A Short Course on Differentiable Dynamical Systems

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Lecture Notes in Mathematics 5

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Contents

Preface

This is a short course on structural stability theory of differentiable dynamical systems. We attempt to present the essence of this theory in a short but self-contained way. We focus on Smale's Ω-stability theorem. We begin with the basic notions of dynamical systems. The first, and probably the most important example is Smale's horseshoe diffeomorphism. This is a structurally stable system for which the limit set, which contains the long run behavior of all orbits, does not reduce simply to finitely many periodic orbits. This contrasts with the classical result of Peixoto for 2-dimensional flows, and presents a striking new phenomenon: chaos can be compatible to structural stability. The analytic condition that makes such a chaotic set structurally stable is *hyperbolicity*, a concept of central importance to the topic. Hyperbolic sets have *stable manifolds*, which handle the way points approach the sets, in forward or backward time. The classical results on hyperbolic periodic orbits are hence generalized to a modern theory of hyperbolic sets. Smale's Ω-stability theorem is the first general result based on this theory. Thus our short course will consist of three parts:

Chapter 1. Preliminaries.

Chapter 2. Hyperbolic sets.

Chapter 3. The Ω-stability theorem of Smale.

The material contained in this short course is the part of structural stability theory that has been best treated in book form, see the survey article [S], and a number of books [Bo4, F5, GMN, I, KH, Ni, PdeM, Ro7, Sh] cited in the references. Our treatment for this topic follows these nice books, with more effort made on uniformity of estimates in Theorem 2.11 through 2.13 to reach Theorem 2.14.

Dynamical systems develops simultaneously in two closely related settings: the discrete case (or the diffeomorphism case) and the continuous case (or the flow case). For simplicity we take the discrete case in our short course throughout.

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Lan Wen

CHAPTER 1

Preliminaries

1.1 Some basic notions in dynamical systems

We review some basic notions in topological dynamical systems. Thus we consider a compact metric space X, together with a homeomorphism $f : X \to X$. This generates a family of *iterates* of f, written as

$$f^n = f \circ f \circ \ldots \circ f \ (n \text{ times}), \ f^0 = id, \ f^{-n} = (f^n)^{-1},$$

where n ranges over the integers. It is obvious that

$$f^n \circ f^m = f^{n+m}$$

for any integers n and m. We call the family $\{f^n\}_{n=-\infty}^{\infty}$ a (topological) *dynamical system*. We even simply call f itself a dynamical system.

For any $x \in X$, the set $\{f^n(x)\}_{-\infty}^{\infty}$ is called the *orbit* of x under f, denoted by $\mathrm{Orb}(x, f)$, or simply $\mathrm{Orb}(x)$. It is easy to see that any two orbits are either identical, or else disjoint. We call x a *periodic point* if there is $n \geq 1$ such that $f^n(x) = x$. The minimal positive integer n that satisfies this equality is called the *period* of x. Periodic points of period 1 are just fixed points. It is easy to see that a point x is periodic if and only if the orbit of x consists of finitely many points. We denote the set of periodic points of f by $\mathrm{Per}(f)$, and the set of fixed points of f by $\mathrm{Fix}(f)$.

A subset $\Lambda \subset X$ is called *invariant* under f if $f(\Lambda) = \Lambda$. Clearly, any orbit is invariant. It is easy to see that Λ is invariant if and only if Λ is a union of orbits. $\mathrm{Per}(f)$ and $\mathrm{Fix}(f)$ are two important invariant set, and so are $\emptyset$ and X.

Theorem 1.1. *If Λ is invariant, so are $\overline{\Lambda}$, $\partial(\Lambda)$, and int(Λ).*

Proof. It is clear that $f\overline{\Lambda} \subset \overline{f\Lambda} = \overline{\Lambda}$. Likewise, $f^{-1}\overline{\Lambda} \subset \overline{\Lambda}$. Hence $f\overline{\Lambda} \supset \overline{\Lambda}$. This proves $f\overline{\Lambda} = \overline{\Lambda}$. The other two are similar. This proves Theorem 1.1.

Since the main point of interest in dynamical systems is the limiting behavior of orbits, closed invariant sets will be important to us. Unfortunately the set $\mathrm{Per}(f)$ of periodic points is generally not closed. This yields many difficulties, as well as many interesting phenomena. Nevertheless the set $\mathrm{Fix}(f)$ of fixed points is always closed.

A point $y \in X$ is called an *ω-limit point* of a point $x \in X$ if there is a subsequence $n_i \to +\infty$ such that $f^{n_i}(x) \to y$. The set of ω-limit points of x is called the *ω-limit set* of x, and is denoted as $\omega(x)$. Reversing time defines the *α-limit set* $\alpha(x)$ of x. Clearly $\alpha(x) = \omega(x, f^{-1})$. Hence one usually states results for $\omega(x)$ only. Note that any periodic point is its own ω (and α)-limit point.

Theorem 1.2. *For any $x \in X$, $\omega(x)$ is non-empty, closed, and invariant. Moreover,*

$$\lim \mathrm{d}(f^n(x), \omega(x)) = 0, \text{as } n \to \infty.$$

Proof. The set $\omega(x)$ is clearly non-empty and closed. To prove the invariance we first prove $f(\omega(x)) \subset \omega(x)$. Take $y \in \omega(x)$. There is a subsequence n_i such that $f^{n_i}(x) \to y$. Then $f^{n_i+1}(x) \to f(y)$. This proves $f(y) \in \omega(x)$. Hence $f(\omega(x)) \subset \omega(x)$. A Similar argument yields $f^{-1}(\omega(x)) \subset \omega(x)$. Thus $\omega(x)$ is invariant.

Now suppose that the limit of distances is not zero. Then there is $\varepsilon_0 > 0$ and a subsequence n_i such that

$$\mathrm{d}(f^{n_i}(x), \omega(x)) \geq \varepsilon_0$$

for all i. Taking a subsequence n_{i_k} further yields $f^{n_{i_k}} \to z \notin \omega(x)$, a contradiction. This proves Theorem 1.2.

The ω-limit set $\omega(x)$ and the α-limit set $\alpha(x)$ depend on the point x. To handle the limiting behavior of all points we need to consider the union of $\omega(x)$ and $\alpha(x)$ for all $x \in X$. This union is usually not closed. This leads us to consider the closure of the union. More precisely, define the *limit set* of Newhouse (1972) to be

$$L(f) = \overline{\bigcup_{x \in M} \omega(x) \cup \alpha(x)}.$$

Thus $L(f)$ contains the long run behavior of all orbits. Clearly,

$$\overline{Per(f)} \subset L(f).$$

Another important closed invariant set is the non-wandering set of Birkhoff we now define. A point is called *wandering* under f if there is a neighborhood V such that $f^n(V) \cap V = \emptyset$ for all $n \neq 0$. A point that is not wandering is called *non-wandering*. Thus x is non-wandering if for any neighborhood V of x, there is $m \neq 0$ such that $f^m(V) \cap V \neq \emptyset$. This is the same as to say that for any neighborhood V of x, there is some orbit that hits V at least twice. The set of non-wandering points of f is called the *non-wandering set* of f, and is denoted by $\Omega(f)$.

Theorem 1.3. *The non-wandering set $\Omega(f)$ is a non-empty closed invariant set that contains $L(f)$.*

We leave the proof to the reader. It is a good exercise to find an f with $\Omega(f)$ strictly larger than $L(f)$. Thus the recurrence exhibited by a non-wandering point is generally weaker than that exhibited by a limit point.

1.2 Topological conjugacy

Two homeomorphisms $f : X \to X$ and $g : X \to X$ are called *topologically conjugate* to each other if there is a homeomorphism $h : X \to X$ such that $hf = gh$. Roughly, such two f and g differ by a continuous change of coordinates. This is clearly an equivalence relation on the space of all homeomorphisms. The homeomorphism h is called a *conjugacy* between f and g. A conjugacy h preserves orbits, that is,

$$h(Orb(x, f)) = Orb(h(x), g)$$

for any $x \in X$. In particular, a conjugacy preserves ω-limit sets, the periodic set, and the non-wandering set. That is, $h(\omega(x, f)) = \omega(h(x), g)$, $h(\text{Per}(f)) = \text{Per}(g)$, and $h(\Omega(f)) = \Omega(g)$.

To classify all homeomorphisms up to conjugacy would be ideal, but is not realistic. Nevertheless for some extremely simple cases it is easy. Let's quickly study the case that X is the unit interval [0, 1]. For simplicity we consider orientation preserving homeomorphisms only. Such an $f : [0, 1] \to [0, 1]$ is simply a strictly increasing continuous function that fixes the two end points of the interval. For more simplicity we consider only those homeomorphisms that have no fixed points in (0, 1). For such an f, when under iteration, either every point goes to the right (if the graph of f is above the diagonal), or else every point goes to the left (if the graph is below). Clearly,

$$\Omega(f) = \text{Fix}(f) = \{0, 1\}.$$

Theorem 1.4. *Any two orientation preserving homeomorphisms of* $[0, 1]$ *without fixed points in* $(0, 1)$ *are topologically conjugate.*

Proof. Take any $x \in (0, 1)$ and any homeomorphism

$$h_0 : [x, f(x)] \longrightarrow [x, g(x)].$$

For each integer n, define

$$h_n : [f^n(x), f^{n+1}(x)] \longrightarrow [g^n(x), g^{n+1}(x)]$$

to be

$$h_n = g^n \circ h_0 \circ f^{-n}.$$

It is easy to see that these h_n glue together to give a homeomorphism $h : [0, 1] \to [0, 1]$ such that $hf = gh$. This proves Theorem 1.4.

Though it is very simple, this example tells us that the construction of a topological conjugacy is quite flexible on wandering domains.

1.3 The notion of structural stability

Now we turn to the notion of our main interest, structural stability. This is formulated in differentiable dynamics.

Let M be a compact manifold without boundary, and $f\colon M \to M$ be a diffeomorphism. Denote by $\mathrm{Diff}^r(M)$ the set of diffeomorphisms of M, endowed with the C^r topology. Roughly, two diffeomorphisms are C^r close if they are close up to their r-th derivatives. Thus a C^r neighborhood of f always contains a C^{r+1} neighborhood of f (when both make sense).

A diffeomorphism $f : M \to M$ is called C^r *structurally stable* if there is a C^r neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M)$ such that every $g \in \mathcal{U}$ is topologically conjugate to f.

Thus f is C^r structurally stable if C^r perturbations can not topologically change the orbit structure of f. Clearly, if f is C^r structurally stable, then it is C^{r+1} structurally stable. Thus C^1 structural stability is the strongest structural stability. The concept of C^0 structural stability is vacant because C^0 perturbations are too damaging. For instance, a C^0 perturbation easily turns an isolated fixed point into a whole neighborhood of fixed points, hence destroys any structural stability. On the other hand, the conjugacy h is allowed to be topological. This is because a differentiable conjugacy would be too restrictive. For instance, a C^r perturbation easily changes the eigenvalues of a fixed point, so no diffeomorphism h would serve as a conjugacy, because a differentiable conjugacy would preserve eigenvalues of fixed points. Thus the notion of structural stability is an appropriate one. It limits to less damaging perturbations, but allows a more powerful conjugacy to fix it up.

For the simplest case considered in the last section, structural stability is easy to characterize.

Theorem 1.5. *For any $r \geq 1$, an orientation preserving diffeomorphism $f : [0,1] \to [0,1]$ without fixed points in $(0,1)$ is C^r structurally stable if and only if $f'(0) \neq 1$ and $f'(1) \neq 1$.*

Proof. "if": Assume f is orientation preserving with $f'(0) \neq 1$ and $f'(1) \neq 1$. It suffices to consider $r = 1$ only. There is a C^1 neighborhood $\mathcal{U}_1$ of f, together with a neighborhood U of $x = 0$ and a neighborhood V of $x = 1$, such that any $g \in \mathcal{U}_1$ is also orientation preserving, and also has a unique fixed point in U which is $x = 0$, and a unique fixed point in V which is $x = 1$. Since f has no fixed points on $[0,1] - U - V$ which is compact, $|f(x) - x|$ assumes a positive minimum on $[0,1] - U - V$. Then there is a C^1 neighborhood $\mathcal{U}_2$ of f such that any $g \in \mathcal{U}_2$ has no fixed points on $[0,1] - U - V$. Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. Then any $g \in \mathcal{U}$ is orientation preserving and has no fixed points in $(0,\,1)$. By Theorem 1.4, g and f are conjugate.

"only if": Assume $f'(0) = 1$. For any $r \geq 1$, we construct a C^r perturbation g of f such that g has more than two fixed points, hence f would not be C^r

structurally stable. Take a C^∞ bump function α: $[0,\ 1] \to [0,\ 1]$ such that $\alpha = 1$ on $[0, 1/3]$, $\alpha = 0$ on $[2/3, 1]$, $0 \leq \alpha \leq 1$ on $[0,\ 1]$. Let

$$K = \max\{|(\alpha f)^{(j)}(x)|\ |x \in [0,1], j = 1, ..., r\}.$$

Without loss of generality we assume the graph of f is above the diagonal. Then for any $\varepsilon > 0$, define

$$g(x) = f(x) - \varepsilon\alpha(x) f(x).$$

It is easy to check that, $x = 0$ and $x = 1$ remain to be fixed points for g, but the graph of g near $x = 0$ is slightly pulled down so that g has also some fixed point a little bit to the right of $x = 0$. Moreover, g is C^r close to f if ε is small. This proves Theorem 1.5.

Though very simple, Theorem 1.5 is instructive. It indicates that fixed points are sensitive to perturbations. To survive from perturbations they need a condition like $f'(0) \neq 1$, the so called *hyperbolicity*. Thus fixed points, or more generally non-wandering points, are important to us not only because they exhibit long run behavior of orbits, but also because they need more attention from the point of view of perturbation. Another interesting feature is that, for this simplest example, C^i structural stability is equivalent to C^j structural stability, for any i and j. This is true also for Peixoto's result on 2-dimensional flows. For general cases, this is unknown so far (it will be true if the C^r stability conjecture is verified).

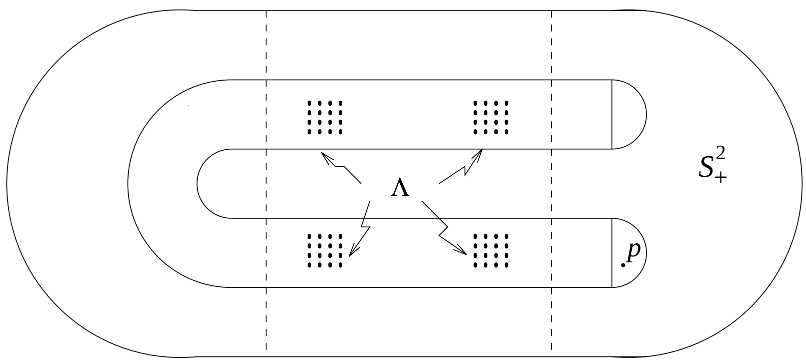
CHAPTER 2

Hyperbolic Sets

This chapter gives a short but comprehensive account of the theory of hyperbolic sets.

2.1 The concept of hyperbolic set

The notion of hyperbolic sets is essential to stability theory. We first give some informal illustrations for the concept of hyperbolic sets. We start with Smale's horseshoe set, the first non-trivial hyperbolic set which appeared historically and inspired the whole theory. As the reader knows, it is the maximal invariant set Λ in a square, for the horseshoe map, see [Sh] and [Sm3].



It is well known that topologically, Λ is a Cantor set with periodic points dense, and with a dense orbit. In fact f restricted to Λ is topologically conjugate to the 2-shift on the symbolic space Σ_2. The horseshoe map can be defined more globally as a diffeomorphism $f : S^2 \to S^2$. The upper semi-sphere S^2_+ is embbeded into itself so that the maximal invariant set in S^2_+ consists of the horseshoe Λ, together with a hyperbolic sink p, as the figure shows. Thus every point $x \in S^2_+ - \{p\} - \Lambda$ gets out of S^2_+ at a certain time. The lower semi-sphere S^2_- can be arranged to have only a hyperbolic source q as the maximal invariant set. One can check that $\Omega(f) = \Lambda \cup \{p\} \cup \{q\}$. (Generally, every point that goes to a sink or a source is wandering.)

The main differentiable dynamical feature on $\Omega(f)$ is a uniform contraction or expansion of vectors, under the tangent map Tf. This will be called hyperbolicity. For instance, all vectors in T_pM get contracted, while all vectors in T_qM get expanded. More complicated are vectors in $T_\Lambda M$. The tangent map Tf contracts vectors of $T_\Lambda M$ that are in the vertical direction, but expands vectors of $T_\Lambda M$ in the horizontal direction, with uniform rates. This is like a saddle, but moving, even non-periodically. While hyperbolic behavior for a fixed saddle, or a periodic saddle, was known a long time ago, the surprising feature that infinitely many orbits, both periodic and non-periodic, fit together so harmonically and even structurally stably with a uniform saddle-like behavior, became known only a few decades ago, through Smale's horseshoe, and another important diffeomorphism found in early sixties, known as Thom's automorphism. In contrast to this, uniform contraction in all directions (sink-like), or uniform expansion in all directions (source-like), can not have such chaotic features. It can occur only on finite sets, which hence must reduce to finitely many periodic orbits.

On the other hand, for the horseshoe map, there is no such hyperbolicity for wandering orbits (It is possible for a wandering orbit to be hyperbolic as well. A wandering transverse heteroclinic orbit is such an example). In fact for a system to be structurally stable, wandering orbits need satisfy only a weaker condition, known as strong transversality.

Now we give the formal definition of hyperbolic set. A compact invariant set Λ of $f : M \to M$ is called *hyperbolic for f* if for each $x \in \Lambda$, the tangent space T_xM has a Tf-invariant (as family) splitting

$$T_xM = E^s(x) \oplus E^u(x)$$

$$Tf(E^s(x)) = E^s(f(x)),\ Tf(E^u(x)) = E^u(f(x)),$$

such that for two uniform constants $C > 0$ and $0 < \lambda < 1$, the following estimates hold:

$$|Tf^n(v)| \leq C\lambda^n|v|,\ \forall x \in \Lambda, v \in E^s(x), n \geq 1,$$

$$|Tf^{-n}(v)| \leq C\lambda^n|v|,\ \forall x \in \Lambda, v \in E^u(x), n \geq 1.$$

The reader can see that Smale's horseshoe is hyperbolic. Actually $E^s(x)$ is the vertical direction, $E^u(x)$ is the horizontal direction. Moreover, the two inequalities behave even simpler, because for the horseshoe map $C = 1$. That is, the contraction and expansion in the horseshoe are immediate:

$$|Tf(v)| \leq \lambda|v|,\ \forall x \in \Lambda, v \in E^s(x),$$

$$|Tf^{-1}(v)| \leq \lambda|v|,\ \forall x \in \Lambda, v \in E^u(x).$$

A standard result of Mather (see for instance Shub (1987)) says that, by changing to a suitable equivalent norm, any hyperbolic set behaves in this immediate way. Such a norm is called *adapted* to Λ. If we are assuming, but not proving hyperbolicity, it is always convenient to use adapted norms.

The definition of hyperbolic set deserves more remarks.

1. A hyperbolic fixed point, or a hyperbolic periodic orbit, is a hyperbolic set. The two general eigenspaces of eigenvalues of norm less than 1 and greater than 1, respectively, serve as E^s and E^u. Note that the eigenvalues have something to do with λ, but not with C. Also note that here a purely algebraic definition by eigenvalues is equivalent to a definition that involves norm. This is because all norms are equivalent for finite dimensional spaces.

2. Any compact invariant subset of a hyperbolic set is hyperbolic. A finite union of hyperbolic sets is hyperbolic.

3. Hyperbolicity allows $E^s = \{0\}$ (source-like), or $E^u = \{0\}$ (sink-like). In this case Λ must be finite. (Try a direct proof)

4. The second inequality can be written as

$$|Tf^n(v)| \geq C^{-1}\lambda^{-n}|v|, \ \forall v \in E^u, n \geq 1.$$

5. Likewise for the first inequality. Hence (the length of) vectors that are neither in E^s nor E^u all go to ∞ in both directions with exponential rates. Thus vectors in E^s and E^u are charactered respectively by the two inequalities in the definition (but not the inequalities in this remark).

6. Hyperbolic splitting is unique. That is, if $F^s(x) \oplus F^u(x)$ is another hyperbolic splitting of Λ, then $E^s(x) = F^s(x)$, and $E^u(x) = F^u(x)$. This can be easily proved either directly, or through some other characterizations of E^s and E^u such as

$$E^s(x) = \{v \in T_xM \mid |Tf^n(v)| \to 0, \ n \to +\infty\},$$

$$E^u(x) = \{v \in T_xM \mid |Tf^{-n}(v)| \to 0, \ n \to +\infty\}.$$

7. Hyperbolic splitting is continuous. That is, $E^s(x)$ and $E^u(x)$ vary continously with $x \in \Lambda$, hence E^s and E^u are subbundles of $T_\Lambda M$. This follows from the uniqueness, by taking a convergent subsequence of $E^s(x_i)$. In particular, $\dim(E^s(x))$ and $\dim(E^u(x))$ are locally constant.

8. One could define the notion of hyperbolic set the same way, with merely the compactness of Λ dropped. Then it is easy to see that the hyperbolic splitting $E^s(x) \oplus E^u(x)$ on Λ extends to the closure $\overline{\Lambda}$. This is proved also by taking convergent subsequences and using uniqueness. Nevertheless most of the hyperbolic sets considered below are compact. As convention we include the compactness of Λ into the definition of hyperbolic set, as we have done. In a few places below we may talk about a *hyperbolic orbit*, which might be non-compact. Of course by this we will mean that the closure of this orbit is a hyperbolic set.

We will present in Section 2.4 six theorems about hyperbolic sets, which form a basis for the stability theorem. These theorems concern persistence of hyperbolicity, expansiveness, embedding stability, shadowing property, stability of isolated hyperbolic sets, and the stable manifolds. These theorems (except the last one) turn out to be ingenious applications of a simple theorem (Theorem 2.1 below) on hyperbolic fixed points in Banach spaces.

2.2 Hyperbolic fixed points in Banach spaces

Let E be a Banach space. Let us call a linear homeomorphism $A : E \to E$ an *automorphism*, for short. An automorphism $A : E \to E$ is called *hyperbolic* if E has an A-invariant splitting

$$E = E^s \oplus E^u$$

$$A(E^s) = E^s,\ A(E^u) = E^u,$$

such that for some suitable norm $|\cdot|$ and some number $0 < \tau < 1$, the following estimates hold:

$$|Av| \leq \tau|v|,\ \forall v \in E^s,$$

$$|A^{-1}v| \leq \tau|v|,\ \forall v \in E^u.$$

The least such number τ is called the *skewness* of A. Note that we have used an adapted norm in the definition, because in this and the next chapter we assume hyperbolicity.

A setting for perturbation that is easy to work with and also general enough is $A+\varphi$, where φ has small Lipschitz constant. For instance, near a fixed point $x = 0$ of f, a C^1 nearby g is of the form: $g = Df(0) + \varphi$, where $\varphi = g - Df(0)$ is Lipschitz small near $x = 0$. Here is a simple theorem in this setting, which will find surprising applications to hyperbolic sets in Section 2.4. We denote components of maps by subscripts, for instance

$$A_s = p_s A,\ \varphi_s = p_s \varphi,\ A_u = p_u A,\ \varphi_u = p_u \varphi,$$

where $p_s : E \to E^s$ and $p_u : E \to E^u$ are the two projections. Also, denote

$$A_{ss} = A_s|E^s,\ A_{uu} = A_u|E^u.$$

Note that

$$A_s x = A_s x_s = A_{ss} x_s,\ \forall x \in E.$$

Finally, we assume the norm to be of box-type, that is,

$$|v| = \max\ \{|v_s|,\ |v_u|\},$$

where $v = v_s + v_u$, $v_s \in E_s$, and $v_u \in E_u$. This is enough to our purpose because the Banach spaces used below are section spaces of $T_\Lambda M$ for some invariant set Λ, and hence have induced norm of box-type.

Theorem 2.1. *Let $A : E \to E$ be a hyperbolic automorphism with skewness τ, and $\varphi : B(0,\ r) \to E$ be a Lipschitz map with Lipschitz constant*

$$Lip\varphi < 1 - \tau.$$

Then $A + \varphi$ has at most one fixed point in $B(0,\ r)$. If, in addition,

$$|\varphi(0)| \leq (1 - \tau - Lip\varphi)r,$$

then $A + \varphi$ does have a fixed point p in $B(0,\ \frac{|\varphi(0)|}{1-\tau-Lip\varphi})$ (which is unique in $B(0, r)$).

Proof. We are solving the equation

$$(A + \varphi)x = x.$$

This is equivalent to

$$A_s x + \varphi_s x = x_s,\ \ A_u x + \varphi_u x = x_u,$$

or,

$$A_{ss} x_s + \varphi_s x = x_s,\ \ A_{uu} x_u + \varphi_u x = x_u,$$

or, what is the same,

$$A_{ss} x_s + \varphi_s x = x_s,\ \ A_{uu}^{-1} x_u - A_{uu}^{-1} \varphi_u x = x_u.$$

This gives a map

$$T : B(0,\ r) \to E$$

$$T(x) = (A_{ss} x_s + \varphi_s x,\ \ A_{uu}^{-1} x_u - A_{uu}^{-1} \varphi_u x).$$

Since T and $A + \varphi$ have the same set of fixed points, to prove $A + \varphi$ has at most one fixed point in $B(0,\ r)$, it suffices to prove T is a contraction. This is easily checked as

$$\begin{aligned}
&|Tx - Ty| \\
&\leq \max\{\tau|x_s - y_s| + Lip\varphi|x_s - y_s|, \tau|x_u - y_u| + \tau Lip\varphi|x_u - y_u|\} \\
&\leq (\tau + Lip\varphi)|x - y|,
\end{aligned}$$

while $\tau + Lip\varphi < 1$.

Now we assume in addition that

$$|\varphi(0)| \leq (1 - \tau - Lip\varphi)r.$$

We prove T in this case maps $B(0,\ r)$ into itself. Take any $x \in B(0,\ r)$. Then, noting that

$$|T(0)| = |(\varphi_s(0),\ -A_{uu}^{-1}\varphi_u(0))| \leq |\varphi(0)|,$$

we have

$$\begin{aligned}|T(x)| &\leq |T(0)| + |T(x) - T(0)| \\ &\leq |\varphi(0)| + (\tau + Lip\varphi)|x| \\ &\leq (1 - \tau - Lip\varphi)r + (\tau + Lip\varphi)r \\ &= r.\end{aligned}$$

This proves T does have a (unique) fixed point p in $B(0,\ r)$. Moreover,

$$|p| \leq |\varphi(0)| + (\tau + Lip\varphi)|p|,$$

hence

$$|p| \leq \frac{|\varphi(0)|}{1 - \tau - Lip\varphi}.$$

This proves Theorem 2.1.

A special case of Theorem 2.1 is when φ is differentiable. Let $f : E \to E$ be a C^1 map, and x be a fixed point of f. We call x a *hyperbolic fixed point* of f if the derivative $Df(x) : E \to E$ is a hyperbolic automorphism. For this definition to make sense f needs be, and often is, defined only near x, say $f : U \to E$.

Theorem 2.2. *Let $x \in U$ be a hyperbolic fixed point of $f : U \to E$. There are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any C^1 map g with $d^1(g, f) \leq \delta_0$ has at most one fixed point in $B(x, \varepsilon_0)$. Moreover, for any $0 < \varepsilon \leq \varepsilon_0$, there is $\delta \leq \delta_0$ such that any g with $d^1(g, f) \leq \delta$ does have a fixed point p in $B(x,\ \varepsilon)$, which is unique in $B(x,\ \varepsilon_0)$.*

Another differentiable version of Theorem 2.1 does not even assume that some unperturbed map f has a fixed point in advance:

Theorem 2.2'. *Let $g : U \to E$ be a C^1 map and $x \in U$ be a point such that $Dg(x)$ is a hyperbolic automorphism. There is $\varepsilon_0 > 0$ such that g has at most one fixed point in $B(x, \varepsilon_0)$. Moreover, for any $0 < \varepsilon \leq \varepsilon_0$, there is $\delta > 0$ such that if, in addition, $g(x)$ is in $B(x, \delta)$, then g does have a fixed point p in $B(x,\ \varepsilon)$, which is unique in $B(x,\ \varepsilon_0)$.*

2.3 Lifting and the induced operator

This section prepares a framework within which some results on hyperbolic sets can be interpreted as results on hyperbolic fixed points in some suitable Banach spaces. This beautiful idea is due to Moser, Mather, Hirsch, Pugh, Anosov, and perhaps others. Let Λ be a compact invariant set of f. As usual, a map $F : T_\Lambda M \to T_\Lambda M$ is called *fiber preserving* over f on Λ if $\pi F = f\pi$, where π is the bundle projection. The tangent map $Tf : T_\Lambda M \to T_\Lambda M$ is fiber preserving over f on Λ, and is linear on fibers. Generally a fiber preserving map F over f on Λ may not be linear on fibers.

Any fiber preserving map induces an operator on section spaces. More precisely, let $\Gamma^b(T_\Lambda M)$ be the Banach space of bounded sections (vector fields) $\gamma : \Lambda \to T_\Lambda M$, with the sup norm, induced by the norm on $T_\Lambda M$. The subspace $\Gamma^0(T_\Lambda M)$ of continuous sections is closed in $\Gamma^b(T_\Lambda M)$, hence also Banach. We will be concerned with both Banach spaces. Briefly, we want uniqueness in Γ^b, but existence in Γ^0 — results that are stronger. A fiber preserving map $F : T_\Lambda M \to T_\Lambda M$ over f induces an operator

$$\begin{aligned} &F_\# : \Gamma^b(T_\Lambda M) \to \Gamma^b(T_\Lambda M), \\ &(F_\# \gamma)(x) = F(\gamma(f^{-1}(x))), \forall x \in \Lambda. \end{aligned}$$

Likewise for Γ^0. This is very natural. Roughly, the map F sends vectors to vectors, covering a bijection f, so F (or more precisely, its induced operator $F_\#$) sends sections to sections. Note that an inverse f^{-1} is involved. This is because we wish to use x (not fx) to denote a general point, and the vector to be defined at x is the F-image of the vector at $f^{-1}(x)$. For the tangent map Tf, the induced operator $(Tf)_\#$ is easily seen to be a linear automorphism, for both Γ^b and Γ^0.

An important fiber preserving map on Λ, which is not linear on fibers, is the *lifting of g along f on Λ*

$$\begin{aligned} \Phi^{fg} &\equiv \Phi^{fg\Lambda} : T_\Lambda M(r_0) \to T_\Lambda M \\ \Phi^{fg}(v) &= exp_{fx}^{-1}\ g\ exp_x(v), \end{aligned}$$

where $T_\Lambda M(r_0)$ denotes the set of vectors in $T_\Lambda M$ with norm less than or equal to r_0. Thus Φ^{fg} sends vectors at $x \in \Lambda$ to vectors at $fx \in \Lambda$, hence is fiber preserving over f on Λ. Roughly, Φ^{fg} is simply the map g applied to the top of vectors v with the base point x taken care of by f. Of course to define such a lifting, g has to be close to f, and r_0 has to be small. More precisely, let $\rho > 0$ be a number such that $exp_x : T_x M(\rho) \to B(x, \rho)$ is a diffeomorphism for any $x \in M$. We fix $r_\rho > 0$ and $\delta_\rho > 0$ below in this and the next section such that for any g, C^0-δ_ρ-close to f, and any two points $x, y \in M$ with $d(x, y) < r_\rho$, the inequality $d(fx, gy) < \rho$ holds, hence the lifting $\Phi^{fg} : T_\Lambda M(r_\rho) \to T_\Lambda M$ can be defined.

We will be mainly concerned with the non-linear operator

$$\begin{aligned} &\Phi^{fg}_\# : \Gamma^b(T_\Lambda M)(r_\rho) \to \Gamma^b(T_\Lambda M) \\ &(\Phi^{fg}_\#(\gamma))(x) = \Phi^{fg}(\gamma(f^{-1}(x)), \forall x \in \Lambda \end{aligned}$$

induced by the lift Φ^{fg}. Likewise for Γ^0. The reader may have noticed that this concerns the conjugacy problem. Roughly, a conjugacy from $f|_\Lambda$ to g corresponds to a fixed point of $\Phi^{fg}_\#$. To make it precise let us call an $h : \Lambda \to M$, not necessarily continuous, nor injective, an *ε-preconjugacy from f to g*, where $0 < \varepsilon \leq r_\rho$, if $hf = gh$ and if $d(h(x),\ x) \leq \varepsilon, \forall x \in \Lambda$. The image $h(\Lambda)$ is easily checked to be an invariant set of g. Note that a map $h : \Lambda \to M$ within r_ρ of the inclusion $\iota_\Lambda : \Lambda \to M$ gives rise to a section $\gamma \in \Gamma^b(T_\Lambda M)(r_\rho)$ through the exponential map, and vice versa:

$$h(x) = exp_x \gamma(x), \text{ or } \gamma(x) = exp_x^{-1} h(x).$$

Besides, h is continuous if and only if γ is.

Theorem 2.3. *A map $h : \Lambda \to M$ is an ε-preconjugacy from f to g if and only if the corresponding section γ is a fixed point of $\Phi_\#^{fg}$ in $\Gamma^b(T_\Lambda M)(\varepsilon)$.*

Proof. This is straightforward. The preconjugacy equation

$$h(x) = ghf^{-1}(x)$$

is equivalent to

$$\gamma(x) = exp_x^{-1}\ g\ exp_{f^{-1}x} \gamma(f^{-1}x),$$

which is the same as

$$\gamma = \Phi_\#^{fg}(\gamma).$$

The number ε is preserved because $d(exp_x v, x) = |v|$. This proves Theorem 2.3.

Let Λ be a hyperbolic set of f. We always fix a Riemannian norm on TM that is adapted to Λ, which determines the induced sup norm on section spaces.

Theorem 2.4. *Let Λ be a hyperbolic set of f. Then*

$$(Tf)_\# : \Gamma^b(T_\Lambda M) \to \Gamma^b(T_\Lambda M)$$

is a hyperbolic automorphism with the same skewness. Likewise for Γ^0.

Proof. Let $T_\Lambda M = E^s \oplus E^u$ be the hyperbolic splitting for Tf. The splitting

$$\Gamma^b(T_\Lambda M) = \Gamma^b(E^s) \oplus \Gamma^b(E^u)$$

is clearly invariant for $(Tf)_\#$. We check the contraction rates. Take any $\gamma \in \Gamma^b(E^s)$. Then

$$\begin{aligned} |(Tf)_\#(\gamma)| &= \sup_{x \in \Lambda}\{|Tf(\gamma(f^{-1}x))|\} \\ &\leq \tau \sup_{x \in \Lambda}\{|\gamma(f^{-1}x)|\} \\ &= \tau|\gamma|. \end{aligned}$$

The rate on $\Gamma^b(E^u)$ is checked similarly. This proves that $(Tf)_\#$ is a hyperbolic automorphism of $\Gamma^b(T_\Lambda M)$ with the same skewness. The proof for Γ^0 is the same. This proves Theorem 2.4.

Note that the zero-section 0_Λ is always a fixed point of $\Phi_{\#}^{ff}$, the induced operator by the self-lifting. A basic analytical observation is the following.

Theorem 2.5. *Let Λ be a hyperbolic set of f with skewness τ. Then 0_Λ is a hyperbolic fixed point of*

$$\Phi_{\#}^{ff} : \Gamma^b(T_\Lambda M)(r_\rho) \to \Gamma^b(T_\Lambda M)$$

with the same skewness. Likewise for Γ^0.

Proof. We prove that

$$D\Phi_{\#}^{ff}(0_\Lambda) = (Tf)_{\#},$$

then Theorem 2.4 will apply. We take the case of Γ^b only. The case for Γ^0 is similar. First we prove Palais' formula

$$(DF_{\#}(\sigma)\gamma)(x) = D_2F(\sigma(f^{-1}x)) \cdot \gamma(f^{-1}x), \forall x \in \Lambda,$$

where F is a fiber preserving map of $T_\Lambda M$ and $D_2F(\sigma(f^{-1}x))$, the partial derivative of F at $\sigma(f^{-1}x)$ along fiber direction, is assumed to be continuous. By definition, this is to prove that

$$|F(\sigma(f^{-1}x)+\gamma(f^{-1}x))-F(\sigma(f^{-1}x))-D_2F(\sigma(f^{-1}x))\cdot\gamma(f^{-1}x)| = o(|\gamma(f^{-1}x)|)$$

holds for $x \in \Lambda$ uniformly. But the left hand side is dominated by

$$\int_0^1 |(D_2F(\sigma(f^{-1}x) + t\gamma(f^{-1}x)) - D_2F(\sigma(f^{-1}x)))| \cdot |\gamma(f^{-1}x)| dt.$$

Since D_2F is continuous and $\gamma(\Lambda)$ is bounded, it follows that D_2F is uniformly continuous on a compact neighborhood of $\gamma(\Lambda)$ in $T_\Lambda M$. This proves Palais' formula. Note that here F takes care of the calculus along fibers, while f takes care of, in fact fixes, the base points.

Now the partial derivative $D_2\Phi^{ff}$ at a zero-vector 0_x along fiber direction is calculated as

$$\begin{aligned} D_2\Phi^{ff}(0_x) &= D(\Phi^{ff}|_{T_xM(r_\rho)})(0_x) \\ &= D(exp_{fx}^{-1}\ f\ exp_x)(0_x) \\ &= id \circ Df(x) \circ id \\ &= Tf|_{T_xM}. \end{aligned}$$

Hence by Palais' formula,

$$D\Phi_{\#}^{ff}(0_\Lambda) = (Tf)_{\#}.$$

This proves Theorem 2.5.

Above is a framework in which a preconjugacy is interpreted as a fixed point of the induced operator. Before proceeding to the next section we point out that the framework needs be relaxed to handle a more general situation. Note that two orbits $\{f^n x\}$ and $\{g^n y\}$ remind us of a preconjugacy. We might wish to simply define a preconjugacy h that takes $f^n x$ to $g^n y$. The problem is that $\{f^n x\}$ may be finite while $\{g^n y\}$ may be infinite. In this case such an h would not be possible (Thanks to C. S. Lin for pointing this out to me). The same problem arises below for the so called pseudo orbits. To handle this we consider the pull-back bundle of TM, through $\{f^n x\}$, to the set $\mathbb{Z}$ of integers . More precisely, let $s : \mathbb{Z} \to M$ be the sequence that gives the orbit $\{f^n x\}$, that is, $s(n) = f^n x$. We think of the pull-back bundle $s^*(TM)$ over the integers as the union

$$\bigcup_{n \in \mathbb{Z}} T_{f^n x} M,$$

and consider the fiber preserving map

$$\Phi^{sg} : s^*(TM)(r_\rho) \to s^*(TM)$$
$$\Phi^{sg}(v) = exp^{-1}_{f^{n+1}x}\ g\ exp_{f^n x}(v),\ \ \forall v \in T_{f^n x}M.$$

The map is essentially g regarded as taking tops of vectors at $f^n x$ to tops of vectors at $f^{n+1}x$. Associated with this are the Banach space of sections

$$\Gamma^b(s^*(TM)) = \{\gamma : \mathbb{Z} \to M \mid \gamma(n) \in T_{f^n x}M, \text{ and } \gamma \text{ is bounded}\}$$

and the induced operator

$$\Phi^{sg}_{\#} : \Gamma^b(s^*(TM))(r_\rho) \to \Gamma^b(s^*(TM))$$
$$(\Phi^{sg}_{\#}(\gamma))(n) = \Phi^{sg}(\gamma(n-1)), \forall n \in \mathbb{Z}.$$

All results obtained above for Φ^{fg} hold for such a more general Φ^{sg}. Thus, even if $\{f^n x\}$ is finite and $\{g^n y\}$ is infinite, an ε-shadowing, that is, the situation $d(f^n x,\ g^n y) \leq \varepsilon$ for all $n \in \mathbb{Z}$, corresponds to a well defined section $\gamma \in \Gamma^b(s^*(TM))(\varepsilon)$ with $\gamma(n) = exp^{-1}_{f^n x}(g^n y)$. Several vectors $\gamma(n)$ may share the same base point $f^n x$ on the manifold. However, pulled back to the integers, they have different base points n. We state two results that correspond to Theorem 2.3 and 2.5.

Theorem 2.3' *The section γ that corresponds to an ε-shadowing between $\{s(n)\} = \{f^n x\}$ and $\{g^n y\}$ is a fixed point of $\Phi^{sg}_{\#}$ in $\Gamma^b(s^*(TM))(\varepsilon)$.*

Theorem 2.5' *Let $\{s(n)\} = \{f^n x\}$ be a hyperbolic orbit of f* with skewness τ. *Then the zero-section 0_s is a hyperbolic fixed point of $\Phi^{sf}_{\#}$ with the same skewness.*

Similar results hold for shadowing with even pseudo orbits, as seen in Theorem 2.10 below.

2.4 Some basic theorems on hyperbolic sets

This section present some basic theorems on hyperbolic sets that are needed for the stability theorem in chapter 3. We first state the theorem on persistence of hyperbolicity, which says if Λ is hyperbolic for f, then a nearby invariant set Δ (if any) of a nearby system g is also hyperbolic and with nearby rates. This can be proved similarly to a theorem on hyperbolic fixed points in Banach spaces.

Theorem 2.6. *Let Λ be a hyperbolic set of f with skewness τ. For any $\tau < \mu < 1$, there are $a > 0$ and $\delta > 0$ such that for any $g \in \mathcal{B}^1(f, \delta)$, any invariant set Δ of g contained in $B(\Lambda, a)$ is hyperbolic with skewness less than μ.*

The next three theorems are about expansiveness, embedding stability, and the shadowing property. We first give simplified versions. Recall a homeomorphism $f : \Lambda \to \Lambda$ is called *expansive* if there is a constant $\zeta > 0$ such that for every pair of different points $x \neq y$, there is an integer m such that $\mathrm{d}(f^m(x),\ f^m(y)) \geq \zeta$. The number ζ is called an expansive constant of f on Λ.

Theorem 2.7. *Let Λ be a hyperbolic set of f. Then $f|_\Lambda$ is expansive.*

Proof. Let τ be the skewness of Λ. We use the uniqueness part of Theorem 2.2. Note that the numbers r and δ_0 in Theorem 2.2 depend on the skewness τ (and f) only. Now any orbit $\{f^n x\}$ in Λ has skewness τ, hence by Theorem 2.5', the 0-section 0_s (here s gives the orbit $\{f^n x\}$) is a hyperbolic fixed point of

$$\Phi_{\#}^{sf} : \Gamma^b(s^*(TM))(r_\rho) \to \Gamma^b(s^*(TM))$$

with skewness τ. According to the uniqueness part of Theorem 2.2, there is $\zeta = \zeta(\tau) > 0$, independent of $x \in \Lambda$, such that 0_s is the unique fixed point of $\Phi_{\#}^{sf}$ in $\Gamma^b(s^*(TM))(\zeta)$.

It is easy to see that ζ is an expansive constant for $f|_\Lambda$. Assume there is $y \in \Lambda$ such that

$$d(f^n x,\ f^n y) \leq \zeta, \text{ for all } n \in \mathbb{Z}.$$

By Theorem 2.3', applied to the special case that $f = g$, this corresponds to a section $\gamma \in \Gamma^b(s^*(TM))(\zeta)$, which is a fixed point of $\Phi_{\#}^{sf}$. But 0_s has just been said to be the unique fixed point of $\Phi_{\#}^{sf}$ in $\Gamma^b(s^*(TM))(\zeta)$, so γ must be 0_s, hence $x = y$. This proves Theorem 2.7.

Thus the ζ-expansiveness of f on Λ is simply the uniqueness of fixed points of $\Phi_{\#}^{sf}$ in $\Gamma^b(s^*(TM))(\zeta)$. Next we prove the embedding stability of hyperbolic sets. To make statements snappier let us denote by $\mathcal{B}^1(f, \delta)$ the set of diffeomorphisms g such that $d^1(g, f) < \delta$, and by $\mathcal{B}^0(\iota_\Delta, \varepsilon)$ the set of continuous maps $h : \Delta \to M$ such that $d^0(h, \iota_\Delta) < \varepsilon$, where Δ is any closed subset of

M. Likewise, $\mathcal{B}^b(\iota_\Delta, \varepsilon)$ denotes the set of maps $h : \Delta \to M$, not necessarily continuous, such that $d(hx, x) < \varepsilon$, $\forall\, x \in \Delta$.

Theorem 2.8. *Let Λ be a hyperbolic set of f. Then for any $\varepsilon > 0$, there is $\delta > 0$, such that for any $g \in \mathcal{B}^1(f, \delta)$, there is an injective map $h \in \mathcal{B}^0(\iota_\Lambda, \varepsilon)$ such that $hf = gh$ on Λ.*

Note that the injective image $\Delta = h(\Lambda)$ will be a compact invariant set of g that is ε-conjugate to $f|_\Lambda$. The set Δ comes as the result of h, but not specified beforehand. For this reason we have used the words embedding stability. On the other hand, in Theorem 2.14 below the set Δ will be specified beforehand.

Proof. Let ζ be the expansive constant of f on Λ, guaranteed by Theorem 2.7. By Theorem 2.5, the 0-section 0_Λ is a hyperbolic fixed point of

$$\Phi_{\#}^{ff} : \Gamma^0(T_\Lambda M)(r_\rho) \to \Gamma^0(T_\Lambda M).$$

Note that this time we consider Γ^0. According to the existence part of Theorem 2.2, for any $0 < \varepsilon \le \zeta/2$, there is $\delta > 0$, such that for any $g \in \mathcal{B}^1(f, \delta)$,

$$\Phi_{\#}^{fg} : \Gamma^0(T_\Lambda M)(r_\rho) \to \Gamma^0(T_\Lambda M)$$

is C^1 close enough to $\Phi_{\#}^{ff}$ so that $\Phi_{\#}^{fg}$ has a fixed point γ in $\Gamma^0(T_\Lambda M)(\varepsilon)$, which gives a continuous ε-preconjugacy

$$h : \Lambda \to M$$

from f to g. (This is because $\Phi_{\#}^{fg}$ depends continuously on g in the C^1 topology (Irwin (1980)). It remains to prove that h is injective. This is just a consequence of the ζ-expansiveness of $f|_\Lambda$. Indeed, assume $h(x) = h(y)$ for some x, $y \in \Lambda$. Then for any integer n,

$$\begin{aligned} d(f^n x,\ f^n y) &\le d(f^n x,\ h(f^n x)) + d(h(f^n x),\ h(f^n y)) + d(h(f^n y),\ f^n y) \\ &= d(f^n x,\ h(f^n x)) + d(g^n(hx),\ g^n(hy)) + d(h(f^n y),\ f^n y) \\ &\le \zeta/2 + 0 + \zeta/2 \\ &= \zeta. \end{aligned}$$

Thus $x = y$ by the ζ-expansiveness of $f|_\Lambda$. This proves h is injective. This proves Theorem 2.8.

A diffeomorphism $f : M \to M$ is called an *Anosov diffeomorphism* if the whole manifold M is a hyperbolic set of f.

Theorem 2.9. *Anosov diffeomorphisms are C^1-structurally stable.*

Proof. Let f be an Anosov diffeomorphism. By Theorem 2.8, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $g \in \mathcal{B}^1(f, \delta)$, there is a continuous injective map $h \in \mathcal{B}^0(\iota_M, \varepsilon)$ such that $hf = gh$ on M. It remains to prove that h is surjective. By invariance of domain, h is an open map. Thus $h(M)$ is both open and closed in M, hence $h(M) = M$ (We always assume M is connected). This proves Theorem 2.9.

Note that what is proved here is more than what the theorem states: a stronger ε-structural stability. The same is true for the stability theorem presented in the next chapter. Now we proceed to the shadowing property of hyperbolic sets. Recall a *δ-pseudo orbit* of f is a sequence $\{x_n\}$ such that $d(fx_n,\ x_{n+1}) \le \delta$, and it is *$\varepsilon$-shadowed* by a true orbit $\{f^n x\}$ if $d(x_n,\ f^n x) \le \varepsilon$.

Theorem 2.10. *Let Λ be a hyperbolic set of f. There is $a > 0$ such that, for any $\varepsilon > 0$, there is $\delta > 0$, such that any δ-pseudo orbit $\{x_n\}$ of f contained in $B(\Lambda, a)$ is ε-shadowed by a true orbit of f.*

Proof. We consider the pull-back bundle again. This time it is pulled back by the pseudo orbit. Thus let $s : \mathbb{Z} \to M$ be the sequence that gives the pseudo orbit, that is, $s(n) = x_n$. Consider the pull-back bundle $s^*(TM)$ over the integers

$$\bigcup_{n\in\mathbb{Z}} T_{x_n}M,$$

and the fiber preserving map

$$\Phi^{sf} : s^*(TM)(r_\rho) \to s^*(TM)$$
$$\Phi^{sf}(v) = exp_{x_{n+1}}^{-1}\ f\ exp_{x_n}(v),\ \forall v \in T_{x_n}M.$$

The map is essentially f regarded as taking tops of vectors at x_n to tops of vectors at x_{n+1}. Thus f takes care of calculus in fibers, while the pseudo orbit $\{x_n\}$ takes care of base points, which remain fixed pointwise when calculus is carried out in fibers. This is why it is all right if the base points are given by a pseudo orbit, rather than a true orbit as in Theorem 2.7. Associated are the Banach space of sections

$$\Gamma^0(s^*(TM)) = \{\gamma : \mathbb{Z} \to M \mid \gamma(n) \in T_{x_n}M,\ \text{and } \gamma \text{ is continuous}\}$$

and the induced operator

$$\Phi^{sf}_{\#} : \Gamma^0(s^*(TM))(r_\rho) \to \Gamma^0(s^*(TM))$$
$$(\Phi^{sf}_{\#}(\gamma))(n) = \Phi^{sf}(\gamma(n-1)), \forall n \in \mathbb{Z}.$$

Analogous to Theorem 2.3', a shadowing between a pseudo orbit $\{x_n\}$ and a true orbit $\{f^n x\}$ of f corresponds to a section γ, which is a fixed point of $\Phi^{sf}_{\#}$. To see such a fixed point does exist, we use Theorem 2.2', treating $\Phi^{sf}_{\#}$ as g there. Extending the hyperbolic splitting on Λ to a neighborhood of Λ, it is not hard to see that $D\Phi^{sf}_{\#}(0_s)$, where 0_s is the zero-section of $s^*(TM)$, is a hyperbolic automorphism of $\Gamma^0(s^*(TM))$ with skewness near that of Λ. Moreover, $|\Phi^{sf}_{\#}(0_s)| \leq \delta$, since s is a δ-pseudo orbit. Thus the existence part of Theorem 2.2' applies and gives such a fixed point of $\Phi^{sf}_{\#}$. More detailed discussion determines the number a. This proves Theorem 2.10.

The reader can see that Theorems 2.7, 2.8, and 2.10 are indeed beautiful interpretations of certain part of Theorem 2.1. Theorem 2.7 deals with the uniqueness part, and Theorem 2.8 and 2.10 each deals with the existence part. Though being very cute and having yielded some good results such as Anosov's theorem, theorems stated this way are not adequate to reach Theorem 2.14 below, which is essential to the stability theorem in the next chapter. More complete statements should include C^1-nearby systems with some uniform estimates. This is perhaps natural from a general point of view of dynamical systems: evolution in time, in space, and in systems. As the reader may predict, proofs for such complete statements will not add much difficulty. One just need re-check the proof of Theorem 2.1 closely. We state such results and sketch the proofs.

Theorem 2.11. *Let Λ be a hyperbolic set of f. There are $a > 0$, $\zeta > 0$, and $\delta > 0$ such that for any $g \in \mathcal{B}^1(f, \delta)$, any invariant set Δ of g contained in $B(\Lambda, a)$ is ζ-expansive.*

Proof. Let τ be the skewness of Λ of f. Fix $\tau < \mu < 1$. Take $a > 0$, $\zeta > 0$, and $\delta > 0$ small such that for any $g \in \mathcal{B}^1(f, \delta)$ and any orbit $\{s(n)\}$ of g contained in $B(\Lambda, a)$, the lifting

$$\Phi^{sg} : s^*(TM)(\zeta) \to s^*(TM)$$

and the induced operator

$$\Phi^{sg}_{\#} : \Gamma^b(s^*(TM))(\zeta) \to \Gamma^b(s^*(TM))$$

can be defined and

$$Lip(\Phi^{sg}_{\#} - D\Phi^{sg}_{\#}(0_s)) < 1 - \mu, \text{ on } \Gamma^b(s^*(TM))(\zeta).$$

(Irwin (1980) again). By Theorem 2.6, we may assume that $a > 0$ and $\delta > 0$ have been chosen so that $\{s(n)\}$ is hyperbolic for g with skewness less than μ, hence By Theorem 2.5', the zero-section 0_s is a hyperbolic fixed point of $\Phi^{sg}_{\#}$ with skewness less than μ. By Theorem 2.1, there is at most one fixed point of $\Phi^{sg}_{\#}$ in $\Gamma^b(s^*(TM))(\zeta)$, which is obviously 0_s. This is equivalent to the ζ-expansiveness of g. This proves Theorem 2.11.

The following is a complete statement for Theorem 2.8.

Theorem 2.12. *Let Λ be a hyperbolic set of f. Then,*

1. *There are* $a_0 > 0$, $\varepsilon_0 > 0$, *and* $\delta_0 > 0$ *such that for any* g_1, g_2 *in* $\mathcal{B}^1(f, \delta_0)$ *and any invariant set* Δ_1 *of* g_1 *contained in* $B(\Lambda, a_0)$, *there is at most one map* $h \in \mathcal{B}^b(\iota_{\Delta_1}, \varepsilon_0)$ *such that* $hg_1 = g_2 h$ *on* Δ_1.

2. *There is* $a_0 > 0$ *such that, for any* $\varepsilon > 0$, *there is* $\delta > 0$, *such that for any* g_1, g_2 *in* $\mathcal{B}^1(f, \delta)$ *and any invariant set* Δ_1 *of* g_1 *contained in* $B(\Lambda, a_0)$, *there is an injective map* $h \in \mathcal{B}^0(\iota_{\Delta_1}, \varepsilon)$ *such that* $hg_1 = g_2 h$ *on* Δ_1.

Proof. Let τ be the skewness of Λ under f. Fix $\tau < \mu < 1$. Take $a_0 > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ small such that for any g_1, g_2 in $\mathcal{B}^1(f, \delta_0)$ and any invariant set Δ_1 of g_1 contained in $B(\Lambda, a_0)$, the lifting

$$\Phi^{g_1 g_2} : T_{\Delta_1} M(\varepsilon_0) \to T_{\Delta_1} M$$

and the induced operator

$$\Phi^{g_1 g_2}_{\#} : \Gamma^b(T_{\Delta_1} M)(\varepsilon_0) \to \Gamma^b(T_{\Delta_1} M)$$

can be defined, and

$$Lip(\Phi^{g_1 g_2}_{\#} - D\Phi^{g_1 g_1}_{\#}(0_{\Delta_1})) < 1 - \mu, \text{ on } \Gamma^b(T_{\Delta_1} M)(\varepsilon_0).$$

By Theorem 2.10, we may assume that $a_0 > 0$ and $\delta_0 > 0$ have been chosen so that Δ_1 is hyperbolic of g_1 with skewness less than μ, hence By Theorem 2.4,

$$D\Phi^{g_1 g_1}_{\#}(0_{\Delta_1}) : \Gamma^b(T_{\Delta_1} M) \to \Gamma^b(T_{\Delta_1} M)$$

is a hyperbolic automorphism with skewness less than μ. By Theorem 2.1, there is at most one fixed point of $\Phi^{g_1 g_2}_{\#}$ in $\Gamma^b(T_{\Delta_1} M)(\varepsilon_0)$, hence at most one ε_0-preconjugacy $h : \Delta_1 \to M$ from g_1 to g_2. This proves option 1.

For option 2, we simply take the same number a_0 determined in option 1. Let $\varepsilon > 0$ be given. We may assume $\varepsilon < \varepsilon_0/2$. Note that

$$|\Phi^{g_1 g_2}_{\#}(0_{\Delta_1})| = d^0(g_1|_{\Delta_1},\ g_2|_{\Delta_1}),$$

because the left hand side is just

$$\sup_{x \in \Delta_1} |exp^{-1}_{g_1 x}\ g_2\ exp_x(0_x)|$$

which is

$$\sup_{x \in \Delta_1} |exp^{-1}_{g_1 x}(g_2 x)| = \sup_{x \in \Delta_1} d(g_1 x, g_2 x).$$

Now take $0 < \delta \leq \delta_0$ small such that for any g_1, g_2 in $\mathcal{B}^1(f, \delta)$ and any invariant set Δ_1 of g_1 contained in $B(\Lambda, a_0)$,

$$|\Phi^{g_1 g_2}_{\#}(0_{\Delta_1})| \leq (1 - \mu - Lip(\Phi^{g_1 g_2}_{\#} - D\Phi^{g_1 g_1}_{\#}(0_{\Delta_1}))|_{\Gamma^0(T_{\Delta_1} M)(\varepsilon_0)})\varepsilon.$$

By Theorem 2.1, there is a fixed point of $\Phi_{\#}^{g_1 g_2}$ in $\Gamma^0(T_{\Delta_1}M)(\varepsilon)$, which corresponds to a continuous map $h \in \mathcal{B}^0(\iota_{\Delta_1}, \varepsilon)$ such that $hg_1 = g_2 h$ on Δ_1. Moreover, h must be injective. This is just a consequence of expansiveness in Theorem 2.11. This proves option 2, and completes the proof of Theorem 2.12.

The statement of Theorem 2.12 is complicated. It would be helpful to keep Theorem 2.1 in mind as model. Thus $D\Phi_{\#}^{g_1 g_1}(0_{\Delta_1})$ corresponds to A in Theorem 2.1, $\Phi_{\#}^{g_1 g_2} - D\Phi_{\#}^{g_1 g_1}(0_{\Delta_1})$ corresponds to φ, ε_0 corresponds to r, fixed point of $\Phi_{\#}^{g_1 g_2}$, or preconjugacy h, corresponds to fixed point of $A + \varphi$, and $|\Phi_{\#}^{g_1 g_2}(0_{\Delta_1})|$ corresponds to $|\varphi(0)|$.

Next is a complete statement for Theorem 2.10. We omit the proof.

Theorem 2.13. *Let Λ be a hyperbolic set of f. Then,*

1. *There are $a_0 > 0$, $b_0 > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any δ_0-pseudo orbit of any $g \in \mathcal{B}^1(f, b_0)$ contained in $B(\Lambda, a_0)$ can be ε_0-shadowed by at most one ture orbit of g.*

2. *There are $a_0 > 0$ and $b_0 > 0$ such that, for any $\varepsilon > 0$, there is $\delta > 0$, such that any δ-pseudo orbit of any $g \in \mathcal{B}^1(f, b_0)$ contained in $B(\Lambda, a_0)$ is ε-shadowed by a true orbit of g.*

Theorem 2.12 and 2.13 both concern the conjugacy problem. In fact, regarding true orbits $g^n x$ as pseudo orbits x_n of f, the uniqueness of shadowing yields a preconjugacy from g to f, hence Theorem 2.13 gives a similar result on embedding stability of hyperbolic sets as Theorem 2.12 does. Theorem 2.12 gives conjugacy from f to g, while Theorem 2.13 does backwards, from g to f. Either one suffices for Theorem 2.14 (we will take Theorem 2.12), which we now proceed to.

Recall that for any subset S of M (not necessarily invariant), the *maximal invariant set* of f in S is the biggest f-invariant set A (could be empty) contained in S. Hence $A = \cap_{n \in \mathbb{Z}} f^n(S)$. A compact invariant set Λ of f is called *isolated* (or *locally maximal*) if Λ is the maximal invariant set in some neighborhood U of Λ in M. In this case U is called an *isolating neighborhood* of Λ. Note that in this case Λ can be expressed also as $\cap_{n \in \mathbb{Z}} f^n(K)$, for any set K with $\Lambda \subset K \subset U$. Any hyperbolic periodic orbit is an isolated invariant set. The horseshoe is too. Sometimes it is missed that a hyperbolic periodic orbit in a horseshoe is an isolated invariant set. A good example for non-isolated hyperbolic invariant sets is the orbit (or its closure) of a transverse homoclinic point x associated with a hyperbolic fixed point p. It is not hard to prove that $\mathrm{Orb}(x) \cup \{p\}$ is hyperbolic. In fact the invariant splitting is given by the tangent planes to the stable and unstable manifold of p at the iterates of x. Exponential rates exist because most of the iterates are near p. Now take m large and ε small, and let z be the intersection point of $W^s_\varepsilon(f^{-m}x)$ with

$W^u_\varepsilon(f^m x)$. Then the orbit of z remains in the ε-neighborhood of, but different from, the orbit of x. Note that this does not contradict the expansiveness of hyperbolic sets. In fact $\{f^n z\}$ does not shadow $\{f^n x\}$. In abuse of language it "shadows" $\{f^n x\}$ "twice". Here the difference between a periodic orbit and a non-periodic one is that, an orbit near a periodic orbit always gives a shadowing, while an orbit near a non-periodic one may not.

Theorem 2.14. *Let Λ be an isolated hyperbolic set of f with U an isolating neighborhood. For any $\varepsilon > 0$, there is $\delta > 0$, such that for any $g \in \mathcal{B}^1(f, \delta)$, the maximal invariant set Δ of g in U is isolated in U, and $g|_\Delta$ is ε-conjugate to $f|_\Lambda$.*

Proof. First we notice a simple topological fact. For U fixed, the maximal invariant set in U varies upper semicontinuously with f in the C^0 topology. That is, if Λ is the maximal invariant set of f in U, then for any $a > 0$, there is $\delta > 0$, such that for any $g \in \mathcal{B}^0(f, \delta)$, the maximal invariant set Δ of g in U is contained in $B(\Lambda, a)$. This is because Λ is the infinite intersection $\cap_{n\in\mathbb{Z}} f^n U$, hence for some large positive integer N we have $\cap_{n=-N}^{N} f^n U \subset B(\Lambda, a/2)$. Then we can take $\delta > 0$ small such that, for any $g \in \mathcal{B}^0(f, \delta)$, $\cap_{n=-N}^{N} g^n U \subset B(\Lambda, a)$, hence $\cap_{n=-\infty}^{\infty} g^n U \subset B(\Lambda, a)$.

Let Λ be an isolated hyperbolic set of f with U an isolating neighborhood. By option 1 of Theorem 2.12, there are $a_0 > 0$, $\varepsilon_0 > 0$, and $\delta_0 > 0$ such that for any g_1, g_2 in $\mathcal{B}^1(f, \delta_0)$ and any invariant set Δ_1 of g_1 contained in $B(\Lambda, a_0)$, there is at most one map $h \in \mathcal{B}^b(\iota_{\Delta_1}, \varepsilon_0)$ such that $hg_1 = g_2 h$ on Δ_1. We may assume that

$$B(\Lambda, a_0 + \varepsilon_0) \subset U.$$

Let $\varepsilon > 0$ be given. We may assume $\varepsilon \le \varepsilon_0/2$. By option 2 of Theorem 2.12, for this $a_0 > 0$, there is $0 < \delta \le \delta_0$ such that, for any g_1, g_2 in $\mathcal{B}^1(f, \delta)$ and any invariant set Δ_1 of g_1 contained in $B(\Lambda, a_0)$, there is an injective map $h \in \mathcal{B}^0(\iota_{\Delta_1}, \varepsilon)$ such that $hg_1 = g_2 h$ on Δ_1. Taking a smaller number δ if necessary, we may assume, by the topological fact just mentioned, that the maximal invariant set of any $g \in \mathcal{B}^1(f, \delta)$ in U is contained in $B(\Lambda, a_0)$.

Now let Δ_1 and Δ_2, respectively, be the maximal invariant sets in U for g_1 and g_2 in $\mathcal{B}^1(f, \delta)$. Thus Δ_1 and Δ_2 are in $B(\Lambda, a_0)$, and there are two continuous injective maps $h_1 : \Delta_1 \to M$ with

$$h_1 g_1 = g_2 h_1 \text{ on } \Delta_1, \text{ and } d^0(h,\ \iota_{\Delta_1}) \le \varepsilon,$$

and $h_2 : \Delta_2 \to M$ with

$$h_2 g_2 = g_1 h_2 \text{ on } \Delta_2, \text{ and } d^0(h,\ \iota_{\Delta_2}) \le \varepsilon.$$

Note that

$$h_1(\Delta_1) \subset \Delta_2,$$

because $h_1(\Delta_1)$ is clearly g_2-invariant, because $h_1(\Delta_1) \subset B(\Delta_1, \varepsilon) \subset B(\Lambda, \varepsilon + a_0) \subset U$, and because Δ_2 is the maximal invariant set of g_2 in U. Likewise,

$$h_2(\Delta_2) \subset \Delta_1.$$

Combine h_1 and h_2 together gives $h_2 h_1 : \Delta_1 \to \Delta_1$ with

$$(h_2 h_1) g_1 = g_1 (h_2 h_1), \text{ and } d^0(h_2 h_1, \iota_{\Delta_1}) \leq \varepsilon + \varepsilon \leq \varepsilon_0.$$

However, the inclusion ι_{Δ_1} is already a conjugacy between g_1 and itself. By uniqueness of preconjugacy in $\mathcal{B}^b(\iota_{\Delta_1}, \varepsilon_0)$,

$$h_2 h_1 = \iota_{\Delta_1}.$$

Likewise,

$$h_1 h_2 = \iota_{\Delta_2}.$$

This proves $g_1|_{\Delta_1}$ and $g_2|_{\Delta_2}$ are ε-conjugate. In particular, one of them could be $f|_\Lambda$. This proves Theorem 2.14.

Another major result about hyperbolic sets is the stable manifold theorem. Recall the *local stable manifold* $W^s_\varepsilon(x)$ *of size* ε for a point x is defined as

$$W^s_\varepsilon(x, f) = \{y \in M | d(f^n y, f^n x) \leq \varepsilon, \ \forall \ n \geq 1\},$$

and likewise, the *local unstable manifold* $W^u_\varepsilon(x)$ *of size* ε

$$W^u_\varepsilon(x, f) = \{y \in M | d(f^{-n} y, f^{-n} x) \leq \varepsilon, \ \forall \ n \geq 1\}.$$

Thus $W^s_\varepsilon(x)$ consists of points that ε-shadow x along forward iterates, and $W^u_\varepsilon(x)$ — backward iterates.

Theorem 2.15. *Let Λ be a hyperbolic set of f with splitting $E^s \oplus E^u$. Assume f is C^r. Then there is $0 < \mu < 1$ and $\varepsilon > 0$ such that for any $x \in \Lambda$,*

1. *$W^s_\varepsilon(x)$ is a C^r embedded disc, tangent to $E^s(x)$ at x.*
2. *$d(fy, fx) \leq \mu d(y, x), \ \forall \ y \in W^s_\varepsilon(x)$.*
3. *The family of discs $W^s_\varepsilon(x)$ varies continuously with x, in the C^r topology.*

Likewise for $W^u_\varepsilon(x)$. Thus points that shadow a hyperbolic orbit $\{f^n x\}$ along forward iterates form a disc of the same dimension as $E^s(x)$. Moreover, in this case, forward shadowing implies forward asymptotic approaching, even exponentially. That the stable manifolds are differentiable is delicate. This differentiability, C^1 at least, is essential to us in what follows.

Local stable manifolds visualize beautifully the notion of ε-shadowing. Thus the ζ-expansiveness is visualized as that for ζ small, $W^s_\zeta(x) \cap W^u_\zeta(x)$ is the single point $\{x\}$. A more complete stable manifold theorem includes C^r

perturbations g, which asserts that points whose positive g-orbit ε-shadows the positive f-orbit of x is a disc D^s_ε that is C^r close to $W^s_\varepsilon(x)$, likewise a disc D^u_ε for backward shadowing. The intersection $D^s_\varepsilon \cap D^u_\varepsilon$ is hence a single point y whose g-orbit ε-shadows the f-orbit of x. The disc D^s_ε is hence in $W^s_{\varepsilon'}(y)$ for some ε'. In this way the correspondence $x \to y$ visualizes the ε-preconjugacy h in Theorem 2.8 and 2.12.

Local stable manifolds yield global stable manifolds through iterates. Recall the (global) stable and unstable sets are defined topologically as

$$\begin{aligned} W^s(x) &= \{y \in M \mid d(f^n(y),\ f^n(x)) \to 0, n \to +\infty\}, \\ W^u(x) &= \{y \in M \mid d(f^{-n}(y),\ f^{-n}(x)) \to 0, n \to +\infty\}. \end{aligned}$$

Unlike the local stable manifolds, the global stable manifolds are equivalence classes of the equivalence relation $x \sim y$ iff $d(f^n(y),\ f^n(x)) \to 0$. Thus if $y \in W^s(x)$, then $x \in W^s(y)$. Through iterates, one gets

$$\begin{aligned} W^s(x) &= \bigcup_{n\geq 0} f^{-n} W^s_\varepsilon(f^n x), \\ W^u(x) &= \bigcup_{n\geq 0} f^n W^u(f^{-n} x). \end{aligned}$$

Thus the global stable manifolds are immersed C^r submanifolds of M. Such a submanifold can wrap around in complicated ways as seen in the horseshoe map. Associated is the celebrated notion of transverse homoclinic points.

Finally we study the way points approach an isolated hyperbolic set. Let Λ be a compact invariant set. The stable set and unstable set of Λ are defined as

$$\begin{aligned} W^s(\Lambda) &= \{y \in M | d(f^n y, \Lambda) \to 0,\ n \to +\infty\}, \\ W^u(\Lambda) &= \{y \in M | d(f^{-n} y, \Lambda) \to 0,\ n \to +\infty\}. \end{aligned}$$

These are points that approach the set Λ under iteration. This is weaker than to approach asymptoticly a point $x \in \Lambda$, which is what the (global) stable manifold $W^s(x)$ or unstable manifold $W^u(x)$ of x means. The next theorem, known as the *In Phase Theorem*, says that for isolated hyperbolic sets these two are the same.

Theorem 2.16. *Let Λ be an isolated hyperbolic set of f. Then*

$$\begin{aligned} W^s(\Lambda) &= \bigcup_{x\in\Lambda} W^s(x), \\ W^u(\Lambda) &= \bigcup_{x\in\Lambda} W^u(x). \end{aligned}$$

Proof. We take W^s, and prove the $\subset$ part only. Let U be an isolating neighborhood of Λ. Take $r > 0$ such that $B(\Lambda, r) \subset U$. By Theorem 2.10, there are a_0 and $\delta > 0$ such that any δ-pseudo orbit of f contained in $B(\Lambda, a_0)$ is $r/2$-shadowed by a true orbit of f. We may assume $a_0 \leq r/2$. Now let $x \in W^s(\Lambda)$. Denote $a = \min(a_0, \delta)$. For a large N, the forward orbit $\{f^{N+n}(x)\}$ of $f^N(x)$ remains in $B(\Lambda, a)$. Take any point $y \in \Lambda$ with $d(y, f^N(x)) \leq a$. Then the backward orbit of $f^{-1}y$ and the forward orbit of $f^N(x)$ together form a δ-pseudo orbit of f contained in $B(\Lambda, a_0)$, hence is $r/2$-shadowed by a true orbit $\{f^n z\}$ of f. Thus $f^N(x) \in W^s_{r/2}(z)$, hence $x \in W^s(z)$. It remains to prove $z \in \Lambda$. But this is obvious because the orbit of z is contained in $B(\Lambda, a_0 + r/2) \subset U$, and Λ is maximal in U. This proves Theorem 2.16.

CHAPTER 3

The Ω-stability theorem of Smale

In this chapter we prove the Ω-stability theorem of Smale (1970), which asserts that Axiom A and no-cycle systems are Ω-stable. Naturally, a diffeomorphism $f: M \to M$ is called Ω-stable if there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, $g|_{\Omega(g)}$ is topologically conjugate to $f|_{\Omega(f)}$.

3.1 Axiom A systems

A diffeomorphism $f: M \to M$ is called an *Axiom A system* if $\Omega(f)$ is hyperbolic and if $\Omega(f) = \overline{Per(f)}$.

The north-south pole map is axiom A, and so is the horseshoe map defined on the 2-sphere. Any Anosov diffeomorphism f is Axiom A. To see this we only need check $\Omega(f) = \overline{Per(f)}$. This follows from the shadowing property. Let x be any non-wandering point x of f. We prove there is a periodic point z arbitrarily close to x. Now there is $y \in M$ and $m \in N$ such that both y and $f^m y$ are arbitrarily close to x. Repeating the finite orbit y, fy, ... , $f^{m-1}y$ in both forward and backward directions gives a periodic pseudo orbit of f, which is shadowed by a unique true orbit $\{f^n z\}$. The m-shift of the pseudo orbit is then shadowed by the m-shift of the true orbit $\{f^{n+m}z\}$. But the pseudo orbit is periodic hence its m-shift is itself. By uniqueness the two true orbits $\{f^n z\}$ and $\{f^{n+m}z\}$ must be the same, which means z is periodic. This proves any Anosov diffeomorphism is Axiom A. Note that this argument does not show that, generally, if $\Omega(f)$ is hyperbolic then $\Omega(f) = \overline{Per(f)}$. The reason is that the (periodic) pseudo orbit in this case is not known to be near a hyperbolic set (which is $\Omega(f)$ in this case), hence the shadowing property may not hold. In fact it remained to be an important problem whether or not the hyperbolicity of $\Omega(f)$ implies $\Omega(f) = \overline{Per(f)}$. Newhouse and Palis (1973) showed this is right in dimension 2. Dankner (1978) constructed later a counterexample in dimension 3.

3.2 The isolation of Axiom A non-wandering sets

One of the main features of an Axiom A system is that the non-wandering set is isolated. We start with the λ-lemma of Palis, which grasps a crucial feature of chaotic dynamics hence has many applications.

Theorem 3.1 (The λ-lemma). *Let $p \in M$ be a hyperbolic periodic point. For any u-disc B in $W^u(p)$, any point $x \in W^s(p)$, any u-disc D transverse to $W^s(p)$ at x, and any $\varepsilon > 0$, there is N such that if $n > N$, $f^n(D)$ contains a u-disc that is C^1-ε close to B.*

Here for simplicity we have denoted $s = \dim W^s(p)$ and $u = \dim W^u(p)$. The λ-lemma says that, no matter how big B is, how small D is, and how weakly transverse (small angle) D to $W^s(p)$ is, the conclusion is always true. Though the geometrical fact seems easy to swallow, the proof is delicate (see the book of de Melo and Palis).

Here is an application of the λ-lemma, which has a beautiful name, the *cloud lemma*.

Theorem 3.2. *Let p and q be hyperbolic periodic points of f. Assume x is a point of transverse intersection of $W^s(p)$ and $W^u(q)$, and y is a point of transverse intersection of $W^s(q)$ and $W^u(p)$. Then x and y are both non-wandering.*

Proof. Replacing f by an iterate if necessary, we may assume p and q are fixed. Take a u-disc D of center x in $W^u(q)$. By the λ-lemma, there is n_1 such that $f^{n_1}(D)$ intersects transversely with $W^s(q)$ at a point z. Take a u-disc D_1 of center z in $f^{n_1}(D)$. By the λ-lemma again, there is n_2 such that $f^{n_2}(D_1)$ contains a u-disc that is arbitrarily C^1-close to D. This proves $x \in \Omega(f)$. The proof for y is similar. This proves Theorem 3.2.

The cloud lemma extends to m hyperbolic periodic points $p_1, p_2, \cdots, p_m$ such that $W^s(p_i)$ and $W^u(p_{i+1})$ intersect transversely at x_i for each i. Then each x_i is non-wandering. The case $m = 1$ is the celebrated transverse homoclinic point. Thus every transverse homoclinic point is non-wandering. In fact Birkhoff-Smale theorem asserts that any transverse homoclinic point is in a horseshoe.

The reader may have noticed that, to guarantee that an individual point of intersection, say x_i, is non-wandering, transversality at the other $x_j, j \neq i$, would be enough. In particular, any homoclinic point, whatever transverse or not, is non-wandering.

The following general fact is a consequence of the stable manifold theorem.

Theorem 3.3. *Let Λ be a hyperbolic set. There are $\varepsilon > 0$ and $\delta > 0$ such that for any two points x and y in Λ with $d(x, y) < \delta$, $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ intersect transversely at a single point.*

Proof. By Theorem 2.15, $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ are tangent to $E^s(x)$ and $E^u(x)$ at x, respectively, and vary continuously with x in the C^1 topology. There is clearly $\varepsilon > 0$ such that, for all $x \in \Lambda$, $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ intersect transversely at the single point x. So there is $\delta(x) > 0$ such that if $y \in \Lambda$ and $d(x, y) < \delta(x)$ then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ intersect transversely at a single point. Since Λ is compact, δ can be chosen independent of x. This proves Theorem 3.3.

We may call such a pair of numbers (ε, δ) an *adapted size* of the hyperbolic set Λ. Note that if (ε, δ) is an adapted size, so is (ε, δ') for any $0 < \delta' \leq \delta$ and, for any $0 < \varepsilon' \le \varepsilon$, there is $0 < \delta'' \le \delta'$ such that (ε', δ'') is an adapted size. We say a hyperbolic set Λ has *local product structure*, if there is an adapted size (ε, δ) of Λ such that, for every pair of points x and y in Λ with $d(x, y) < \delta$, the single point of transverse intersection of $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ remains in Λ.

Theorem 3.4. *If f is Axiom A, then $\Omega(f)$ has local product structure.*

Proof. Take any adapted size (ε, δ) of $\Omega(f)$. Let x and y be in $\Omega(f)$ with $d(x, y) < \delta$. If x and y are both periodic, the two intersection points are non-wandering, by the cloud lemma. Since $\Omega(f) = \overline{P(f)}$, the theorem follows.

The following is the well known shadowing lemma of Bowen.

Theorem 3.5. *If a hyperbolic set Λ has local product structure, then for every $\varepsilon > 0$, there is $\delta > 0$, such that any δ-pseudo orbit in Λ can be ε-shadowed by a true orbit in Λ.*

This theorem can be found in many text books. The proof is an elegant geometrical construction of shadowing using local product structure. It assumes more than plain hyperbolicity of Λ as Theorem 2.10 does. It asserts more too that the shadowing orbit is in Λ. It yields as a consequence the following

Theorem 3.6. *A hyperbolic set has local product structure if and only if it is isolated.*

Proof. Let Λ be a hyperbolic set with local product structure. Let $\zeta > 0$ be the expansiveness constant of Λ. By the shadowing lemma, there is $\delta > 0$ such that any δ-pseudo orbit in Λ can be $\zeta/2$-shadowed by a true orbit in Λ. Take $0 < a < \zeta/2$ small enough such that for any orbit $\{f^n x\}$ in $B(\Lambda, a)$, letting x_n to be the closest point to $f^n x$ in Λ, $\{x_n\}$ form a δ-pseudo orbit. Then there is $z \in \Lambda$ such that $\{f^n z\}$ $\zeta/2$-shadows $\{x_n\}$. Since $\{f^n x\}$ and $\{f^n z\}$ both $\zeta/2$-shadow the same sequence $\{x_n\}$, they ζ-shadow each other. By ζ-expansiveness, $x = z$. Thus $x \in \Lambda$. This proves Λ is isolated.

Let Λ be an isolated hyperbolic set. By definition there is $a > 0$ such that if an orbit $\{f^n x\}$ is contained in $B(\Lambda, a)$, then the orbit is actually in Λ. Now let (ε, δ) be an adapted size of Λ. We may assume $\varepsilon \le a$. Let z be an intersection of $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$, $x, y \in \Lambda$. Then z ε-shadows the positive orbit of x and the negative orbit of y. Then $\{f^n z\}$ is contained in $B(\Lambda, a)$. Thus $z \in \Lambda$. This proves Λ has local product structure. This proves Theorem 3.6.

As a consequence of Theorem 3.4 and 3.6 we obtain the main result of this section.

Theorem 3.7. *If f is Axiom A, then $\Omega(f)$ is isolated.*

This important theorem is first proved by Hirsch-Palis-Pugh-Shub. The above proof using shadowing lemma (Theorem 3.5) is due to Bowen. The next theorem says, for Axiom A systems, every point $x \in M$ shares the same asymptotic behavior with some non-wandering point.

Theorem 3.8. *If f is Axiom A, then*

$$M = \bigcup_{z\in\Omega(f)} W^s(z) = \bigcup_{z\in\Omega(f)} W^u(z).$$

Proof. Since every point $x \in M$ approaches to the non-wandering set for both forward and backward time, this is an immediate corollary of the In Phase Theorem 2.16. This proves Theorem 3.8.

Thus if f is Axiom A, $W^s(x)$ is an immersed submanifold of M for any $x \in M$. This gives a decomposition of M into immersed submanifolds. Likewise for the unstable manifolds. The decomposition can be very complicated. The horseshoe map on S^2 can be taken as illustration. While two stable manifolds never intersect as different equivalence classes, a stable manifold may well intersect an unstable manifold. An Axiom A system f is said to satisfy the *strong transversality condition* if $W^s(x)$ and $W^u(y)$ intersect transversely for every pair of points $x, y \in M$. Now we can state without proof the celebrated stability theorem of Robbin and Robinson, which is a central result of Dynamical Systems obtained in the early seventies.

Theorem 3.9. *If f satisfies Axiom A and the strong transversality condition, then f is C^1 structurally stable.*

3.3 Spectral decomposition

Another main feature for Axiom A systems is that the non-wandering set decomposes into a finitely many disjoint transitive sets. This is the spectral decomposition theorem of Smale. Here a compact invariant set Λ is called *transitive* if there is a point $x \in \Lambda$ such that the orbit of x is dense in Λ. Any transitive set can not be decomposed into two disjoint closed invariant sets. (Though some non-transitive set can not be decomposed into two disjoint closed invariant sets either.) We insert Birkhoff's theorem on transitivity.

Theorem 3.10. *A compact invariant set Λ is transitive if and only if for any two open sets U and V in Λ, there is an integer m such that $f^m(U) \cap V \neq \emptyset$.*

Note that the condition formulated in the theorem amounts to say that the orbit of any open set U (by this we simply mean the union of orbits of x for all $x \in U$) is dense.

Proof. The "only if" part is obvious. we prove the "if" part. Take a countable basis $U_1, U_2, \ldots$ of Λ. Then for any $i \geq 1$, $\cup_{n\in\mathbb{Z}} f^n(U_i)$ is both open and dense in Λ. By Baire's theorem,

$$B = \bigcap_{i\geq 1} \bigcup_{n\in\mathbb{Z}} f^n(U_i)$$

is dense in Λ. It is easy to see that every $x \in B$ has orbit dense in Λ. This proves Theorem 3.10.

Now we prove the spectral decomposition theorem.

Theorem 3.11. *If f satisfies Axiom A, then the non-wandering set decomposes uniquely into a finite union of disjoint transitive sets*

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k.$$

These sets Ω_i are called the *basic sets* of f.

Proof. First we prove the uniqueness of the decomposition. Assume there is another such decomposition

$$\Omega(f) = \Delta_1 \cup \cdots \cup \Delta_l.$$

Take $x_i \in \Omega_i$ such that the orbit of x_i is dense in Ω_i. There is a unique j with $x_i \in \Delta_j$, hence

$$\Omega_i = \overline{Orb(x_i)} \subset \Delta_j.$$

This proves $k \leq l$. Similarly, $l \leq k$. Thus $k = l$. Since $\cup\Omega_i = \cup\Delta_j$, the above inclusion $\Omega_i \subset \Delta_j$ is actually an equality. This proves uniqueness.

Consider the binary relation $\sim$ on the set $P = P(f)$ of periodic points defined as $x \sim y$ if and only if $W^s(x)$ and $W^u(y)$ intersect transversely, and $W^u(x)$ and $W^s(y)$ intersect transversely. The relation is reflective, symmetric and, by the λ-lemma, transitive, hence an equivalence relation. Thus P decomposes into equivalence classes P_i. By Theorem 3.3, there is $\delta > 0$ such that if two periodic points x and y are within δ, then $x \sim y$. Hence the following three properties are clear.

(1). There are only finitely many equivalence classes P_i, say N of them. In fact,

$$d(\overline{P}_i,\ \overline{P}_j) \geq \delta, i \neq j.$$

(2). $\Omega(f) = \cup_{i=1}^N \overline{P}_i$.

(3). For any i, there is a unique j such that $f(\overline{P}_i) = \overline{P}_j$. The correspondence $i \to j$ is an N-permutation.

We only prove (3). Note that if $x \sim y$, then $fx \sim fy$. Hence for any i, there is a unique j such that $fP_i \subset P_j$. But $fP = P$. So $fP_i = P_j$. Then $f(\overline{P}_i) = \overline{P}_j$, and $i \to j$ is an N-permutation.

Note that $\overline{P}_i$ is not invariant in general. But the N-permutation is a product of cyclic permutations. Putting together the $\overline{P}_i$'s that are associated with the same cyclic permutation gives a finitely disjoint union of compact invariant sets

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k.$$

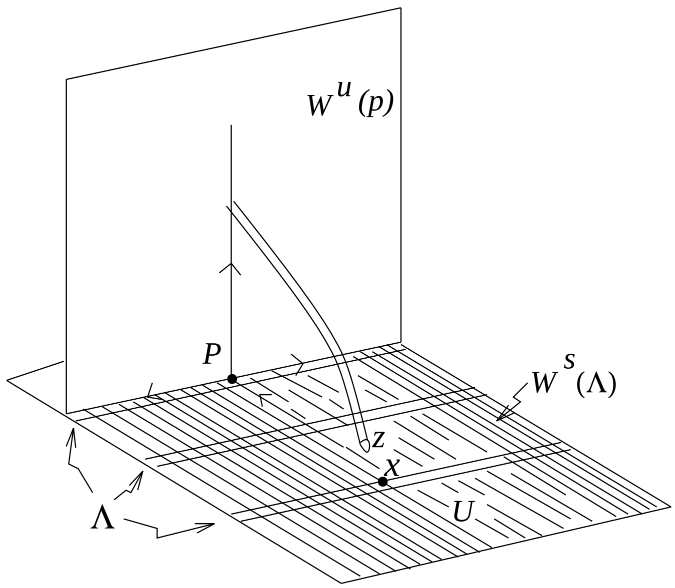
It remains to prove each Ω_i is transitive. Let Ω_i be the (cyclic) union

$$\Omega_i = \overline{P}_{i_1} \cup \cdots \cup \overline{P}_{i_r},$$

such that $f(\overline{P}_{i_p}) = \overline{P}_{i_{p+1}}$. We use Birkhoff's theorem. For any open set U of $\overline{P}_{i_p}$ and any open set V of $\overline{P}_{i_q}$, $1 \leq p \leq q \leq r$, we need to show there is $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$. Let $W = f^{q-p}(U)$. We show there is m such that $f^m(W) \cap V \neq \emptyset$. But this is clear if we take two periodic points $x \in W$ and $y \in V$, and note that $x \sim y$. This proves Theorem 3.11.

We end this section with an example that shows Axiom A systems are not C^1-dense in $\mathrm{Diff}^1(M)$. The example is taken from Newhouse's lecture notes [GMN]. The idea is to have a basic set Λ such that, C^1 persistently, there are homoclinic tangencies $z \in W^s(x) \cap W^u(y) - \Lambda$ for some $x, y \in \Lambda$. Such a z obstructs the hyperbolicity of the non-wandering set. Here we use a more flexible notion of *basic set* which may not known to come from the spectral decomposition of an Axiom A system. It is simply defined to be an isolated transitive hyperbolic set. It is easy to see using shadowing that periodic points are dense in a basic set. An analogous fact to the case of a hyperbolic periodic point is that, any homoclinic point of a basic set Λ is non-wandering. Here by a *homoclinic point of a basic set* Λ we mean a point $z \in W^s(\Lambda) \cap W^u(\Lambda) - \Lambda$. By the In Phase Theorem, $z \in W^s(x) \cap W^u(y) - \Lambda$ for some x and y in Λ. Of course generally we can not expect homoclinic points to be non-wandering, without constraints put on the set Λ. In general any wandering point approaches to $\Omega(f)$ for both forward and backward time, hence is in $W^s(\Omega(f)) \cap W^u(\Omega(f)) - \Omega(f)$. This would not be interesting. An extreme example could be the north-south pole map on the sphere with $\Omega(f) = \{S, N\}$.

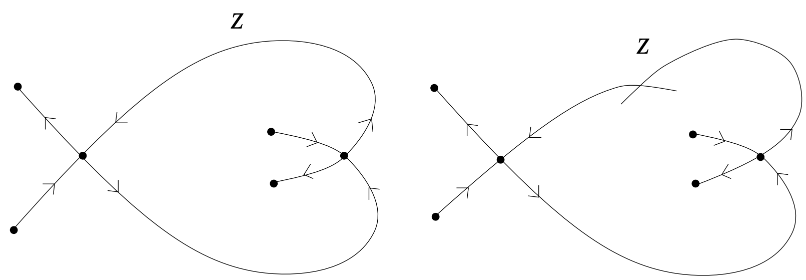
To make the example more intuitive, we need a non-trivial hyperbolic attractor in lower dimension. Plykin's attractor in the plane is a good choice. Here by an *attractor* Λ we mean a transitive set that has a neighborhood U such that $f(\overline{U}) \subset U$ and $\Lambda = \cap_{n \geq 0} f^n U$. Note that in this case the intersection of positive iterates of U is the same as the intersection of all iterates. Thus an attractor Λ is always isolated. Moreover, $W^s(\Lambda)$ always contains a neighborhood of Λ. If the attractor is hyperbolic, it is a basic set, and by the In Phase Theorem, the family $W^s(x), x \in \Lambda$, fill out a neighborhood of Λ. A classical hyperbolic attractor is the solenoid of Smale. It comes from a map f that maps a neighborhood U of the circle S^1 in R^3 into itself that stretches the S^1 direction, while shrinks the normal direction. Bob Williams has extended S^1 to any branched 1-manifold. This gives an amazing variety of 1-dimensional hyperbolic attractors. Moreover, Williams shows any 1-dimensional hyperbolic attractor arises this way (Later Williams extended his theory to higher dimensional attractors). Let Λ be the Plykin's hyperbolic attractor in the plane, which is locally a Cantor set cross an interval, as the figure shows. Thus the family of stable intervals $W^s(x)$, $x \in \Lambda$, fill out a neighborhood U of Λ in the plane.



Take a periodic point $p \in \Lambda$. Add one more dimension to the problem so that $W^u(p)$ has dimension 2 (the map is still denoted by f). Push $W^u(p)$ with a finger till it has immerged a little bit under U so that $W^u(p)$ intersects U at a small circle. Thus $W^u(p)$ is tangent to a (actually two) stable interval $W^s(x)$ at some homoclinic tangency z, for some $x \in \Lambda$. This feature persists under C^1 perturbations.

3.4 No-cycle condition, filtration, and the Ω-stability theorem

Only Axiom A does not guarantee Ω-stability. The following example is due to Palis and Smale. It is a diffeomorphism f on S^2 with the non-wandering set $\Omega(f)$ of six hyperbolic fixed points as the figure shows.



Thus f is Axiom A. There are two saddle-connections that form a "cycle". A small C^r perturbation (any r) near a wandering point z on one of the connection (f composed with a small rotation localized by a bump function) may make the connection transverse at z, hence make all points on the other connection non-wandering, thus an Ω-explosion.

We define cycles formally. Let $\Lambda_1, \cdots, \Lambda_k$ be a finitely many disjoint compact invariant sets of f. Define a binary relation $\to$ as

$$\Lambda_i \to \Lambda_j \text{ if and only if } W^u(\Lambda_i) \cap W^s(\Lambda_j) - \bigcup_{l=1}^{k} \Lambda_l \neq \emptyset.$$

Roughly, $\Lambda_i \to \Lambda_j$ means there is some point outside these Λ_l's that goes from Λ_i to Λ_j. Note that this binary relation is not reflexive, nor symmetric, nor transitive. We say $\Lambda_{i_1}, \cdots, \Lambda_{i_m}$ form a *cycle* if

$$\Lambda_{i_1} \to \Lambda_{i_2} \to \cdots \to \Lambda_{i_m} \to \Lambda_{i_1}.$$

We say $\Lambda_1, \cdots, \Lambda_k$ satisfy the *no-cycle condition* if there are no cycles between them. Note that an Axiom A system does not have 1-cycle to any of its basic set. This is because points on the 1-cycle would be homoclinic to this basic set and hence non-wandering, but the whole non-wandering set has been decomposed into basic sets.

When $\{\Lambda_i\}$ satisfy the no-cycle condition, one may reindex so that $\Lambda_i \to \Lambda_j$ implies $i > j$. We call such an ordering of $\{\Lambda_i\}$ a *filtration ordering.* The idea is simple. Let us call Λ_i a *lower extreme* if there is no Λ_j such that $\Lambda_i \to \Lambda_j$. Here we think of the binary relation $\to$ as going downwards, hence use the term "lower". Since there are no cycles between $\Lambda_1, \cdots, \Lambda_k$, there must be some lower extremes (Otherwise one would trace out a cycle since there are only finitely many Λ_i). Collect all lower extremes of $\Lambda_1, \cdots, \Lambda_k$ and put them in any order as $\Lambda_1, ..., \Lambda_{k_1}$. The rest of the Λ_i still have no cycles, hence still have some lower extremes (respecting to the rest of the Λ_i). Put them as $\Lambda_{k_1+1}, ..., \Lambda_{k_2}$, etc. This gives a filtration ordering.

Generally, if the union of finitely many disjoint compact invariant sets Λ_i is sufficiently large, even without the no-cycle condition, it yields a decomposition of M:

Theorem 3.12. *Let $\Lambda_1, ..., \Lambda_k$ be finitely many disjoint compact invariant sets of f, whose union contains the limit set $L(f)$. Then*

$$M = \bigcup_{i=1}^{k} W^s(\Lambda_i) = \bigcup_{i=1}^{k} W^u(\Lambda_i).$$

The conclusion is wearker than Theorem 3.8. It concludes that every point approaches a (single) set Λ_i in forward or backward time, rather than asymptotic to a point of Λ_i. Nevertheless it does not assume Axiom A as Theorem 3.8 does.

Proof. For each i, take a neighborhood U_i of Λ_i such that for any $i \neq j$,

$$\overline{U_i} \cap \overline{U_j} = \emptyset, \ (f\overline{U_i}) \cap \overline{U_j} = \emptyset.$$

Let $x \in M$. Since $\omega(x) \subset \cup_{i=1}^k \Lambda_i$, there is N such that $f^n x \in \cup_{i=1}^k U_i$ for all $n \geq N$. Let $f^N x \in \Lambda_l$. Then $f^n x \in U_l$ for all $n \geq N$, since $(f\overline{U_i}) \cap \overline{U_j} = \emptyset$, for $i \neq j$. Then $\omega(x) \subset \Lambda_l$. Hence $x \in W^s(\Lambda_l)$. Likewise for W^u. This proves Theorem 3.12.

If, moreover, the no-cycle condition is added, it turns out to yield (see Theorem 3.15 below) a strong global dynamics, — the filtration structure we now define.

A finite nested sequence

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

of compact subsets M_i of M is called a *filtration* of f, if

$$fM_i \subset intM_i.$$

In this case we call $M_i - M_{i-1}$ the i-th *layer* of the filtration, and denote

$$K_i = K_i(f) = \bigcap_{n=-\infty}^{\infty} f^n(M_i - M_{i-1})$$

the maximal invariant set of f in the i-th layer.

Theorem 3.13. *Let $\{M_i\}_{i=0}^k$ be a filtration of f. Then K_i are mutually disjoint isolated invariant sets of f, with the i-th layer $M_i - M_{i-1}$ an isolating neighborhood of K_i. The union of K_i contains the chain recurrent set $R(f)$ of f, and $\{K_i\}$ satisfies the no-cycle condition.*

Recall a point $x \in M$ is called *chain recurrent* if for any $\varepsilon > 0$, there is an ε-chain going from x to x itself. The chain recurrent set $R(f)$ is compact and f-invariant, and it contains $\Omega(f)$.

Proof. Since $fM_i \subset intM_i$, it follows that

$$f^{n+1}(M_i) \subset f^n(intM_i).$$

Then

$$\bigcap_{n=-\infty}^{\infty} f^n M_i = \bigcap_{n=-\infty}^{\infty} f^n(intM_i).$$

Therefore K_i can be equivalently expressed to be

$$\begin{aligned} K_i &= \bigcap_{n=-\infty}^{\infty} f^n(M_i - M_{i-1}) \\ &= \bigcap_{n=-\infty}^{\infty} f^n(M_i - intM_i) \\ &= \bigcap_{n=-\infty}^{\infty} f^n(intM_i - M_i). \end{aligned}$$

From the second expression we see K_i is compact. From the third expression we see $M_i - M_{i-1}$ is an isolating neighborhood of K_i.

Now let $x \in M$ be a point whose orbit is not contained in any single layer, say $x \notin M_i$, but $f^N x \in M_i$. Then $f^{N+1}(x) \in fM_i \subset intM_i$. It is then easy to see x can not be chain recurrent. This proves that the union of K_i contains the chain recurrent set $R(f)$ of f. It is also easy to see that $\{K_i\}$ satisfies the no-cycle condition. This proves Theorem 3.13.

The filtration structure is C^0 robust. More precisely,

Theorem 3.14. *Let $\{M_i\}_{i=0}^k$ be a filtration of f. Then*

(1). There is a C^0 neighborhood $\mathcal{U}$ of f such that, for any $g \in \mathcal{U}$, $\{M_i\}_{i=0}^k$ is also a filtration of g.

(2). For any neighborhood U_i of $K_i(f)$, there is a C^0 neighborhood $\mathcal{U}$ of f such that, for any $g \in \mathcal{U}$, $K_i(g) \subset U_i$ for any $i = 1, ..., k$.

The proof is obvious and omitted. Now we proceed to a converse of Theorem 3.13.

Theorem 3.15. *Let $\Lambda_1, ..., \Lambda_k$ be mutually disjoint compact invariant sets of f, the union of which contains the limit set $L(f)$ of f. If Λ_i satisfy the no-cycle condition and the ordering by indices is a filtration ordering, then there is a filtration $\{M_i\}_{i=0}^k$ of f such that $K_i = \Lambda_i$.*

Proof. The proof is a prototype of global arguments in topological dynamics. The idea is to analyze through stable and unstable sets the basins of certain attracting sets, as seen in Step 4 below. Steps 1 through 3 are preparations. For each i, take a neighborhood U_i of Λ_i such that for any $i \neq j$,

$$\overline{U_i} \cap \overline{U_j} = \emptyset, \ (f\overline{U_i}) \cap \overline{U_j} = \emptyset.$$

Step 1. If $f^n z \in \overline{U_i}$ for all $n \geq 0$, then $z \in W^s(\Lambda_i)$.

Proof. The same as in the proof of Theorem 3.12.

Step 2. If $\overline{W^u(\Lambda_i)} \cap W^u(\Lambda_j) \neq \emptyset$, then $\overline{W^u(\Lambda_i)} \cap \Lambda_j \neq \emptyset$.

This means that if $W^u(\Lambda_i)$ approaches $W^u(\Lambda_j)$, it climbs up to Λ_j.

Proof. $\overline{W^u(\Lambda_i)}$ is closed and invariant, so whenever it contains a point $x \in W^u(\Lambda_j)$, it contains $\alpha(x)$, which is contained in Λ_j. This proves Step 2.

Step 3. If $\overline{W^u(\Lambda_i)} \cap \Lambda_j \neq \emptyset$, and if $i \neq j$, then $\overline{W^u(\Lambda_i)} \cap (W^s(\Lambda_j) - \Lambda_j) \neq \emptyset$.

This means that if $W^u(\Lambda_i)$ approaches Λ_j, and if $i \neq j$, then it climbs up to $W^s(\Lambda_j)$.

Proof. Let $x \in \overline{W^u(\Lambda_i)} \cap \Lambda_j$. Then there are $x_k \in W^u(\Lambda_i) \cap U_j, k = 1, 2, ...,$ such that $x_k \to x$. But $i \neq j$, so for each k, there is n_k such that

$$x_k, f^{-1}x_k, ..., f^{-n_k+1}(x_k) \in U_j,$$

but

$$f^{-n_k}(x_k) \notin U_j.$$

Then

$$z_k = f^{-n_k}(x_k) \in f^{-1}U_j - U_j \subset f^{-1}\overline{U_j} - U_j.$$

Since $x \in \Lambda_j$, it follows that

$$n_k \to +\infty.$$

Take a limit point z of $\{z_k\}$ in the compact set $f^{-1}\overline{U_j} - U_j$. Then $z \in \overline{W^u(\Lambda_j)} - \Lambda_j$. Since for all k, the positive orbit of z_k, up to the n_k-th iterate, is contained in $\overline{U_j}$, it follows that $f^n z \in \overline{U_j}$ for all $n \geq 0$. By Step 1,

$$z \in W^s(\Lambda_j).$$

This proves Step 3.

Step 4. For any $i = 1, 2, ..., k$,

(1) $\bigcup_{l \leq i} W^u(\Lambda_l)$ is compact and invariant.

(2) $\bigcup_{l \leq i} W^s(\Lambda_l)$ is an (invariant) open neighborhood of $\bigcup_{l \leq i} W^u(\Lambda_l)$.

(3) For any compact set Q_i with

$$\bigcup_{l\le i} W^u(\Lambda_l) \subset Q_i \subset \bigcup_{l\le i} W^s(\Lambda_l),$$

one has

$$\bigcap_{n\ge 0} f^n(Q_i) = \bigcup_{l\le i} W^u(\Lambda_l).$$

This means roughly that the W^u-union from the bottom is a compact attracting set with basin the corresponding W^s-union from the bottom. Note that here l, but not i, serves as the dummy index.

Proof. (1) The invariance is obvious. We prove the compactness. Take any $m \le i$, we prove

$$\overline{W^u(\Lambda_m)} \subset \bigcup_{l\le i} W^u(\Lambda_l).$$

Suppose not, then there is $j > m$ such that

$$\overline{W^u(\Lambda_m)} \cap W^u(\Lambda_j) \ne \emptyset.$$

We show in this case $W^u(\Lambda_m)$ would climb up higher and higher without end, a contradiction.

We may assume j to be the maximal index that satisfies the last inequality. Of course $j \ne m$. By Step 2 and 3, there is $z \in \overline{W^u(\Lambda_m)} \cap (W^s(\Lambda_j) - \Lambda_j)$. By Theorem 3.12, $z \in W^u(\Lambda_r)$ (hence $\overline{W^u(\Lambda_m)} \cap W^u(\Lambda_r) \ne \emptyset$) for some r. Then $\Lambda_r \to \Lambda_j$. Hence $r > j$ by the filtration ordering. This contradicts that j is the maximal index with this property. This proves (1).

(2) Replacing f by f^{-1} it follows that $\bigcup_{l\ge i+1} W^s(\Lambda_l)$ is closed in M. By Theorem 3.12, $\bigcup_{l\le i} W^s(\Lambda_l)$ is open in M. Now for any $m \le i$,

$$W^u(\Lambda_m) \cap (\bigcup_{l>i} W^s(\Lambda_l)) = \emptyset$$

by the filtration ordering. This proves (2).

(3) The "$\supset$" part is obvious. We prove the "$\subset$" part. Let $x \in \cap_{n\ge 0} f^n(Q_i)$. Then $f^{-n}x \in Q_i$ for any $n \ge 0$. But Q_i is compact and disjoint from $\cup_{l>i}\Lambda_l$ (even $\cup_{l\le i} W^s(\Lambda_i)$ is disjoint from $\cup_{l>i}\Lambda_l$), it follows that $x \notin \cup_{l>i} W^u(\Lambda_l)$. This proves (3), and finishes Step 4.

Step 5. If a compact invariant set P of f has a compact neighborhood Q with $\cap_{n\ge 0} f^n Q = P$, then P has a compact neighborhood V with $V \subset intQ$ such that $fV \subset intV$.

While this result seems somewhat natural, the proof is delicate. We omit the proof (See [Sh]).

Now we complete the proof of Theorem 3.15. For each $i = 1, 2, ..., k$, take a compact set Q_i with

$$\bigcup_{l \le i} W^u(\Lambda_l) \subset intQ_i \subset Q_i \subset \bigcup_{l \le i} W^s(\Lambda_l).$$

By Step 4,

$$\bigcap_{n \ge 0} f^n(Q_i) = \bigcup_{l \le i} W^u(\Lambda_l).$$

By Step 5, the compact invariant set $\cup_{l \le i} W^u(\Lambda_l)$ has a compact neighborhood V_i with $V_i \subset intQ_i$ such that

$$fV_i \subset intV_i.$$

Let

$$M_i = \bigcup_{l \le i} V_l.$$

It is easy to see that $\{M_i\}$ is a filtration of f. We prove $K_i = \Lambda_i$.

Since $\Lambda_i \subset M_i - M_{i-1}$ (It is obvious that $\Lambda_i \subset M_i$. On the other hand, Λ_i is disjoint from $\cup_{l \le i-1} W^s(\Lambda_l)$, which contains M_{i-1}), and since Λ_i is invariant, it follows that $\Lambda_i \subset K_i$. It remains to prove $\Lambda_i \supset K_i$. Take any $x \in K_i$. Then $f^n x \in M_i - M_{i-1}$ for all integer n. By Theorem 3.12, there is j such that $x \in W^s(\Lambda_j)$. It is easy to see $j = i$. Likewise $x \in W^u(\Lambda_i)$. Then $x \in W^s(\Lambda_i) \cap W^u(\Lambda_i)$. By the no-cycle condition, $x \in \Lambda_i$. This proves $\Lambda_i \supset K_i$, and completes the proof of Theorem 3.15.

Now we proceed to the central issue of this short course, the Ω-stability theorem of Smale.

Theorem 3.16. *If f satisfies Axiom A and the basic sets have no cycles, then f is C^r Ω-stable, for any $r \ge 1$.*

Proof. It suffices to prove that f is C^1 Ω-stable. Let $\Omega_1, ..., \Omega_k$ be the basic sets of f. Since there is no cycle between Ω_i, by Theorem 3.15, there is a filtration

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

of f with

$$K_i(f) = \Omega_i.$$

By Theorem 3.13, Ω_i is isolated with isolating neighborhood $M_i - M_{i-1}$.

By Theorem 3.14, there is a C^0 neighborhood $\mathcal{U}^0$ of f such that for any $g \in \mathcal{U}^0$, $\{M_i\}_{i=0}^k$ is also a filtration of g. Denote

$$K_i(g) = \bigcap_{n=-\infty}^{\infty} g^n(M_i - M_{i-1}).$$

Let $\varepsilon > 0$ be given. By Theorem 2.14, there is $\delta > 0$ such that for any $g \in \mathcal{B}^1(f, \delta)$ and any $i = 1, 2, ..., k$, there is a homeomorphism

$$h_i = h_i(g) : K_i(f) \to K_i(g)$$

such that

$$d(h_i, id) < \varepsilon \ \text{ and } \ h_i f = g h_i.$$

These h_i together give an ε-conjugacy

$$h : \Omega(f) \to \bigcup_{i=1}^{k} K_i(g).$$

It remains to prove

$$\bigcup_{i=1}^{k} K_i(g) = \Omega(g).$$

The "$\supset$" part is given by Theorem 3.13. The "$\subset$" part is given by

$$\begin{aligned} \bigcup_{i=1}^{k} K_i(g) &= h(\Omega(f)) = h(\overline{P(f)}) \\ &= \overline{h(P(f))} \subset \overline{P(g)} \subset \Omega(g). \end{aligned}$$

This proves Theorem 3.16.

Note that to prove the "$\subset$" part we could not have argued as that h is a conjugacy from $\Omega(f)$ to $\cup_{i=1}^{k} K_i(g)$ and that a conjugacy "preserves the non-wandering set", hence $\cup_{i=1}^{k} K_i(g) \subset \Omega(g)$. This is because h is defined merely on $\Omega(f)$, hence the non-wandering set it preserves is $\Omega(f|_{\Omega(f)})$, which is generally different from $\Omega(f)$ as the concept of non-wandering point depends on neighborhoods. On the other hand, the concept of periodic point does not, hence the above last second "$\subset$ " goes through.

3.5 More about spectral decompositions and cycles

In this section we present some further development on the topic of spectral decompositions and cycles. First we remark that the spectral decomposition is really for $\overline{P}$.

Theorem 3.17. *If $\overline{P}$ is hyperbolic, then $\overline{P}$ decomposes uniquely into a finitely many disjoint transitive sets $\overline{P}_1 \cup \overline{P}_2 \cup \cdots \cup \overline{P}_k$. Also, $\overline{P}$ is isolated.*

Proof. The proof for decomposition is the same as in Theorem 3.9. The proof for isolation is a refinement of the proof of Theorem 3.4 as we now indicate. By Theorem 3.6 we only check the local product structure. Let p and q be two periodic points that are close enough so that their stable and unstable manifolds intersect transversely at x and y. We need to show x and y are in $\overline{P}$. This is a refinement of the cloud lemma, and is guaranteed by Birkhoff-Smale theorem on homoclinic points, which asserts that any transverse homoclinic point is in a horseshoe, see Guckenheimor(1980). This proves Theorem 3.17.

We know by Dankner's example that Ω-hyperbolicity does not imply $\Omega = \overline{P}$. The following results of Newhouse (Theorem 3.18 and 3.19) and Franke-Selgrade (Theorem 3.20) give some important contrast.

Theorem 3.18. *If $L(f)$ is hyperbolic, then $L(f) = \overline{P(f)}$.*

Proof. This can be proved by shadowing. Let $x \in L$. There is $y \in M$ and some integer N such that y and $f^N(y)$ are both close to x and the whole finite orbit $y, fy, \cdots, f^N(y)$ remains near L. Repeating this finite orbit gives a periodic pseudo orbit near L. Then shadowing works to give a periodic point z near x. This proves Theorem 3.18.

Note that the proof is like the proof that for Anosov diffeomorphisms $\Omega = \overline{P}$. We have noticed that this method does not prove that Ω-hyperbolicity implies $\Omega = \overline{P}$. Now it does prove that L-hyperbolicity implies $L = \overline{P}$. Combined with Theorem 3.17, this shows if $L(f)$ is hyperbolic, then f has an L-decomposition $L_1 \cup L_2 \cup \cdots \cup L_k$, which is just the $\overline{P}$-decomposition.

Theorem 3.19. *If L is hyperbolic, and if there are no cycles in the L-decomposition, then $\overline{P(f)} = L(f) = \Omega(f) = R(f)$. In particular, f is Axiom A and no-cycle.*

Proof. Let $L_1, ..., L_k$ be the basic sets in the L-decomposition of f. By Theorem 3.15, there is a filtration $\{M_i\}$ of f such that $K_i = L_i$. By Theorem 3.13, $L(f)$ contains $R(f)$, hence $L(f) = \Omega(f) = R(f)$. Combining with Theorem 3.18 this proves Theorem 3.19.

Theorem 3.20. *If $R(f)$ is hyperbolic, then $\overline{P(f)} = L(f) = \Omega(f) = R(f)$. Actually $R(f)$ is hyperbolic if and only if f is Axiom A and no-cycle.*

Proof. Let $R(f)$ be hyperbolic. Then $L = \overline{P}$ is hyperbolic and hence decomposes into

$$L_1 \cup L_2 \cup \cdots \cup L_k,$$

each transitive. By Theorem 3.19, it suffices to prove there are no cycles between the L_i. Suppose there is a cycle, say

$$L_1, L_2, \ldots, L_m; \quad x_1, x_2, \cdots, x_m,$$

It is easy to see every x_i is chain recurrent (since L_i is transitive). Now $R(f)$ is assumed to be hyperbolic, so the stable and unstable manifolds of x_i intersect transversely at x_i. Then it is not hard to see (In Phase Theorem, λ-lemma, Birkhoff-Smale theorem) each x_i is in $\overline{P}$, a contradiction.

Conversely, if f is Axiom A and no cycle, then $\overline{P} = L = \Omega$. Then Theorem 3.19 applies. This proves Theorem 3.20.

A delicate problem is: Can any Axiom A system be C^r approximated by an Axiom A and no-cycle system? Newhouse and Palis (1973) prove that this is true for any $r \geq 1$ if M is 2-dimensional. Patterson (1988) constructed a C^2 counterexample in 3-dimension. The case $r = 1$ for dimension 3 or higher is unknown.

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