+g(t)M_t} dt. \tag{7.27}$$

Consequently, if η is a martingale and we define a probability measure $\mathbf{P}^*$ by $\frac{d\mathbf{P}^*}{d\mathbf{P}} = \eta_T^{-1}$, then $\mathbf{P}^*$ is the unique probability measure such that

$$P(t,s) = C_t^{-1}\mathbf{E}^*[C_s \mid \mathcal{F}_t]. \tag{7.28}$$

C_t can be solved from (7.27) and the result is

$$C_t = \exp\Big\{ \int_0^t \frac{f'(\tau)+g'(\tau)M_\tau}{f(\tau)+g(\tau)M_\tau} d\tau \Big\}.$$

Then from (7.28) we can derive an explicit expression for the short rate process $(r(t))$:

$$r(t) = -\frac{f'(t)+g'(t)M_t}{f(t)+g(t)M_t}. \tag{7.29}$$

In particular, it is readily verifiable that $r(t) = f(t,t)$, where $f(t,s)$ is the forward interest rates.

The main advantage of the Flesaker-Hughston model is that we can use directly the supermartingale A to express the price at time t of a interest rate derivative ξ with maturity $s < T$ as

$$V_t = A_t^{-1}\mathbf{E}[\xi A_s \mid \mathcal{F}_t], \quad \forall t \in [0,s]. \tag{7.30}$$

This equation enables us to obtain closed-form expressions for the prices of some interest rate derivatives, such as caps and swaptions. We refer the reader to Rutkowski (1997) for this subject.

7.5 Pricing interest rate derivatives

For pricing an interest rate derivative with maturity date T there are two possible choices of the numeraire: the bank account or the T-bond. When a derivative is written on interest rates and the model for the interest rates is represented by an Itô SDE (such as the Vasicek model or the CIR model), we choose the bank account as the numeraire. We are going to show that in this case the value of an interest derivative can be expressed in terms of the solution of a PDE. We assume the short rate process obeys a one-factor model specified by (7.14). Consider an interest rate derivative with maturity $\tau \le T$, which has the dividend rate $h(t, r(t))$ at any time $t \le \tau$ and the terminal payoff $g(\tau, r(\tau))$. By the definition of the equivalent martingale measure $\mathbf{P}^*$, the value at time t of the derivative is given by

$$F(t,r(t)) = \mathbf{E}^*\left[\int_t^\tau \phi_{t,s} h(s,r(s))ds + \phi_{t,\tau} g(\tau, r(\tau)) \Big| \mathcal{F}_t\right], \tag{7.31}$$

where $\phi_{t,s} = \exp\{-\int_t^s r(u)du\}$. Under suitable conditions, the Feynman-Kac formula assures that F solves the following PDE

$$\mathcal{D}F(t,x) - xF(t,x) + h(t,x) = 0, \quad (t,x) \in [0,\tau) \times \mathbf{R}^d \tag{7.32}$$

subject to the boundary condition

$$F(\tau, x) = g(\tau, x), \quad x \in \mathbf{R}^d, \tag{7.33}$$

where

$$\mathcal{D}F(t,x) = F_t(t,x) + F_x(t,x)\mu(t,x) + \frac{1}{2}F_{xx}(t,x)\sigma(t,x)^2. \tag{7.34}$$

In particular, the value at time t of the T-bond is given by $P(t,T) = f(t, r(t))$, where f solves equation (7.32) with $h = 0$ subject to the boundary condition $f(\tau, x) = 1$.

Now we assume that the term structure of interest rates is represented by the HJM model. In this case we can take the T-bond as the numeraire. More precisely, let $P(t,T)$ be the bond price at time t. We define $\alpha_t = P(t,T)/P(0,T)$ as the normalised T-bond with $\alpha_0 = 1$. We take (α_t) as the numeraire. We denote by β_t the value process of the bank account. Now we seek a probabilty measure $\mathbf{Q}$ equivalent to $\mathbf{P}^*$ such that the value process of the bank account discounted by the numeraire α_t is a $\mathbf{Q}$-martingale. To this end we define a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}^*} = \frac{\alpha_T}{\beta_T} = \frac{1}{P(0,T)\beta_T}.$$

Since the associated $\mathbf{P}^*$-martingale is obviously

$$L_t = \mathbf{E}^*\Big[\frac{d\mathbf{Q}}{d\mathbf{P}^*}\Big|\mathcal{F}_t\Big] = \frac{P(t,T)}{P(0,T)\beta_t} = \frac{\alpha_t}{\beta_t},$$

(β_t/α_t) is a $\mathbf{Q}$-martingale. Now by (7.16),

$$dL_t = L_t H(t,T) dB_t^*.$$

Thus by Girsanov's theorem,

$$\hat{B}_t = B_t^* - \int_0^t H(u,T)du$$

is a d-dimensional $\mathbf{Q}$-Brownian motion. If an interest derivative with maturity T has only the terminal payoff ξ, then by the Bayes rule its value at time t is given by

$$V_t = \beta_t \mathbf{E}^*[\beta_T^{-1}\xi \mid \mathcal{F}_t] = \beta_t L_t^{-1}\mathbf{E}_{\mathbf{Q}}[L_T\beta_T^{-1}\xi \mid \mathcal{F}_t] = P(t,T)\mathbf{E}_{\mathbf{Q}}[\xi \mid \mathcal{F}_t].$$

The following are some examples of best-known interest derivatives, to which the above mentioned valuation methodes are applicable.

(1) A *European call option* with exercise price K is a contract having the terminal payoff $(P(\tau,T)-K)^+$.

(2) An *interest-rate swap* is a contract between two *counterparties* (referred to as A and B), to exchange a series of cash payments. A agrees to pay B interest at a fixed rate and receive interest at a floating rate. The same notional principal is used in determining the size of the payments, and there is no exchange of principal. From A's point of view, it is a derivative which pays dividends at a rate $h(t,r(t)) = r(t) - r^*$, where r^* is the fixed interest rate agreed upon at time zero. It is easy to see that the value of the swap at time t is $V_t = 1 - P(t,\tau) - r^* \int_t^\tau P(t,s)ds$.

(3) An *interest rate cap* is a financial instrument that effectively places a maximum amount on the interest payments on floating-rate debt. In other words, a cap is a loan at a variable interest rate that is capped at some level $\bar{r}$. If the short rate $r(t)$ is assumed to obey equation (7.14), then per unit of the principal amount of the loan, the value of the cap is given by (7.31)–(7.32) with $h(t,x)$ =min$(x,\bar{r})$ and $g(\tau,x) = 1$.

(4) An *interest rate floor* is a financial instrument which effectively places a minimum amount on the interest payments on floating-rate debt. It is a contingent claim with a "floored" rate $\max(r(t),\underline{r})$ as the dividend rate and with the terminal payoff 1.

(5) An *interest rate collar* is a long position on a cap and a short position on a floor with the same settlement dates and reset interval.

7.6 Forward price and futures price

Consider a forward contract with maturity T written on one unit of a particular asset whose price process is (S_t). Assume that the short-term interest rate process $r(t)$ is bounded and the discounted process $(\widetilde{S}_t)$ is a martingale under the equivalent martingal measure $\mathbf{P}^*$. Let F_t be the forward price at time t of the underlying asset.

Then by definition the payoff at maturity T of this contract is equal to $S_T - F_t$. Since the value at time t of this forward contract should be zero, we have

$$0 = \mathbf{E}^*\Big[\exp\{-\int_t^T r(s)ds\}(S_T - F_t)\Big|\mathcal{F}_t\Big].$$

Consequently,

$$F_t = \frac{\mathbf{E}^*\Big[\exp\{-\int_t^T r(s)ds\}S_T\Big|\mathcal{F}_t\Big]}{\mathbf{E}^*\Big[\exp\{-\int_t^T r(s)ds\}\Big|\mathcal{F}_t\Big]},$$

which gives

$$F_t = \frac{S_t}{P(t,T)} \tag{7.35}$$

Now we study the futures price. Consider a futures contract with maturity T written on one unit of a particular asset whose price process is (S_t). Let Φ_t be the futures price at time t of the underlying asset. Assume that the settlements during the time period $(t, T]$ take place at the time $t_1 < t_2 \cdots < t_N = T$. Since the value at time t of the futures contract is zero, we must have

$$0 = \mathbf{E}^*\Big[\sum_{i=1}^N \exp\{-\int_0^{t_i} r(s)ds\}(\Phi_{t_i} - \Phi_{t_{i-1}}) \mid \mathcal{F}_t\Big],$$

where $t_0 = t, \Phi_T = S_T$. In order to get an approximation of the value Φ_t, we consider a continuous settlement which is purely fictitious. In this case, we should have

$$0 = \mathbf{E}^*\Big[\int_t^T Y_s d\Phi_s\Big|\mathcal{F}_t\Big],$$

where $Y_s = \exp\{-\int_0^s r(\tau)d\tau\}$. It means that the stochastic integral $\int_0^t Y_s d\Phi_s$ is a $\mathbf{P}^*$-martingale. Since there are constants $k_1, k_2 > 0$ such that $k_2 \leq Y \leq k_1$, Φ_t is also a martingale. Therefore we have

$$\Phi_t = \mathbf{E}^*[\Phi_T \mid \mathcal{F}_t] = \mathbf{E}^*[S_T \mid \mathcal{F}_t]. \tag{7.36}$$

From (7.35) and (7.36) we see that if r and S_T are independent, then the forward price and the futures price are the same. For example, if r is a deterministic function, then this is the case.

CHAPTER 8

The Fundamental Theorem of Asset Pricing

In this chapter we follow Yan (1997) to present the fundamental theorem of asset pricing. In order to fully understand the content of this chapter an advanced knowledge of semimartingales and stochastic integrals is required.

8.1 Introduction

In the early 70's Black and Scholes (1973) made a breakthrough in option pricing theory by deriving the celebrated Black-Scholes formula for pricing European options via a "hedge approach". This work was further elaborated and extended by Merton (1973). Cox and Ross (1976) is the overture of a modern theory of option pricing—the risk-neutral valuation or arbitrge pricing. A key step in this direction was made in Harrison and Kreps (1979). They remarked that the hedge approach is not mathematically rigorous unless one excludes doubling-like strategies. Harrisson and Kreps imposed some "admissibility" condition on the trading strategy and showed that the existence of an equivalent martingale measure for the deflated price processes implies the absence of arbitrage. Since then many attempts have been devoted to show the converse statement. Harrisson and Pliska (1981) solved this problem in discrete-time and finite-state case. This result is referred to as the *fundamental theorem of asset pricing*. In the general state and discrete-time with finite and infinite horizon case, this problem has been solved by Dalang-Morton-Willinger (1990) and Schachermayer (1994) respectively. However, in the continuous-time case and the discrete-time with infinite horizon case the absence of arbitrage is no longer a sufficient condition for the existence of an equivalent martingale measure. A "no-free-lunch" condition, slightly stronger than no-arbitrage condition, was introduced by Kreps (1981). Under a mild but irrelevant separability assumption Kreps proved that if the deflated price process is bounded then the market is fair if and only if the market has no free-lunch. See Schachermayer (1994) for a transparent proof of this result. Without knowing this result of Kreps, the problem was attacked by Stricker (1990), who discovered that a result of Yan (1980) (or more precisely, the method of its proof) is an appropriate tool for solving the problem. The result of Stricker was re-examinated and extended by Delbaen (1992), Kusuoka (1993), Lakner (1993), Delbaen and Schachermeyer (1994), Frittelli and Lakner (1994).

In this chapter we consider a semimartingale model for a market. The market is said to be *fair* if there exists an equivalent martingale measure for the deflated price process. In section 8.2 we show that the fairness of a market is invariant under the change of numeraire and give a characterization of self-financing strategies. In section 8.3, by augmenting the original market with a new asset we show that the characterization of the fairness of a market can be reduced to the case, where the deflated price process is bounded. By using a theorem of Delbaen and Schachermayer (1994) we obtain an intrinsic characterization of the fairness of a market. A theorem of Delbaen (1992) implies a more elegant result. In section 8.4 we show that a fair market has no arbitrage with allowable strategies and the arbitrage pricing of replicatable contingent claims is independent of the choice of numeraire and equivalent martingale measure.

8.2 The characterization of self-financing strategies and fair market

We fix a finite time-horizon $[0, T]$ and consider a security market which consists of $m+1$ assets whose price processes $(S_t^i), i = 0, \cdots, m$ are assumed to be strictly positive semimartingales, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ satisfying the usual conditions. Moreover, we assume that $\mathcal{F}_0$ is the trivial σ-algebra. For notational convenience, we take asset 0 as the numeraire asset. We set $\gamma_t \hat{=} (S_t^0)^{-1}$ and call γ_t the *deflator* at time t. We set $S_t = (S_t^1, \cdots, S_t^m)$ and $\widetilde{S}_t = (\widetilde{S}_t^1, \cdots, \widetilde{S}_t^m)$, where $\widetilde{S}_t^i = \gamma_t S_t^i, 1 \leq i \leq m$. We call $(\widetilde{S}_t)$ the *deflated* price process of the assets. Note that the deflated price process of asset 0 is the constant 1.

The continuous trading is modeled by a stochastic integral. In order to be able to define a trading strategy we need the notion of integration w.r.t. a vector-valued semimartingale (see Jacod (1980)). Such integral is defined globally and not componentwise. A basic fact is that a vector-valued predictable process H is integrable w.r.t. a vector-valued semimartingale X if and only if the sequence $(I_{[|H| \leq n]} H).X$ converges in the semimartingale topology. In this case the limit gives the integral $H.X$. Consequently, if $H = (H^0, \cdots, H^m)$ is integrable w.r.t. a semimartingle $(X^0, \cdots, X^m)$ and H^0 is integrable w.r.t. X^0, then we have

$$H.(X^0, \cdots, X^m) = H^0.X^0 + (H^1, \cdots, H^m).(X^1, \cdots, X^m). \tag{8.1}$$

A *trading strategy* is a $\mathbf{R}^{m+1}$-valued $\mathcal{F}_t$-predictable process $\phi = \{\theta^0, \theta\}$, where

$$\theta(t) = (\theta^1(t), \cdots, \theta^m(t)),$$

such that ϕ is integrable w.r.t semimartingale (S^0, S) with $S = (S^1, \cdots, S^m)$. $\theta^i(t)$ represents the numbers of units of asset i held at time t. This notion of trading strategy is not very realistic. However it is convenient for mathematical studies. The wealth $V_t(\phi)$ at time t of a trading stratey $\phi = \{\theta^0, \theta\}$ is

$$V_t(\phi) = \theta^0(t) S_t^0 + \theta(t) \cdot S_t, \tag{8.2}$$

where $\theta(t) \cdot S_t = \sum_{i=1}^m \theta^i(t) S_t^i$. The deflated wealth at time t is $\widetilde{V}_t(\phi) = V_t(\phi)\gamma_t$. A trading strategy $\{\theta^0, \theta\}$ is said to be *self-financing*, if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi(u) d(S_u^0, S_u). \tag{8.3}$$

In this chapter we always use notation $\int_0^t H_u dX_u$ or $(H.X)_t$ to stand for the intergral of H w.r.t. X over the interval $(0, t]$. In particular, we have $(H.X)_0 = 0$.

It is easy to see that for any given $\mathbf{R}^m$-valued predictable process θ which is integrable w.r.t (S_t) and a real number x there exists a real-valued predictable process (θ_t^0) such that $\{\theta^0, \theta\}$ is a self-financing strategy with initial wealth x.

A process $\{\theta^0, \theta\}$ is said to be *elementary*, if there exist a finite partition of $[0, T]$: $0 = t_0 \leq t_1 < \cdots < t_n = T$ and a sequence of $\mathbf{R}^{m+1}$-valued random variables $(\xi_1, \cdots, \xi_n)$, with each ξ_i being $\mathcal{F}_{t_{i-1}}$-measurable, such that

$$\theta^i(t) = \sum_{k=1}^{n} \xi_k^i I_{(t_{k-1}, t_k]}(t), \quad t \in [0, T], \ 0 \leq i \leq m.$$

If we take stopping times $t_k' s$ instead of deterministic times, the corresponding process is said to be *simple*. If furthermore $(\xi_1, \cdots, \xi_n)$ are elementary random variables (i.e. taken only a finite number of values), the corresponding process is said to be *very simple.*

Definition 8.1 A security market is said to be *fair* if there exists a probability measure $\mathbf{Q}$ equivalent to the "objective" probability measure $\mathbf{P}$ such that the deflated price processes $(\widetilde{S}_t)$ is a (vector-valued) $\mathbf{Q}$-martingale. We call such a $\mathbf{Q}$ an *equivalent martingale measure* for the market.

We denote by $\mathcal{M}^j$ the set of all equivalent martingale measures for the market, if asset j is taken as the numeraire asset.

The following theorem shows that the definition of fair market does not depend on the choice of numeraire.

Theorem 8.2 The fairness of a market is invariant under the change of numeraire.

Proof Assume that $\mathcal{M}^0 \neq \emptyset$. For a $\mathbf{P}^* \in \mathcal{M}^0$ we define a probability measure $\mathbf{Q}$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}^*} = \frac{S_0^0}{S_0^j} (S_T^0)^{-1} S_T^j. \tag{8.4}$$

We denote $\mathbf{Q}$ by $h_j(\mathbf{P}^*)$. We are going to show that h_j is a bijection from $\mathcal{M}^0$ onto $\mathcal{M}^j$. Let $\gamma_t' = (S_t^j)^{-1}$ and put

$$\widehat{S}_t^i = \gamma_t' S_t^i, \quad 0 \leq i \leq m.$$

Since $(S_t^0)^{-1} S_t^j = \widetilde{S}_t^j$ is a $\mathbf{P}^*$-martingale, we must have

$$M_t := \mathbf{E}^* \Big[\frac{d\mathbf{Q}}{d\mathbf{P}^*} \Big| \mathcal{F}_t \Big] = \frac{S_0^0}{S_0^j} (S_t^0)^{-1} S_t^j, \quad 0 \leq t \leq T. \tag{8.5}$$

From the fact that

$$M_t \widehat{S}_t^i = M_t (S_t^j)^{-1} S_t^i = \frac{S_0^j}{S_0^0} \widetilde{S}_t^i, \quad 0 \leq i \leq m$$

we know that $\mathbf{Q} \in \mathcal{M}^j$. The theorem is proved.

A strategy is said to be *admissible*, if its wealth process is non-negative. A strategy is said to be *tame*, if its deflated wealth process is bounded from below by some real constant. The weakness of the notion of tame strategy is that it is not invariant under the change of numeraire. Moreover, all bounded elementary or simple strategies are not tame. We propose below to extend the notion of tame strategy to a notion of "allowable strategy".

Definition 8.3 A strategy $\phi = \{\theta^0, \theta\}$ is said to be *allowable*, if there exists a positive constant c such that the wealth $(V_t(\phi))$ at any time t is bounded from below by $-c\sum_{i=0}^m S_t^i$.

It is easy to see that all bounded elementary or simple strategies are allowable, and the notion of allowable strategy does not involve the numeraire.

Definition 8.4 A market is said to have no arbitrage with allowable strategies if there exists no allowable self-financing strategy with initial wealth zero and a non-negative terminal wealth V_T such that $\mathbf{P}\big(V_T > 0\big) > 0$.

A key point of arbitrage pricing of contingent claims is the following characterization of the self-financing strategy.

Theorem 8.5 A strategy $\phi = \{\theta^0, \theta\}$ is self-financing if and only if its wealth process (V_t) satisfies

$$d\widetilde{V}_t = \theta(t)d\widetilde{S}_t, \tag{8.6}$$

where $\widetilde{V}_t = V_t\gamma_t$. In particular, the deflated wealth process of an allowable self-financing strategy is a local $\mathbf{Q}$-martingale and a $\mathbf{Q}$-supermartingale for any $\mathbf{Q} \in \mathcal{M}^0$.

Proof Assume that $\phi = \{\theta^0, \theta\}$ is a self-financing strategy. First of all, by Itô's formula, we have

$$d(1, \widetilde{S}_t) = d(\gamma_t S_t^0, \gamma_t S_t) = \gamma_t d(S_t^0, S_t) + (S_t^0, S_t)d\gamma_t + d([S^0, \gamma]_t, [S, \gamma]_t). \tag{8.7}$$

Secondly, by (8.1) we have

$$\phi(t)d(1, \widetilde{S}_t) = \theta(t)d\widetilde{S}_t. \tag{8.8}$$

Thirdly, by (8.3) we have

$$\Delta V_t = \theta^0(t)\Delta S_t^0 + \theta(t)\cdot\Delta S_t,$$

which together with (8.2) implies

$$V_{t-} = \theta^0(t)S_{t-}^0 + \theta(t)\cdot S_{t-}. \tag{8.9}$$

Finally, applying Itô's formula to the product $V_t\gamma_t$ we get from (8.3) and (8.7)-(8.9)

$$\begin{aligned} d\widetilde{V}_t &= V_{t-}(\phi)d\gamma_t + \gamma_{t-}dV_t + d[V,\gamma]_t \\ &= (\theta^0(t)S_{t-}^0 + \theta(t)\cdot S_{t-})d\gamma_t + \gamma_{t-}(\theta^0(t), \theta(t))d(S_t^0, S_t) \\ &\quad +\theta^0(t)d[S^0,\gamma]_t + \theta(t)\cdot d[S,\gamma]_t) \\ &= \theta(t)d\widetilde{S}_t. \end{aligned}$$

Similarly, we can prove the "if" part.

Now assume that $\phi = \{\theta^0, \theta\}$ is an allowable self-financing strategy. By definition there exists a positive constant c such that $V_t(\phi) \geq -c\sum_{i=0}^m S_t^i,\ t \in [0, T]$. Put

$$\theta_1^i = \theta^i + c, \quad 0 \leq i \leq m, \quad \phi_1 = \{\theta_1^0, \theta_1\}.$$

Then by (8.6) we have $d\widetilde{V}_t(\phi_1) = \theta_1(t)d\widetilde{S}_t$ and

$$\widetilde{V}_t(\phi_1) = \widetilde{V}_t(\phi) + c\sum_{i=0}^m \widetilde{S}_t^i \geq 0.$$

By a theorem of Ansel and Stricker (1994), $(\widetilde{V}_t(\phi_1))$ is a local $\mathbf{Q}$-martingale and $\mathbf{Q}$-supermartingle. Thus so is $(\widetilde{V}_t)$ because $(\sum_{i=0}^m \widetilde{S}_t^i)$ is a $\mathbf{Q}$-martingale.

As a corollary we obtain:

Theorem 8.6 A fair market has no arbitrage with allowable strategies.

Proof Let $\mathbf{Q} \in \mathcal{M}^0$. Let $\{\theta, \theta^0\}$ be an allowable self-financing strategy with initial wealth zero. By Theorem 8.5 the deflated wealth process of ϕ is a $\mathbf{Q}$-supermartingale. Therefore, we must have $\mathbf{E}_\mathbf{Q}[\widetilde{V}_T] \leq 0$. So the market has no arbitrage with allowable strategies.

8.3 The fundamental theorem of asset pricing

The fairness of a security market is the basis of the so-called "pricing by arbitrage". By the *fundamental theorem of asset pricing* we mean a characterization of the fairness of a market. Roughly speaking, such a characterization states that the market is fair if and only if the market has no "free-lunch". In the literature several notions of "free-lunch" have been introduced in different circumstances. A common feature of these notions is that they involve an appropriate topological closure of the set $V - L_+^\infty$, where V is the set of all achievable gains by a certain bounded elementary (or simple) strategy. If the deflated price process is a bounded (vector-valued) semimartingale, several characterizations of the fairness are available.

Now we introduce a new asset, indexed by $m+1$, whose price process is:

$$S_t^{m+1} = \sum_{i=0}^m S_t^i. \tag{8.10}$$

We augment the market with this new asset. It is readily seen that the new market is fair if and only if the old one is fair. According to Theorem 8.2 in order to characterize the fairness of the new market one can choose asset $m+1$ as the numeraire asset. In doing so the deflated price process becomes bounded. This trick not only reduces the problem to the easy case but also leads to an *intrinsic* characterization of the fairness of a market in the sense that no numeraire asset is involved.

In the following we consider the augmented market and choose the new asset as the numeraire asset. We denote by (X_t^i) the deflated price process of asset i (i.e. $X_t^i = (S_t^{m+1})^{-1} S_t^i$) and set $X_t = (X_t^0, \cdots, X_t^m)$.

A theorem of Lakner (1993, Theorem 8.1) implies immediately the following characterization of the fairness of a market.

Therore 8.7 Let process (X_t) be defined as above. Put

$$V = \{(H.X)_T : H \text{ is a very simple process}\}. \tag{8.11}$$

Then the (original) market is fair if and only if

$$\overline{V - L_+^\infty} \cap L_+^\infty = \{0\}, \tag{8.12}$$

where $\overline{V - L_+^\infty}$ is the closure of $V - L_+^\infty$ in the $\sigma(L^\infty, L^1(\mathbf{P}))$-topology.

Remark Condition (8.12) can be interpreted as "no-free-lunch" in a certain sense. In fact, if condition (8.12) is violated, then there is an $f_0 \in L_+^\infty \setminus \{0\}$ and a net $(\phi_\alpha)_{\alpha \in J}$ of very simple self-financing strategies with initial wealth 0 such that at the terminal time the agent "throws away" the amount of money $h_\alpha S_T^{m+1}$ with $h_\alpha \in L_+^\infty$

the random variable $(S_T^{m+1})^{-1}V_T(\phi_\alpha) - h_\alpha$ becomes close to f_α w.r.t $\sigma(L^\infty, L^1(\mathbf{P}))$-topology. On the other hand, according to Schachermayer (1994) the Kreps' "no-free-lunch" condition can be stated as follows:

$$\overline{(V_0 - L_+^0) \cap L^\infty} \cap L_+^\infty = \{0\}, \tag{8.13}$$

where

$$V_0 = \{(H.X)_T : H \text{ is an elementary process}\}. \tag{8.14}$$

So the economic meaning of Lakner's no-free-lunch condition (8.12) is more convincing than the Kreps' one. We refer the reader to Kusuoka (1993) for another "no-free-lunch" condition which is similar to condition (8.12). In view of the economic meaning of no-free-lunch, an equivalent martingale measure is also called a *risk-neutral probability measure.*

As pointed out in Delbaen and Schachermayer (1994) the drawback of a variant of Kreps' theorem is twofold. First it is stated in terms of nets or topological closure, a highly non intuitive concept. Second it involves the use of very risky positions. The main theorem of Delbaen and Schachermayer (1994) remedies this drawback. By using this theorem we obtain the following intrinsic characterization of the fairness of a market.

Theorem 8.8 The market is fair if and only if there is no sequence (ϕ_n) of allowable self-financing strategies with initial wealth 0 such that $V_T(\phi_n) \geq -\frac{1}{n}\sum_{i=0}^m S_T^i$ a.s., for all $n \geq 1$ and such that $V_T(\phi_n)$ a.s. tends to a non-negative random variable ξ satisfying $\mathbf{P}(\xi > 0) > 0$.

Proof Consider the market augmented with asset $m+1$ and choose asset $m+1$ as the numeraire asset. Let $\phi = \{\phi^0, \cdots, \phi^m\}$ be an "admissible" integrand for the vector semimartingale $X = (X^0, \cdots, X^m)$, in the sense of Delbaen and Schachermayer (1994) that there is a positive constant c such that $(\phi.X)_T \geq -c$. We can introduce a predictable process ϕ^{m+1} such that ϕ together with ϕ^{m+1} constitutes a self-financing strategy with initial wealth 0 for the augmented market. By Theorem 8.5 we have

$$(S^{m+1})_T^{-1} V_T(\phi, \phi^{m+1}) = \int_0^T \phi(t) dX_t. \tag{8.15}$$

On the other hand, we have

$$V_t(\phi, \phi^{m+1}) = V_t(\phi) + \phi_t^{m+1} S_t^{m+1}, \quad 0 \leq t \leq T.$$

Thus, if we put

$$\phi_t'^i = \phi_t^i + \phi_t^{m+1}, \quad 0 \leq i \leq m,$$

then we have $V_t(\phi') = V_t(\phi, \phi^{m+1})$. Consequently, by (8.15) ϕ' is an allowable strategy for the original market. It is easy to see that ϕ' is self-financing and its initial wealth is 0. Conversely, for any allowable strategy ϕ for the original market, $\{\phi, 0\}$ is a self-financing strategy for the augmented market and we have

$$(S_t^{m+1})^{-1} V_t(\phi) = (S_t^{m+1})^{-1} V_t(\phi, 0) = \int_0^t \phi(t) dX_t.$$

Thus, by the vector versions of Theorem 1.1 and Collorary 3.7 of Delbaen and Schachermayer (1994) we can conclude the theorem.

Remark According to Delbaen and Schachermayer (1994) the condition in Theorem 8.8 is called the condition of *no free lunch vith vanishing risk.*

If the asset price process is continuous, a theorem of Delbaen (1992) (Theorem 5.1) gives us a more elegant characterization of the fairness of a market.

Theorem 8.9 Assume that the asset price process is continuous. Then the market is fair if and only if the following condition is satisfied:

If (ϕ_n) is a sequence of very simple self-financing strategies such that $V_0(\phi_n) = 0$, $|V(\phi_n)| \leq S^{m+1}, \forall n \geq 1$ and $V_T(\phi_n)^- \to 0$ in probability, then $V_T(\phi_n)^+ \to 0$ in probability.

8.4 Arbitrage pricing of contingent claims in a fair market

In this section we will study the problem of the pricing of European contingent claims in a fair market. By a (*European*) *contingent claim* we mean a non-negative $\mathcal{F}_T$-measurable random variable. Let ξ be a contingent claim. One raises naturally a question: what is a "fair" price process of ξ? Assume that $\gamma_T \xi$ is $\mathbf{P}^*$-integrable for some $\mathbf{P}^* \in \mathcal{M}^0$. We put

$$V_t = \gamma_t^{-1} \mathbf{E}^*[\gamma_T \xi \,|\, \mathcal{F}_t]. \tag{8.16}$$

If we consider (V_t) as the price process of an asset, then the market augmented with this asset is still fair, because the deflated price process of this asset is a $\mathbf{P}^*$-martingale. So it seems that (V_t) can be considered as a candidate for a"fair" price process of ξ. However this definition of "fair" price depends on the choice of equivalent martingale measure. We will show that for replicatable contingent claims (see Definition 8.10) this definition is reasonable.

Definition 8.10 Let $\mathbf{P}^* \in \mathcal{M}^0$. A European contingent claim ξ is said to be $\mathbf{P}^*$-*replicatable* (or *attainable*) if $\gamma_T \xi$ is $\mathbf{P}^*$-integrable and there exists an admissible self-financing strategy ϕ such that its terminal wealth is equal to ξ and its deflated wealth process is a $\mathbf{P}^*$-martingale (i.e. $\mathbf{E}^*[\gamma_T \xi] = \gamma_0 V_0(\phi)$). Such a strategy is called a $\mathbf{P}^*$-*hedging strategy* for ξ.

The following theorem shows that the "fair" price process of a replicatable contingent claim is uniquely determined.

Theorem 8.11 Let $\mathbf{P}^*, \mathbf{P}' \in \mathcal{M}^0$ and ξ be $\mathbf{P}^*$- and $\mathbf{P}'$-replicatable. Let (V_t) (resp. (U_t)) be the wealth process of a $\mathbf{P}^*$- (resp. $\mathbf{P}'$-)hedging strategy for ξ. Then (V_t) and (U_t) are the same. Moreover, V_t is given by (8.16), and we have

$$V_t = \operatorname{essinf}_{\mathbf{Q} \in \mathcal{M}^0} \gamma_t^{-1} \mathbf{E}_{\mathbf{Q}}[\gamma_T \xi | \mathcal{F}_t]. \tag{8.17}$$

Proof Put $\widetilde{V}_t = \gamma_t V_t$, $\widetilde{U}_t = \gamma_t U_t$. Then $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale and a $\mathbf{P}'$-supermartingale and $(\widetilde{U}_t)$ is a $\mathbf{P}'$-martingale and a $\mathbf{P}^*$-supermartingale. Note that $U_T = V_T = \xi$ and we have

$$\mathbf{E}^*[\widetilde{V}_T | \mathcal{F}_t] = \widetilde{V}_t \geq \mathbf{E}'[\widetilde{V}_T | \mathcal{F}_t] = \mathbf{E}'[\widetilde{U}_T | \mathcal{F}_t] = \widetilde{U}_t.$$

Thus we have $V_t \geq U_t$, a.s.. Similarly, we have $U_t \geq V_t$, a.s.. Hence $V = U$. The last assertion of the theorem is obvious.

Remark According to Theorem 8.11, for a $\mathbf{P}^*$-replicatable contingent claim ξ it is natural to define its *"fair" price* at time t by (8.16). We call this method of pricing the *arbitrage pricing* (or *pricing by arbitrage*, or *risk-neutral valuation*).

The following theorem shows that the arbitrage pricing of replicatable contingent claims is independent of the choice of numeraire.

Theorem 8.12 Let $\mathbf{P}^* \in \mathcal{M}^0$ and ξ be a $\mathbf{P}^*$-replicatable contingent claim and ϕ be a fair hedging strategy for ξ. Then for any $0 \le j \le m$ ξ is an $h_j(\mathbf{P}^*)$-replicatable contingent claim, and its "fair" price process remains the same.

Proof We keep the notations in the proof of Theorem 8.2. We have by (8.16)

$$\mathbf{E}_{\mathbf{Q}}[\gamma_T' \xi] = \mathbf{E}^*[M_T \gamma_T' \xi] = \frac{S_0^0}{S_0^j} \mathbf{E}^*[\gamma_T \xi] = \gamma_0' V_0.$$

This implies that a $\mathbf{P}^*$-hedging strategy for ξ is also a $\mathbf{Q}$-hedging strategy for ξ. So ξ is a $\mathbf{Q}$-replicatable contingent claim. Moreover, by the Bayes rule we have

$$\begin{aligned}(\gamma_t')^{-1}\mathbf{E}_{\mathbf{Q}}[\gamma_T' \xi | \mathcal{F}_t] &= (\gamma_t')^{-1} M_t^{-1} \mathbf{E}^*[M_T \gamma_T' \xi | \mathcal{F}_t] \\ &= \gamma_t^{-1} \mathbf{E}^*[\gamma_T \xi | \mathcal{F}_t].\end{aligned}$$

This proves that the "fair" price process of ξ is invariant under the change of numeraire.

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