AN INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Lecture Notes Series

Contents

PREFACE

The notes presented here are based on lectures delivered over the years by the author at the Université Pierre et Marie Curie, Paris, at the University of Stuttgart, and at City University of Hong Kong. Their aim is to give a thorough introduction to the basic theorems of Differential Geometry.

In the first chapter, we review the basic notions arising when a three-dimensional open set is equipped with curvilinear coordinates, such as the metric tensor, Christoffel symbols, and covariant derivatives. We then prove that the vanishing of the Riemann curvature tensor is sufficient for the existence of isometric immersions from a simply-connected open subset of $\mathbb{R}^n$ equipped with a Riemannian metric into a Euclidean space of the same dimension. We also prove the corresponding uniqueness theorem, also called rigidity theorem.

In the second chapter, we study basic notions about surfaces, such as their two fundamental forms, the Gaussian curvature, Christoffel symbols, and covariant derivatives. We then prove the fundamental theorem of surface theory, which asserts that the Gauß and Codazzi-Mainardi equations constitute sufficient conditions for two matrix fields defined in a simply-connected open subset of $\mathbb{R}^3$ to be the two fundamental forms of a surface in a three-dimensional Euclidean space. We also prove the corresponding rigidity theorem.

In addition to such "classical" theorems, we also include in both chapters very recent results, which have not yet appeared in book form, such as the continuity of a surface as a function of its fundamental forms.

The treatment is essentially self-contained and proofs are complete. The prerequisites essentially consist in a working knowledge of basic notions of analysis and functional analysis, such as differential calculus, integration theory and Sobolev spaces, and some familiarity with ordinary and partial differential equations.

These notes use some excerpts from Chapters 1 and 2 of my book "Mathematical Elasticity, Volume III: Theory of Shells", published in 2000 by North-Holland, Amsterdam; in this respect, I am indebted to Arjen Sevenster for his kind permission to reproduce these excerpts. Otherwise, the major part of these notes was written during the fall of 2004 at City University of Hong Kong; this part of the work was substantially supported by a grant from the Research Grants Council of Hong Kong Special Administrative Region, China [Project No. 9040869, CityU 100803].

Hong Kong, January 2005

Chapter 1

THREE-DIMENSIONAL DIFFERENTIAL GEOMETRY

1.1 CURVILINEAR COORDINATES

To begin with, we list some notations and conventions that will be consistently used throughout.

All spaces, matrices, etc., considered here are *real*.

Latin indices and exponents vary in the set $\{1, 2, 3\}$, except when they are used for indexing sequences, and the summation convention with respect to repeated indices or exponents is systematically used in conjunction with this rule. For instance, the relation

$$\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x)$$

means that

$$\boldsymbol{g}_i(x) = \sum_{j=1}^{3} g_{ij}(x)\boldsymbol{g}^j(x) \text{ for } i = 1, 2, 3.$$

Kronecker's symbols are designated by δ_i^j, δ_{ij}, or δ^{ij} according to the context.

Let $\mathbf{E}^3$ denote a *three-dimensional Euclidean space*, let $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$ denote the Euclidean inner product and exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{E}^3$, and let $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a} \in \mathbf{E}^3$. The space $\mathbf{E}^3$ is endowed with an *orthonormal basis* consisting of three vectors $\widehat{\boldsymbol{e}}^i = \widehat{\boldsymbol{e}}_i$. Let $\widehat{x}_i$ denote the *Cartesian coordinates* of a point $\widehat{x} \in \mathbf{E}^3$ and let $\widehat{\partial}_i := \partial/\partial\widehat{x}_i$.

In addition, let there be given a *three-dimensional vector space* in which three vectors $\boldsymbol{e}^i = \boldsymbol{e}_i$ form a basis. *This space will be identified with* $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

Let there be given an *open* subset $\widehat{\Omega}$ of $\mathbf{E}^3$ and assume that there exist an *open* subset Ω of $\mathbb{R}^3$ and an *injective* mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ such that $\boldsymbol{\Theta}(\Omega) = \widehat{\Omega}$. Then each point $\widehat{x} \in \widehat{\Omega}$ can be unambiguously written as

$$\widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

and the three coordinates x_i of x are called the **curvilinear coordinates** of $\widehat{x}$ (Figure 1.1-1). Naturally, *there are infinitely many ways of defining curvilinear coordinates in a given open set* $\widehat{\Omega}$, depending on how the open set Ω and the mapping $\boldsymbol{\Theta}$ are chosen!

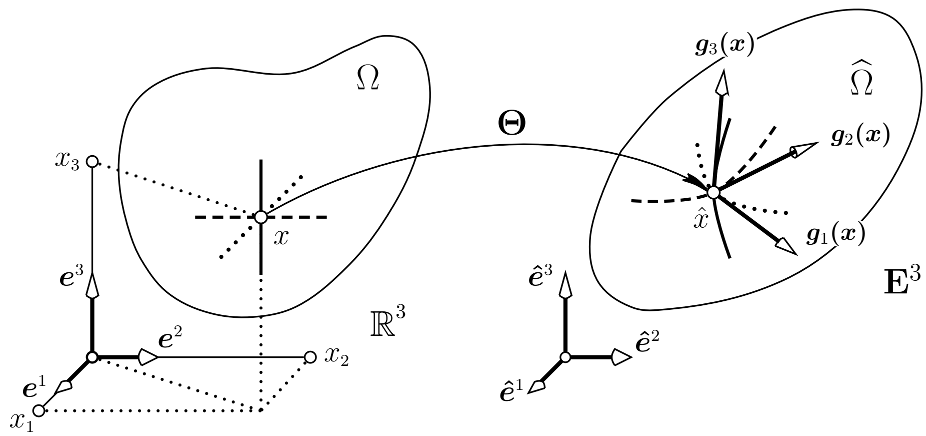

Figure 1.1-1: *Curvilinear coordinates and covariant bases in an open set* $\widehat{\Omega} \subset \mathbf{E}^3$. The three coordinates x_1, x_2, x_3 of $x \in \Omega$ are the curvilinear coordinates of $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$. If the three vectors $\boldsymbol{g}_i(x) = \partial_i \boldsymbol{\Theta}(x)$ are linearly independent, they form the covariant basis at $\widehat{x} = \boldsymbol{\Theta}(x)$ and they are tangent to the coordinate lines passing through $\widehat{x}$.

Examples of curvilinear coordinates include the well-known *cylindrical* and *spherical coordinates* (Figure 1.1-2).

In a different, but equally important, approach, an open subset Ω of $\mathbb{R}^3$ together with a mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ are instead *a priori* given.

If $\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\Theta}$ is injective, the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is *open* by the *invariance of domain theorem* (for a proof, see, e.g., Nirenberg [1974, Corollary 2, p. 17] or Zeidler [1986, Section 16.4]), and curvilinear coordinates inside $\widehat{\Omega}$ are unambiguously defined in this case.

If $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and the three vectors $\partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all $x \in \Omega$, the set $\widehat{\Omega}$ is again *open* (for a proof, see, e.g., Schwartz [1992] or Zeidler [1986, Section 16.4]), but curvilinear coordinates may be defined only locally in this case: Given $x \in \Omega$, all that can be asserted (by the local inversion theorem) is the existence of an open neighborhood V of x in Ω such that the restriction of $\boldsymbol{\Theta}$ to V is a $\mathcal{C}^1$-diffeomorphism, hence an injection, of V onto $\boldsymbol{\Theta}(V)$.

1.2 METRIC TENSOR

Let Ω be an open subset of $\mathbb{R}^3$ and let

$$\boldsymbol{\Theta} = \Theta_i \widehat{\boldsymbol{e}}^i : \Omega \to \mathbf{E}^3$$

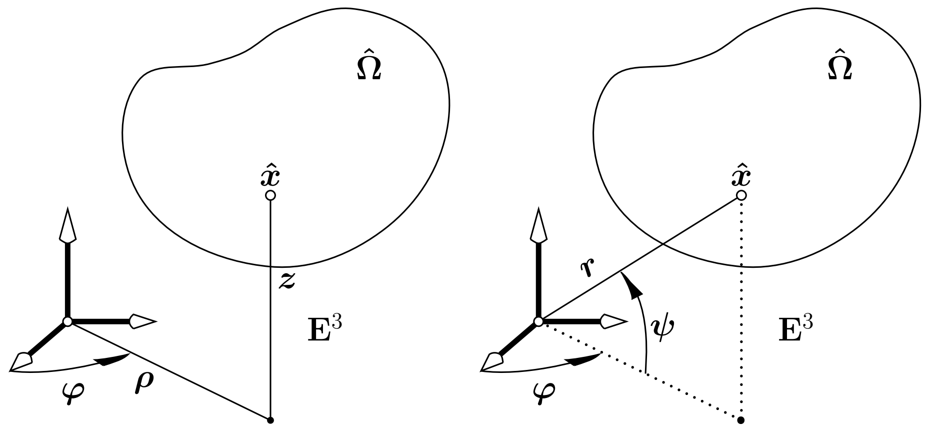

Figure 1.1-2: *Two familiar examples of curvilinear coordinates.* Let the mapping $\mathbf{\Theta}$ be defined by

$$\mathbf{\Theta} : (\varphi, \rho, z) \in \Omega \to (\rho \cos \varphi, \rho \sin \varphi, z) \in \mathbf{E}^3.$$

Then (φ, ρ, z) are the cylindrical coordinates of $\widehat{x} = \mathbf{\Theta}(\varphi, \rho, z)$. Note that $(\varphi + 2k\pi, \rho, z)$ or $(\varphi + \pi + 2k\pi, -\rho, z)$, $k \in \mathbb{Z}$, are also cylindrical coordinates of the same point $\widehat{x}$ and that φ is not defined if $\widehat{x}$ is the origin of $\mathbf{E}^3$.

Let the mapping $\mathbf{\Theta}$ be defined by

$$\mathbf{\Theta} : (\varphi, \psi, r) \in \Omega \to (r \cos \psi \cos \varphi, r \cos \psi \sin \varphi, r \sin \psi) \in \mathbf{E}^3.$$

Then (φ, ψ, r) are the spherical coordinates of $\widehat{x} = \mathbf{\Theta}(\varphi, \psi, r)$. Note that $(\varphi + 2k\pi, \psi + 2\ell\pi, r)$ or $(\varphi + 2k\pi, \psi + \pi + 2\ell\pi, -r)$ are also spherical coordinates of the same point $\widehat{x}$ and that φ and ψ are not defined if $\widehat{x}$ is the origin of $\mathbf{E}^3$.

be a mapping that is *differentiable at a point* $x \in \Omega$. If $\boldsymbol{\delta x}$ is such that $(x + \boldsymbol{\delta x}) \in \Omega$, then

$$\mathbf{\Theta}(x + \boldsymbol{\delta} x) = \mathbf{\Theta}(x) + \boldsymbol{\nabla}\mathbf{\Theta}(x)\boldsymbol{\delta x} + o(\boldsymbol{\delta x}),$$

where the 3×3 matrix $\boldsymbol{\nabla}\mathbf{\Theta}(x)$ is defined by

$$\boldsymbol{\nabla}\mathbf{\Theta}(x) := \begin{pmatrix} \partial_1\Theta_1 & \partial_2\Theta_1 & \partial_3\Theta_1 \\ \partial_1\Theta_2 & \partial_2\Theta_2 & \partial_3\Theta_2 \\ \partial_1\Theta_3 & \partial_2\Theta_3 & \partial_3\Theta_3 \end{pmatrix}(x).$$

Let the three vectors $\boldsymbol{g}_i(x) \in \mathbb{R}^3$ be defined by

$$\boldsymbol{g}_i(x) := \partial_i\mathbf{\Theta}(x) = \begin{pmatrix} \partial_i\Theta_1 \\ \partial_i\Theta_2 \\ \partial_i\Theta_3 \end{pmatrix}(x),$$

i.e., $\boldsymbol{g}_i(x)$ *is the* i-th *column vector of the matrix* $\boldsymbol{\nabla}\mathbf{\Theta}(x)$ and let $\boldsymbol{\delta x} = \delta x^i \boldsymbol{e}_i$. Then the expansion of $\mathbf{\Theta}$ about x may be also written as

$$\mathbf{\Theta}(x + \boldsymbol{\delta x}) = \mathbf{\Theta}(x) + \delta x^i \boldsymbol{g}_i(x) + o(\boldsymbol{\delta x}).$$

If in particular $\boldsymbol{\delta x}$ is of the form $\boldsymbol{\delta x} = \delta t \boldsymbol{e}_i$, where $\delta t \in \mathbb{R}$ and $\boldsymbol{e}_i$ is one of the basis vectors in $\mathbb{R}^3$, this relation reduces to

$$\mathbf{\Theta}(x + \delta t \boldsymbol{e}_i) = \mathbf{\Theta}(x) + \delta t \boldsymbol{g}_i(x) + o(\delta t).$$

A mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an **immersion at** $x \in \Omega$ if it is differentiable at x and the matrix $\boldsymbol{\nabla\Theta}(x)$ is invertible or, equivalently, if the three vectors $\boldsymbol{g}_i(x) = \partial_i \boldsymbol{\Theta}(x)$ are linearly independent.

Assume from now on in this section that *the mapping* $\boldsymbol{\Theta}$ *is an immersion at* x. Then *the three vectors* $\boldsymbol{g}_i(x)$ constitute the **covariant basis** at the point $\widehat{x} = \boldsymbol{\Theta}(x)$.

In this case, the last relation thus shows that *each vector* $\boldsymbol{g}_i(x)$ *is tangent to the* i-th **coordinate line** *passing through* $\widehat{x} = \boldsymbol{\Theta}(x)$, defined as the image by $\boldsymbol{\Theta}$ of the points of $\widehat{\Omega}$ that lie on the line parallel to $\boldsymbol{e}_i$ passing through x (there exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the i-th coordinate line is given by $t \in \,]t_0, t_1[\, \to \boldsymbol{f}_i(t) := \boldsymbol{\Theta}(x + t\boldsymbol{e}_i)$ in a neighborhood of $\widehat{x}$; hence $\boldsymbol{f}'_i(0) = \partial_i \boldsymbol{\Theta}(x) = \boldsymbol{g}_i(x)$); see Figures 1.1-1 and 1.1-2.

Returning to a general increment $\boldsymbol{\delta x} = \delta x^i \boldsymbol{e}_i$, we also infer from the expansion of $\boldsymbol{\Theta}$ about x that (recall that we use the summation convention):

$$\begin{aligned}|\boldsymbol{\Theta}(x + \boldsymbol{\delta x}) - \boldsymbol{\Theta}(x)|^2 &= \boldsymbol{\delta x}^T \boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x) \boldsymbol{\delta x} + o\big(|\boldsymbol{\delta x}|^2\big)\\ &= \delta x^i \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \delta x^j + o\big(|\boldsymbol{\delta x}|^2\big).\end{aligned}$$

In other words, the principal part with respect to $\boldsymbol{\delta x}$ of the length between the points $\boldsymbol{\Theta}(x + \boldsymbol{\delta x})$ and $\boldsymbol{\Theta}(x)$ is $\{\delta x^i \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \delta x^j\}^{1/2}$. This observation suggests to define a matrix $(g_{ij}(x))$ of order three, by letting

$$g_{ij}(x) := \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) = (\boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x))_{ij}.$$

The elements $g_{ij}(x)$ of this symmetric matrix are called the **covariant components** of the **metric tensor** at $\widehat{x} = \boldsymbol{\Theta}(x)$.

Note that *the matrix* $\boldsymbol{\nabla\Theta}(x)$ *is invertible* and that *the matrix* $(g_{ij}(x))$ *is positive definite*, since the vectors $\boldsymbol{g}_i(x)$ are assumed to be linearly independent.

The three vectors $\boldsymbol{g}_i(x)$ being linearly independent, *the nine relations*

$$\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$$

unambiguously define three linearly independent vectors $\boldsymbol{g}^i(x)$. To see this, let *a priori* $\boldsymbol{g}^i(x) = X^{ik}(x) \boldsymbol{g}_k(x)$ in the relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$. This gives $X^{ik}(x) g_{kj}(x) = \delta^i_j$; consequently, $X^{ik}(x) = g^{ik}(x)$, where

$$(g^{ij}(x)) := (g_{ij}(x))^{-1}.$$

Hence $\boldsymbol{g}^i(x) = g^{ik}(x) \boldsymbol{g}_k(x)$. These relations in turn imply that

$$\begin{aligned}\boldsymbol{g}^i(x) \cdot \boldsymbol{g}^j(x) &= \big(g^{ik}(x) \boldsymbol{g}_k(x)\big) \cdot \big(g^{j\ell}(x) \boldsymbol{g}_\ell(x)\big)\\ &= g^{ik}(x) g^{j\ell}(x) g_{k\ell}(x) \;=\; g^{ik}(x) \delta^j_k \;=\; g^{ij}(x),\end{aligned}$$

and thus the vectors $\boldsymbol{g}^i(x)$ are *linearly independent* since the matrix $(g^{ij}(x))$ is positive definite. We would likewise establish that $\boldsymbol{g}_i(x) = g_{ij}(x) \boldsymbol{g}^j(x)$.

The three vectors $\boldsymbol{g}^i(x)$ form the **contravariant basis** at the point $\widehat{x} = \boldsymbol{\Theta}(x)$ and the elements $g^{ij}(x)$ of the symmetric positive definite matrix $(g^{ij}(x))$ are the **contravariant components** of the **metric tensor** at $\widehat{x} = \boldsymbol{\Theta}(x)$.

To conclude this section, we record for convenience the fundamental relations that exist between the vectors of the covariant and contravariant bases and the covariant and contravariant components of the metric tensor:

$$g_{ij}(x) = \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \quad \text{and} \quad g^{ij}(x) = \boldsymbol{g}^i(x) \cdot \boldsymbol{g}^j(x),$$
$$\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x) \quad \text{and} \quad \boldsymbol{g}^i(x) = g^{ij}(x)\boldsymbol{g}_j(x).$$

1.3 VOLUMES, AREAS, AND LENGTHS IN CURVILINEAR COORDINATES

We now review fundamental formulas showing how *volume, area*, and *length elements* at a point $\widehat{x} = \boldsymbol{\Theta}(x)$ in the set $\widehat{\Omega} = \boldsymbol{\Theta}(\Omega)$ can be expressed either in terms of the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ or in terms of the matrix $(g_{ij}(x))$ or of its inverse matrix $(g^{ij}(x))$.

These formulas thus highlight the crucial rôle played by the matrix $(g_{ij}(x))$ for computing "metric" notions at the point $\widehat{x} = \boldsymbol{\Theta}(x)$. Indeed, the "metric tensor" well deserves its name!

A **domain** in $\mathbb{R}^n$ is a bounded, open, and connected subset D of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, the set D being locally on one side of its boundary. All relevant details needed here about domains are found in Nečas [1967] or Adams [1975].

Given a domain $D \subset \mathbb{R}^3$ with boundary Γ, we let $\mathrm{d}x$ denote the *volume element* in D, $\mathrm{d}\Gamma$ denote the *area element* along Γ, and $\boldsymbol{n} = n_i\widehat{\boldsymbol{e}}^i$ denote the *unit* ($|\boldsymbol{n}| = 1$) *outer normal vector* along Γ ($\mathrm{d}\Gamma$ is well defined and $\boldsymbol{n}$ is defined $\mathrm{d}\Gamma$-almost everywhere since Γ is assumed to be Lipschitz-continuous).

Note also that the assumptions made on the mapping $\boldsymbol{\Theta}$ in the next theorem guarantee that, if D is a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$, then $\{\widehat{D}\}^- \subset \widehat{\Omega}$, $\{\boldsymbol{\Theta}(D)\}^- = \boldsymbol{\Theta}(\overline{D})$, and the boundaries $\partial\widehat{D}$ of $\widehat{D}$ and ∂D of D are related by $\partial\widehat{D} = \boldsymbol{\Theta}(\partial D)$ (see, e.g., Ciarlet [1988, Theorem 1.2-8 and Example 1.7]).

If $\mathbf{A}$ is a square matrix, $\mathbf{Cof}\,\mathbf{A}$ denotes the *cofactor matrix* of $\mathbf{A}$. Thus $\mathbf{Cof}\,\mathbf{A} = (\det \mathbf{A})\mathbf{A}^{-T}$ if $\mathbf{A}$ is invertible.

A mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an **immersion** if it is an immersion at each $x \in \Omega$, i.e., if $\boldsymbol{\Theta}$ is differentiable in Ω and the three vectors $\boldsymbol{g}_i(x) = \partial_i\boldsymbol{\Theta}(x)$ are linearly independent at each $x \in \Omega$.

Theorem 1.3-1. *Let Ω be an open subset of $\mathbb{R}^3$, let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be an injective and smooth enough immersion, and let $\widehat{\Omega} = \boldsymbol{\Theta}(\Omega)$.*

(a) *The volume element $\mathrm{d}\widehat{x}$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ is given in terms of the volume element $\mathrm{d}x$ at $x \in \Omega$ by*

$$\mathrm{d}\widehat{x} = |\det \boldsymbol{\nabla}\boldsymbol{\Theta}(x)|\,\mathrm{d}x = \sqrt{g(x)}\,\mathrm{d}x, \text{ where } g(x) := \det(g_{ij}(x)).$$

(b) *Let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$. The area element $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \partial\widehat{D}$ is given in terms of the area element $\mathrm{d}\Gamma(x)$ at $x \in \partial D$ by*

$$\mathrm{d}\widehat{\Gamma}(\widehat{x}) = |\,\mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\boldsymbol{n}(x)|\,\mathrm{d}\Gamma(x) = \sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x),$$

where $\boldsymbol{n}(x) := n_i(x)\boldsymbol{e}^i$ *denotes the unit outer normal vector at* $x \in \partial D$.

(c) *The length element* $\mathrm{d}\widehat{\ell}(\widehat{x})$ *at* $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ *is given by*

$$\mathrm{d}\widehat{\ell}(\widehat{x}) = \left\{\boldsymbol{\delta x}^T \boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x)\boldsymbol{\delta x}\right\}^{1/2} = \left\{\delta x^i g_{ij}(x)\delta x^j\right\}^{1/2},$$

where $\boldsymbol{\delta x} = \delta x^i \boldsymbol{e}_i$.

Proof. The relation $\mathrm{d}\widehat{x} = |\det \boldsymbol{\nabla\Theta}(x)|\,\mathrm{d}x$ between the volume elements is well known. The second relation in (a) follows from the relation $g(x) = |\det \boldsymbol{\nabla\Theta}(x)|^2$, which itself follows from the relation $(g_{ij}(x)) = \boldsymbol{\nabla\Theta}(x)^T\boldsymbol{\nabla\Theta}(x)$.

Indications about the proof of the relation between the area elements $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ and $\mathrm{d}\Gamma(x)$ given in (b) are found in Ciarlet [1988, Theorem 1.7-1] (in this formula, $\boldsymbol{n}(x) = n_i(x)\boldsymbol{e}^i$ is identified with the column vector in $\mathbb{R}^3$ with $n_i(x)$ as its components). Using the relations $\mathbf{Cof}(\mathbf{A}^T) = (\mathbf{Cof\,A})^T$ and $\mathbf{Cof}(\mathbf{AB}) = (\mathbf{Cof\,A})(\mathbf{Cof\,B})$, we next have:

$$\begin{aligned}|\,\mathbf{Cof}\,\boldsymbol{\nabla\Theta}(x)\boldsymbol{n}(x)|^2 &= \boldsymbol{n}(x)^T\,\mathbf{Cof}\left(\boldsymbol{\nabla\Theta}(x)^T\boldsymbol{\nabla\Theta}(x)\right)\boldsymbol{n}(x)\\ &= g(x)n_i(x)g^{ij}(x)n_j(x).\end{aligned}$$

Either expression of the length element given in (c) recalls that $\mathrm{d}\widehat{\ell}(\widehat{x})$ is by definition the principal part with respect to $\boldsymbol{\delta x} = \delta x^i\boldsymbol{e}_i$ of the length $|\boldsymbol{\Theta}(x+\boldsymbol{\delta x}) - \boldsymbol{\Theta}(x)|$, whose expression precisely led to the introduction of the matrix $(g_{ij}(x))$ in Section 1.2. $\square$

The relations found in Theorem 1.3-1 are used in particular for computing *volumes, areas,* and *lengths* inside $\widehat{\Omega}$ by means of integrals inside Ω, i.e., in terms of the *curvilinear coordinates* used in the open set $\widehat{\Omega}$ (Figure 1.3-1):

Let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$, let $\widehat{D} := \boldsymbol{\Theta}(D)$, and let $\widehat{f} \in L^1(\widehat{D})$ be given. Then

$$\int_{\widehat{D}} \widehat{f}(\widehat{x})\,\mathrm{d}\widehat{x} = \int_D (\widehat{f}\circ\boldsymbol{\Theta})(x)\sqrt{g(x)}\,\mathrm{d}x.$$

In particular, the *volume* of $\widehat{D}$ is given by

$$vol\,\widehat{D} := \int_{\widehat{D}} \mathrm{d}\widehat{x} = \int_D \sqrt{g(x)}\,\mathrm{d}x.$$

Next, let $\Gamma := \partial D$, let Σ be a $\mathrm{d}\Gamma$-measurable subset of Γ, let $\widehat{\Sigma} := \boldsymbol{\Theta}(\Sigma) \subset \partial\widehat{D}$, and let $\widehat{h} \in L^1(\widehat{\Sigma})$ be given. Then

$$\int_{\widehat{\Sigma}} \widehat{h}(\widehat{x})\,\mathrm{d}\widehat{\Gamma}(\widehat{x}) = \int_\Sigma (\widehat{h}\circ\boldsymbol{\Theta})(x)\sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x).$$

In particular, the *area* of $\widehat{\Sigma}$ is given by

$$area\,\widehat{\Sigma} := \int_{\widehat{\Sigma}} \mathrm{d}\widehat{\Gamma}(\widehat{x}) = \int_\Sigma \sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x).$$

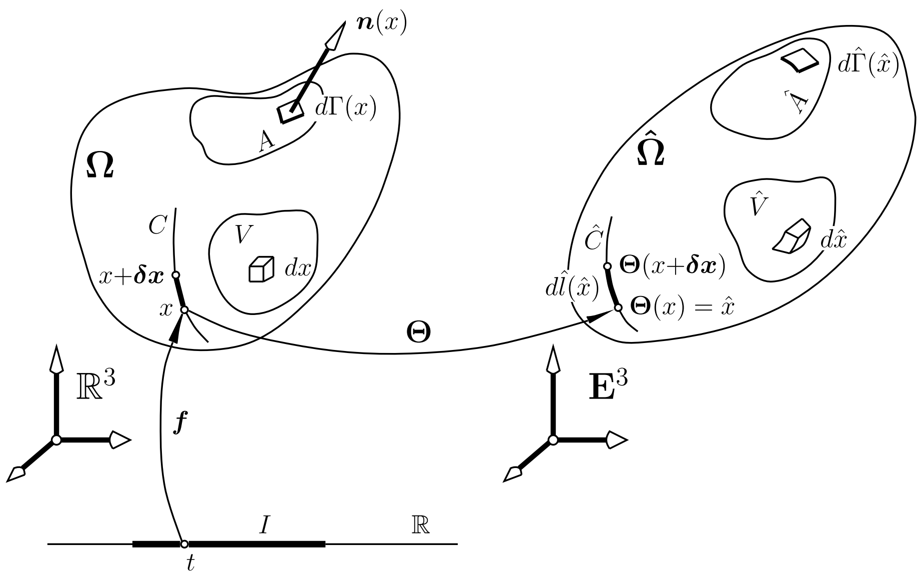

Figure 1.3-1: *Volume, area, and length elements in curvilinear coordinates.* The elements $d\widehat{x}, d\widehat{\Gamma}(\widehat{x})$, and $d\widehat{\ell}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ are expressed in terms of dx, $d\Gamma(x)$, and $\boldsymbol{\delta x}$ at $x \in \Omega$ by means of the covariant and contravariant components of the metric tensor; cf. Theorem 1.3-1. Given a domain D such that $\overline{D} \subset \Omega$ and a $d\Gamma$-measurable subset Σ of ∂D, the corresponding relations are used for computing the volume of $\widehat{D} = \boldsymbol{\Theta}(D) \subset \widehat{\Omega}$, the area of $\widehat{\Sigma} = \boldsymbol{\Theta}(\Sigma) \subset \partial\widehat{D}$, and the length of a curve $\widehat{C} = \boldsymbol{\Theta}(C) \subset \widehat{\Omega}$, where $C = \boldsymbol{f}(I)$ and I is a compact interval of $\mathbb{R}$.

Finally, consider a curve $C = \boldsymbol{f}(I)$ in Ω, where I is a compact interval of $\mathbb{R}$ and $\boldsymbol{f} = f^i \boldsymbol{e}_i : I \to \Omega$ is a smooth enough injective mapping. Then the *length* of the curve $\widehat{C} := \boldsymbol{\Theta}(C) \subset \widehat{\Omega}$ is given by

$$\textit{length}\,\widehat{C} := \int_I \Big|\frac{d}{dt}(\boldsymbol{\Theta} \circ \boldsymbol{f})(t)\Big|\,dt = \int_I \sqrt{g_{ij}(\boldsymbol{f}(t))\frac{df^i}{dt}(t)\frac{df^j}{dt}(t)}\,dt.$$

This relation shows in particular that *the lengths of curves inside the set* $\boldsymbol{\Theta}(\Omega)$ *are precisely those induced by the Euclidean metric of the space* $\mathbf{E}^3$.

1.4 COVARIANT DERIVATIVES OF A VECTOR FIELD AND CHRISTOFFEL SYMBOLS

Suppose that a *vector field* is defined in an open subset $\widehat{\Omega}$ of $\mathbf{E}^3$ by means of its *Cartesian components* $\widehat{v}_i : \widehat{\Omega} \to \mathbb{R}$, i.e., this field is defined by its values $\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ at each $\widehat{x} \in \widehat{\Omega}$, where the vectors $\widehat{\boldsymbol{e}}^i$ constitute the orthonormal basis of $\mathbf{E}^3$; see Figure 1.4-1.

Suppose now that the open set $\widehat{\Omega}$ is equipped with *curvilinear coordinates* from an open subset Ω of $\mathbb{R}^3$, by means of an injective mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ satisfying $\boldsymbol{\Theta}(\Omega) = \widehat{\Omega}$.

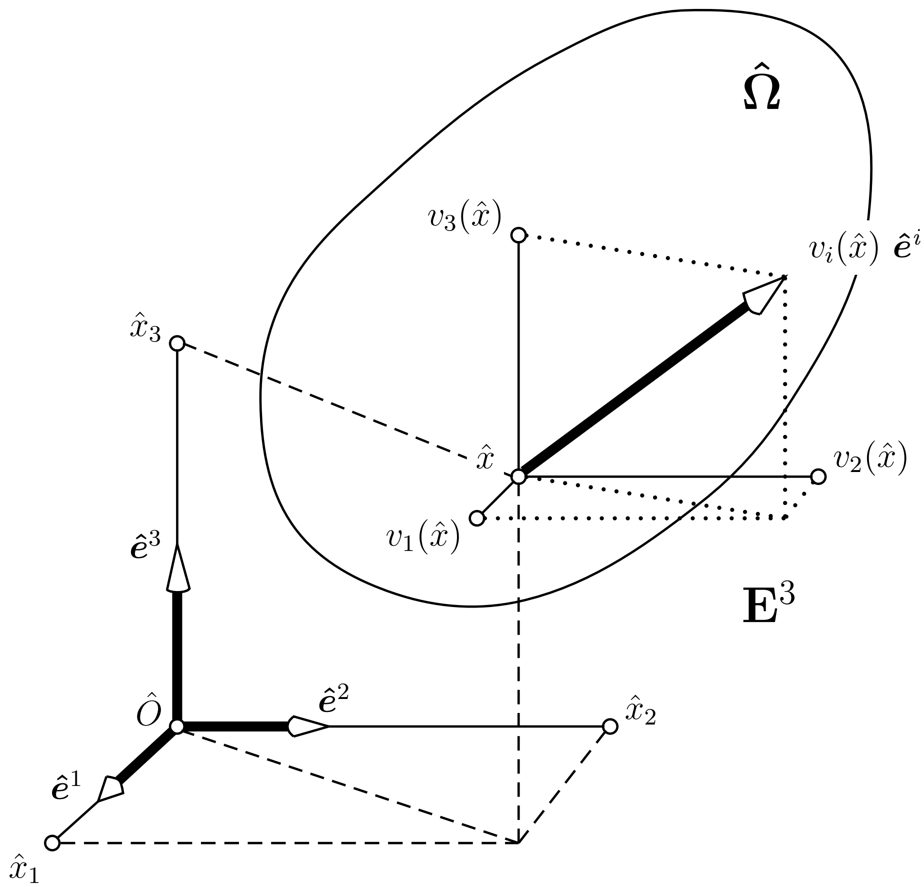

Figure 1.4-1: *A vector field in Cartesian coordinates.* At each point $\widehat{x} \in \widehat{\Omega}$, the vector $\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ is defined by its Cartesian components $\widehat{v}_i(\widehat{x})$ over an orthonormal basis of $\mathbf{E}^3$ formed by three vectors $\widehat{\boldsymbol{e}}^i$.

How to define appropriate components of the same vector field, but this time in terms of these curvilinear coordinates? It turns out that the proper way to do so consists in defining three functions $v_i : \overline{\Omega} \to \mathbb{R}$ by requiring that (Figure 1.4-2)

$$v_i(x)\boldsymbol{g}^i(x) := \widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

where the three vectors $\boldsymbol{g}^i(x)$ form the *contravariant basis* at $\widehat{x} = \boldsymbol{\Theta}(x)$ (Section 1.2). Using the relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$ and $\widehat{\boldsymbol{e}}^i \cdot \widehat{\boldsymbol{e}}_j = \delta^i_j$, we immediately find how the old and new components are related, viz.,

$$v_j(x) = v_i(x)\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \cdot \boldsymbol{g}_j(x),$$
$$\widehat{v}_i(\widehat{x}) = \widehat{v}_j(\widehat{x})\widehat{\boldsymbol{e}}^j \cdot \widehat{\boldsymbol{e}}_i = v_j(x)\boldsymbol{g}^j(x) \cdot \widehat{\boldsymbol{e}}_i.$$

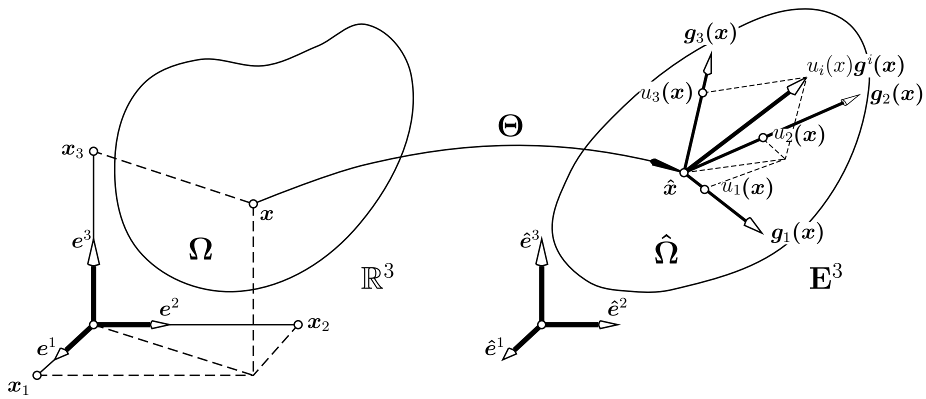

Figure 1.4-2: *A vector field in curvilinear coordinates.* Let there be given a vector field in Cartesian coordinates defined at each $\widehat{x} \in \widehat{\Omega}$ by its Cartesian components $\widehat{v}_i(\widehat{x})$ over the vectors $\widehat{\boldsymbol{e}}^i$ (Figure 1.4-1). In curvilinear coordinates, the same vector field is defined at each $x \in \Omega$ by its covariant components $v_i(x)$ over the contravariant basis vectors $\boldsymbol{g}^i(x)$ in such a way that $v_i(x)\boldsymbol{g}^i(x) = \widehat{v}_i(\widehat{x})\boldsymbol{e}^i$, $\widehat{x} = \boldsymbol{\Theta}(x)$.

The three components $v_i(x)$ are called the **covariant components** *of the vector* $v_i(x)\boldsymbol{g}^i(x)$ *at* $\widehat{x}$, and the three functions $v_i : \Omega \to \mathbb{R}$ defined in this fashion are called the **covariant components** *of the vector field* $v_i\boldsymbol{g}^i : \Omega \to \mathbf{E}^3$.

Suppose next that we wish to compute a partial derivative $\widehat{\partial}_j\widehat{v}_i(\widehat{x})$ at a point $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ in terms of the partial derivatives $\partial_\ell v_k(x)$ and of the values $v_q(x)$ (which are also expected to appear by virtue of the chain rule). Such a task is required for example if we wish to write a system of partial differential equations whose unknown is a vector field (such as the equations of nonlinear or linearized elasticity) in terms of *ad hoc* curvilinear coordinates.

As we now show, carrying out such a transformation naturally leads to a fundamental notion, that of *covariant derivatives of a vector field*.

Theorem 1.4-1. *Let* Ω *be an open subset of* $\mathbb{R}^3$ *and let* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *be an immersion that is also a* $\mathcal{C}^2$*-diffeomorphism of* Ω *onto* $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$. *Given a vector field* $\widehat{v}_i\widehat{\boldsymbol{e}}^i : \widehat{\Omega} \to \mathbb{R}^3$ *in Cartesian coordinates with components* $\widehat{v}_i \in \mathcal{C}^1(\widehat{\Omega})$, *let* $v_i\boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ *be the same field in curvilinear coordinates, i.e., that defined by*

$$\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i = v_i(x)\boldsymbol{g}^i(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega.$$

Then $v_i \in \mathcal{C}^1(\Omega)$ *and for all* $x \in \Omega$,

$$\widehat{\partial}_j\widehat{v}_i(\widehat{x}) = \big(v_{k\|\ell}[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\big)(x),\ \widehat{x} = \boldsymbol{\Theta}(x),$$

where

$$v_{i\|j} := \partial_j v_i - \Gamma^p_{ij}v_p \text{ and } \Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i\boldsymbol{g}_j,$$

and

$$[\boldsymbol{g}^i(x)]_k := \boldsymbol{g}^i(x) \cdot \widehat{\boldsymbol{e}}_k$$

denotes the i-th component of $\boldsymbol{g}^i(x)$ *over the basis* $\{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{e}}_3\}$.

Proof. The following convention holds throughout this proof: The simultaneous appearance of $\widehat{x}$ and x in an equality means that they are related by $\widehat{x} = \boldsymbol{\Theta}(x)$ and that the equality in question holds for all $x \in \Omega$.

(i) *Another expression of* $[\boldsymbol{g}^i(x)]_k := \boldsymbol{g}^i(x) \cdot \widehat{\boldsymbol{e}}_k$.

Let $\boldsymbol{\Theta}(x) = \Theta^k(x)\widehat{\boldsymbol{e}}_k$ and $\widehat{\boldsymbol{\Theta}}(\widehat{x}) = \widehat{\Theta}^i(\widehat{x})\boldsymbol{e}_i$, where $\widehat{\boldsymbol{\Theta}} : \widehat{\Omega} \to \mathbf{E}^3$ denotes the inverse mapping of $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$. Since $\widehat{\boldsymbol{\Theta}}(\boldsymbol{\Theta}(x)) = x$ for all $x \in \Omega$, the chain rule shows that the matrices $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) := (\partial_j \Theta^k(x))$ (the row index is k) and $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Theta}}(\widehat{x}) := (\widehat{\partial}_k \widehat{\Theta}^i(\widehat{x}))$ (the row index is i) satisfy

$$\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Theta}}(\widehat{x})\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{I},$$

or equivalently,

$$\widehat{\partial}_k \widehat{\Theta}^i(\widehat{x})\partial_j \Theta^k(x) = \begin{pmatrix} \widehat{\partial}_1 \widehat{\Theta}^i(\widehat{x}) & \partial_2 \widehat{\Theta}^i(\widehat{x}) & \partial_3 \widehat{\Theta}^i(\widehat{x}) \end{pmatrix} \begin{pmatrix} \partial_j \Theta^1(x) \\ \partial_j \Theta^2(x) \\ \partial_j \Theta^3(x) \end{pmatrix} = \delta^i_j.$$

The components of the above column vector being precisely those of the vector $\boldsymbol{g}_j(x)$, the components of the above row vector must be those of the vector $\boldsymbol{g}^i(x)$ since $\boldsymbol{g}^i(x)$ is uniquely defined for each exponent i by the three relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$, $j = 1, 2, 3$. Hence the k-th component of $\boldsymbol{g}^i(x)$ over the basis $\{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{e}}_3\}$ can be also expressed in terms of the inverse mapping $\widehat{\boldsymbol{\Theta}}$, as:

$$[\boldsymbol{g}^i(x)]_k = \widehat{\partial}_k \widehat{\Theta}^i(\widehat{x}).$$

(ii) *The functions* $\Gamma^q_{\ell k} := \boldsymbol{g}^q \cdot \partial_\ell \boldsymbol{g}_k \in \mathcal{C}^0(\Omega)$.

We next compute the derivatives $\partial_\ell \boldsymbol{g}^q(x)$ (the fields $\boldsymbol{g}^q = g^{qr}\boldsymbol{g}_r$ are of class $\mathcal{C}^1$ on Ω since $\boldsymbol{\Theta}$ is assumed to be of class $\mathcal{C}^2$). These derivatives will be needed in (iii) for expressing the derivatives $\widehat{\partial}_j \widehat{u}_i(\widehat{x})$ as functions of x (recall that $\widehat{u}_i(\widehat{x}) = u_k(x)[\boldsymbol{g}^k(x)]_i$). Recalling that the vectors $\boldsymbol{g}^k(x)$ form a basis, we may write *a priori*

$$\partial_\ell \boldsymbol{g}^q(x) = -\Gamma^q_{\ell k}(x)\boldsymbol{g}^k(x),$$

thereby unambiguously defining functions $\Gamma^q_{\ell k} : \Omega \to \mathbb{R}$. To find their expressions in terms of the mappings $\boldsymbol{\Theta}$ and $\widehat{\boldsymbol{\Theta}}$, we observe that

$$\Gamma^q_{\ell k}(x) = \Gamma^q_{\ell m}(x)\delta^m_k = \Gamma^q_{\ell m}(x)\boldsymbol{g}^m(x) \cdot \boldsymbol{g}_k(x) = -\partial_\ell \boldsymbol{g}^q(x) \cdot \boldsymbol{g}_k(x).$$

Hence, noting that $\partial_\ell(\boldsymbol{g}^q(x) \cdot \boldsymbol{g}_k(x)) = 0$ and $[\boldsymbol{g}^q(x)]_p = \widehat{\partial}_p \widehat{\Theta}^q(\widehat{x})$, we obtain

$$\Gamma^q_{\ell k}(x) = \boldsymbol{g}^q(x) \cdot \partial_\ell \boldsymbol{g}_k(x) = \widehat{\partial}_p \widehat{\Theta}^q(\widehat{x})\partial_{\ell k}\Theta^p(x) = \Gamma^q_{k\ell}(x).$$

Since $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbf{E}^3)$ and $\widehat{\boldsymbol{\Theta}} \in \mathcal{C}^1(\widehat{\Omega}; \mathbb{R}^3)$ by assumption, the last relations show that $\Gamma^q_{\ell k} \in \mathcal{C}^0(\Omega)$.

(iii) *The partial derivatives* $\widehat{\partial}_i \widehat{v}_i(\widehat{x})$ *of the Cartesian components of the vector field* $\widehat{v}_i \widehat{\boldsymbol{e}}^i \in \mathcal{C}^1(\widehat{\Omega}; \mathbb{R}^3)$ *are given at each* $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ *by*

$$\widehat{\partial}_j \widehat{v}_i(\widehat{x}) = v_{k\|\ell}(x)[\boldsymbol{g}^k(x)]_i [\boldsymbol{g}^\ell(x)]_j,$$

where

$$v_{k\|\ell}(x) := \partial_\ell v_k(x) - \Gamma^q_{\ell k}(x) v_q(x),$$

and $[\boldsymbol{g}^k(x)]_i$ *and* $\Gamma^q_{\ell k}(x)$ *are defined as in* (i) *and* (ii).

We compute the partial derivatives $\widehat{\partial}_j \widehat{v}_i(\widehat{x})$ as functions of x by means of the relation $\widehat{v}_i(\widehat{x}) = v_k(x)[\boldsymbol{g}^k(x)]_i$. To this end, we first note that a differentiable function $w : \Omega \to \mathbb{R}$ satisfies

$$\widehat{\partial}_j w(\widehat{\boldsymbol{\Theta}}(\widehat{x})) = \partial_\ell w(x) \widehat{\partial}_j \widehat{\Theta}^\ell(\widehat{x}) = \partial_\ell w(x)[\boldsymbol{g}^\ell(x)]_j,$$

by the chain rule and by (i). In particular then,

$$\begin{aligned}\widehat{\partial}_j \widehat{v}_i(\widehat{x}) &= \widehat{\partial}_j v_k(\widehat{\boldsymbol{\Theta}}(\widehat{x}))[\boldsymbol{g}^k(x)]_i + v_q(x)\widehat{\partial}_j[\boldsymbol{g}^q(\widehat{\boldsymbol{\Theta}}(\widehat{x}))]_i \\ &= \partial_\ell v_k(x)[\boldsymbol{g}^\ell(x)]_j [\boldsymbol{g}^k(x)]_i + v_q(x)\big(\partial_\ell [\boldsymbol{g}^q(x)]_i\big)[\boldsymbol{g}^\ell(x)]_j \\ &= \left(\partial_\ell v_k(x) - \Gamma^q_{\ell k}(x) v_q(x)\right)[\boldsymbol{g}^k(x)]_i [\boldsymbol{g}^\ell(x)]_j,\end{aligned}$$

since $\partial_\ell \boldsymbol{g}^q(x) = -\Gamma^q_{\ell k}(x)\boldsymbol{g}^k(x)$ by (ii). $\square$

The functions

$$v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$$

defined in Theorem 1.4-1 are called the **first-order covariant derivatives** of the vector field $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$.

The functions

$$\Gamma^p_{ij} = \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j$$

are called the **Christoffel symbols of the first kind**.

The following result summarizes properties of covariant derivatives and Christoffel symbols that are constantly used.

Theorem 1.4-2. *Let the assumptions on the mapping* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *be as in* Theorem 1.4-1, *and let there be given a vector field* $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ *with covariant components* $v_i \in \mathcal{C}^1(\Omega)$.

(a) *The first-order covariant derivatives* $v_{i\|j} \in \mathcal{C}^0(\Omega)$ *of the vector field* $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$*, which are defined by*

$$v_{i\|j} := \partial_j v_i - \Gamma^p_{ij} v_p, \text{ where } \Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j,$$

can be also defined by the relations

$$\partial_j (v_i \boldsymbol{g}^i) = v_{i\|j} \boldsymbol{g}^i \iff v_{i\|j} = \left\{\partial_j (v_k \boldsymbol{g}^k)\right\} \cdot \boldsymbol{g}_i.$$

(b) *The Christoffel symbols* $\Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j = \Gamma^p_{ji} \in \mathcal{C}^0(\Omega)$ *satisfy the relations*

$$\partial_i \boldsymbol{g}^p = -\Gamma^p_{ij} \boldsymbol{g}^j \text{ and } \partial_j \boldsymbol{g}_q = \Gamma^i_{jq} \boldsymbol{g}_i.$$

Proof. It remains to verify that the covariant derivatives $v_{i\|j}$, defined in Theorem 1.4-1 by

$$v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p,$$

may be equivalently defined by the relations

$$\partial_j (v_i \boldsymbol{g}^i) = v_{i\|j} \boldsymbol{g}^i.$$

These relations unambiguously define the functions $v_{i\|j} = \{\partial_j (v_k \boldsymbol{g}^k)\} \cdot \boldsymbol{g}_i$ since the vectors $\boldsymbol{g}^i$ are linearly independent at all points of Ω by assumption. To this end, we simply note that, by definition, the Christoffel symbols satisfy $\partial_i \boldsymbol{g}^p = -\Gamma^p_{ij} \boldsymbol{g}^j$ (cf. part (ii) of the proof of Theorem 1.4-1); hence

$$\partial_j (v_i \boldsymbol{g}^i) = (\partial_j v_i) \boldsymbol{g}^i + v_i \partial_j \boldsymbol{g}^i = (\partial_j v_i) \boldsymbol{g}^i - v_i \Gamma^i_{jk} \boldsymbol{g}^k = v_{i\|j} \boldsymbol{g}^i.$$

To establish the other relations $\partial_j \boldsymbol{g}_q = \Gamma^i_{jq} \boldsymbol{g}_i$, we note that

$$0 = \partial_j (\boldsymbol{g}^p \cdot \boldsymbol{g}_q) = -\Gamma^p_{ji} \boldsymbol{g}^i \cdot \boldsymbol{g}_q + \boldsymbol{g}^p \cdot \partial_j \boldsymbol{g}_q = -\Gamma^p_{qj} + \boldsymbol{g}^p \cdot \partial_j \boldsymbol{g}_q.$$

Hence

$$\partial_j \boldsymbol{g}_q = (\partial_j \boldsymbol{g}_q \cdot \boldsymbol{g}^p) \boldsymbol{g}_p = \Gamma^p_{qj} \boldsymbol{g}_p.$$

$\square$

Remark. The Christoffel symbols Γ^p_{ij} can be also defined solely in terms of the components of the metric tensor; see the proof of Theorem 1.5-1. $\square$

If the affine space $\mathbf{E}^3$ is identified with $\mathbb{R}^3$ and $\boldsymbol{\Theta} = \boldsymbol{id}_\Omega$, the relation $\partial_j (v_i \boldsymbol{g}^i)(x) = (v_{i\|j} \boldsymbol{g}^i)(x)$ (Theorem 1.4-2 (a)), reduces to $\widehat{\partial}_j (\widehat{v}_i(\widehat{x}) \widehat{\boldsymbol{e}}^i) = (\widehat{\partial}_j \widehat{v}_i(\widehat{x})) \widehat{\boldsymbol{e}}^i$. In this sense, *a covariant derivative of the first order constitutes a generalization of a partial derivative of the first order in Cartesian coordinates.*

1.5 NECESSARY CONDITIONS SATISFIED BY THE METRIC TENSOR; THE RIEMANN CURVATURE TENSOR

It is remarkable that *the components $g_{ij} : \Omega \to \mathbb{R}$ of the metric tensor of an open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$* (Section 1.2), defined by a smooth enough immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$, *cannot be arbitrary functions.*

As shown in the next theorem, they must satisfy relations that take the form:

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

where the functions Γ_{ijq} and Γ^p_{ij} have simple expressions in terms of the functions g_{ij} and of some of their partial derivatives (as shown in the next proof, it so happens that the functions Γ^p_{ij} as defined in Theorem 1.5-1 coincide with the Christoffel symbols introduced in the previous section; this explains why they are denoted by the same symbol). Note that, according to the rule governing Latin indices and exponents, these relations are meant to hold for all $i, j, k, q \in \{1, 2, 3\}$.

Theorem 1.5-1. *Let Ω be an open set in $\mathbb{R}^3$, let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be an immersion, and let*

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$$

denote the covariant components of the metric tensor of the set $\boldsymbol{\Theta}(\Omega)$. Let the functions $\Gamma_{ijq} \in \mathcal{C}^1(\Omega)$ and $\Gamma^p_{ij} \in \mathcal{C}^1(\Omega)$ be defined by

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}),$$
$$\Gamma^p_{ij} := g^{pq}\Gamma_{ijq} \text{ where } (g^{pq}) := (g_{ij})^{-1}.$$

Then, necessarily,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega.$$

Proof. Let $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}$. It is then immediately verified that the functions Γ_{ijq} are also given by

$$\Gamma_{ijq} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q.$$

For each $x \in \Omega$, let the three vectors $\boldsymbol{g}^j(x)$ be defined by the relations $\boldsymbol{g}^j(x) \cdot \boldsymbol{g}_i(x) = \delta^j_i$. Since we also have $\boldsymbol{g}^j = g^{ij}\boldsymbol{g}_i$, the last relations imply that $\Gamma^p_{ij} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p$. Therefore,

$$\partial_i \boldsymbol{g}_j = \Gamma^p_{ij}\boldsymbol{g}_p$$

since $\partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p)\boldsymbol{g}_p$. Differentiating the same relations yields

$$\partial_k \Gamma_{ijq} = \partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q + \partial_i \boldsymbol{g}_j \cdot \partial_k \boldsymbol{g}_q,$$

so that the above relations together give

$$\partial_i \boldsymbol{g}_j \cdot \partial_k \boldsymbol{g}_q = \Gamma^p_{ij}\boldsymbol{g}_p \cdot \partial_k \boldsymbol{g}_q = \Gamma^p_{ij}\Gamma_{kqp}.$$

Consequently,

$$\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_k \Gamma_{ijq} - \Gamma^p_{ij}\Gamma_{kqp}.$$

Since $\partial_{ik}\boldsymbol{g}_j = \partial_{ij}\boldsymbol{g}_k$, we also have

$$\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_j \Gamma_{ikq} - \Gamma^p_{ik}\Gamma_{jqp},$$

and thus the required necessary conditions immediately follow. □

Remark. The vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^j$ introduced above form the covariant and contravariant bases and the functions g^{ij} are the contravariant components of the metric tensor (Section 1.2). □

As shown in the above proof, the necessary conditions $R_{qijk} = 0$ thus simply constitute a re-writing of the relations $\partial_{ik}\boldsymbol{g}_j = \partial_{ki}\boldsymbol{g}_j$ in the form of the equivalent relations $\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_{ki}\boldsymbol{g}_j \cdot \boldsymbol{g}_q$.

The functions

$$\Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q = \Gamma_{jiq}$$

and

$$\Gamma_{ij}^p = g^{pq}\Gamma_{ijq} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p = \Gamma_{ji}^p$$

are the **Christoffel symbols of the first**, and **second, kinds**. We saw in Section 1.4 that the same Christoffel symbols Γ_{ij}^p also naturally appear in a different context (that of covariant differentiation).

Finally, the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}$$

are the **covariant components** of the **Riemann curvature tensor** of the set $\boldsymbol{\Theta}(\Omega)$. The relations $R_{qijk} = 0$ found in Theorem 1.4-1 thus express that the *Riemann curvature tensor of the set* $\boldsymbol{\Theta}(\Omega)$ (equipped with the metric tensor with covariant components g_{ij}) *vanishes.*

1.6 EXISTENCE OF AN IMMERSION DEFINED ON AN OPEN SET IN $\mathbb{R}^3$ WITH A PRESCRIBED METRIC TENSOR

Let $\mathbb{M}^3, \mathbb{S}^3$, and $\mathbb{S}^3_>$ denote the sets of all square matrices of order three, of all symmetric matrices of order three, and of all symmetric positive definite matrices of order three.

As in Section 1.2, the matrix representing the Fréchet derivative at $x \in \Omega$ of a differentiable mapping $\boldsymbol{\Theta} = (\Theta_\ell) : \Omega \to \mathbf{E}^3$ is denoted

$$\boldsymbol{\nabla\Theta}(x) := (\partial_j \Theta_\ell(x)) \in \mathbb{M}^3,$$

where ℓ is the row index and j the column index (equivalently, $\boldsymbol{\nabla\Theta}(x)$ is the matrix of order three whose j-th column vector is $\partial_j \boldsymbol{\Theta}$).

So far, we have considered that we are given an open set $\Omega \subset \mathbb{R}^3$ and a smooth enough immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$, thus allowing us to define a matrix field

$$\mathbf{C} = (g_{ij}) = \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta} : \Omega \to \mathbb{S}^3_>,$$

where $g_{ij} : \Omega \to \mathbb{R}$ are the covariant components of the *metric tensor* of the open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$.

We now turn to the *reciprocal questions*:

Given an open subset Ω of $\mathbb{R}^3$ and a smooth enough matrix field $\mathbf{C} = (g_{ij}) : \Omega \to \mathbb{S}^3_>$, when is $\mathbf{C}$ the metric tensor field of an open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$? Equivalently, *when does there exist an immersion* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *such that*

$$\mathbf{C} = \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta} \text{ in } \Omega,$$

or equivalently, such that

$$g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \text{ in } \Omega?$$

If such an immersion exists, to what extent is it unique?

The answers to these questions turn out to be remarkably simple: *If* Ω *is simply-connected, the necessary conditions*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega$$

found in Theorem 1.4-1 *are also sufficient for the existence of such an immersion. If* Ω *is connected, this immersion is unique up to isometries in* $\mathbf{E}^3$.

Whether the immersion found in this fashion is *globally injective* is a different issue, which accordingly should be resolved by different means.

This result comprises two essentially distinct parts, a *global existence result* (Theorem 1.6-1) and a *uniqueness result* (Theorem 1.4-1). Note that these two results are established under *different assumptions* on the set Ω and on the smoothness of the field (g_{ij}).

In order to put these results in a wider perspective, let us make a brief incursion into *Riemannian Geometry*. For detailed treatments, see classic texts such as Choquet-Bruhat, de Witt-Morette & Dillard-Bleick [1977], Marsden & Hughes [1983], or Gallot, Hulin & Lafontaine [2004].

Considered as a three-dimensional manifold, an open set $\Omega \subset \mathbb{R}^3$ equipped with an immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ becomes an example of a *Riemannian manifold* $(\Omega; (g_{ij}))$, i.e., a *manifold*, the set Ω, equipped with a *Riemannian metric*, the symmetric positive-definite matrix field $(g_{ij}) : \Omega \to \mathbb{S}^3_>$ defined in this case by $g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω. More generally, a **Riemannian metric on a manifold** is a twice covariant, symmetric, positive-definite tensor field acting on vectors in the tangent spaces to the manifold (these tangent spaces coincide with $\mathbb{R}^3$ in the present instance).

This particular Riemannian manifold $(\Omega; (g_{ij}))$ possesses the remarkable property that *its metric is the same as that of the surrounding space* $\mathbf{E}^3$. More specifically, $(\Omega; (g_{ij}))$ is **isometrically immersed** in the Euclidean space $\mathbf{E}^3$, in the sense that there exists an immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ that satisfies the relations $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$. Equivalently, the length of any curve in the Riemannian manifold $(\Omega; (g_{ij}))$ is the same as the length of its image by $\boldsymbol{\Theta}$ in the Euclidean space $\mathbf{E}^3$ (see Theorem 1.3-1).

The first question above can thus be rephrased as follows: *Given an open subset* Ω *of* $\mathbb{R}^3$ *and a positive-definite matrix field* $(g_{ij}) : \Omega \to \mathbb{S}^3_>$, *when is the Riemannian manifold* $(\Omega; (g_{ij}))$ **flat**, in the sense that it can be locally isometrically immersed in a Euclidean space of the *same* dimension (three)?

The answer to this question can then be rephrased as follows (compare with the statement of Theorem 1.6-1 below): *Let* Ω *be a simply-connected open subset of* $\mathbb{R}^3$. *Then a Riemannian manifold* $(\Omega; (g_{ij}))$ *with a Riemannian metric* (g_{ij}) *of class* $\mathcal{C}^2$ *in* Ω *is flat if and only if its Riemannian curvature tensor vanishes in* Ω. Recast as such, this result becomes a special case of the **fundamental theorem on flat Riemannian manifolds**, which holds for a general finite-dimensional Riemannian manifold.

The answer to the second question, viz., the issue of uniqueness, can be rephrased as follows (compare with the statement of Theorem 1.7-1 in the next section): *Let Ω be a connected open subset of $\mathbb{R}^3$. Then the isometric immersions of a flat Riemannian manifold $(\Omega; (g_{ij}))$ into a Euclidean space $\mathbf{E}^3$ are unique up to isometries of $\mathbf{E}^3$.* Recast as such, this result likewise becomes a special case of the so-called **rigidity theorem**; cf. Section 1.7.

Recast as such, these two theorems together constitute a special case (that where the dimensions of the manifold and of the Euclidean space are both equal to three) of the **fundamental theorem of Riemannian Geometry**. This theorem addresses the same *existence* and *uniqueness* questions in the more general setting where Ω is replaced by a p-dimensional manifold and $\mathbf{E}^3$ is replaced by a $(p+q)$-dimensional Euclidean space (the "fundamental theorem of surface theory", established in Sections 2.8 and 2.9, constitutes another important special case). When the p-dimensional manifold is an open subset of $\mathbb{R}^p$, an outline of a self-contained proof is given in Szopos [2005].

Another fascinating question (which will not be addressed here) is the following: Given again an open subset Ω of $\mathbb{R}^3$ equipped with a symmetric, positive-definite matrix field $(g_{ij}) : \Omega \to \mathbb{S}^3$, assume this time that *the Riemannian manifold $(\Omega; (g_{ij}))$ is no longer flat*, i.e., its Riemannian curvature tensor no longer vanishes in Ω. *Can such a Riemannian manifold still be isometrically immersed, but this time in a higher-dimensional Euclidean space*? Equivalently, do there exist a Euclidean space $\mathbf{E}^d$ with $d > 3$ and an immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^d$ such that $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω?

The answer is yes, according to the following beautiful **Nash theorem**, so named after Nash [1954]: *Any p-dimensional Riemannian manifold equipped with a continuous metric can be isometrically immersed in a Euclidean space of dimension $2p$ with an immersion of class $\mathcal{C}^1$; it can also be isometrically immersed in a Euclidean space of dimension $(2p+1)$ with a globally injective immersion of class $\mathcal{C}^1$.*

Let us now humbly return to the question of existence raised at the beginning of this section, i.e., when the manifold is an open set in $\mathbb{R}^3$.

Theorem 1.6-1. *Let Ω be a connected and simply-connected open set in $\mathbb{R}^3$ and let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}),$$
$$\Gamma^p_{ij} := g^{pq} \Gamma_{ijq} \text{ with } (g^{pq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that

$$\mathbf{C} = \boldsymbol{\nabla} \boldsymbol{\Theta}^T \boldsymbol{\nabla} \boldsymbol{\Theta} \text{ in } \Omega.$$

Proof. The proof relies on a simple, yet crucial, observation. When a smooth enough immersion $\boldsymbol{\Theta} = (\Theta_\ell) : \Omega \to \mathbf{E}^3$ is *a priori* given (as it was so far), its components Θ_ℓ satisfy the relations $\partial_{ij}\Theta_\ell = \Gamma^p_{ij}\partial_p\Theta_\ell$, which are nothing but another way of writing the relations $\partial_i \boldsymbol{g}_j = \Gamma^p_{ij}\boldsymbol{g}_p$ (see the proof of Theorem 1.5-1). This observation thus suggests to begin by solving (see part (ii)) the system of partial differential equations

$$\partial_i F_{\ell j} = \Gamma^p_{ij} F_{\ell p} \text{ in } \Omega,$$

whose solutions $F_{\ell j} : \Omega \to \mathbb{R}$ then constitute natural candidates for the partial derivatives $\partial_j \Theta_\ell$ of the unknown immersion $\boldsymbol{\Theta} = (\Theta_\ell) : \Omega \to \mathbf{E}^3$ (see part (iii)).

To begin with, we establish in (i) relations that will in turn allow us to re-write the sufficient conditions

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega$$

in a slightly different form, more appropriate for the existence result of part (ii). Note that the positive definiteness of the symmetric matrices (g_{ij}) is not needed for this purpose.

(i) *Let Ω be an open subset of $\mathbb{R}^3$ and let there be given a field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3)$ of symmetric invertible matrices. The functions $\Gamma_{ijq}, \Gamma^p_{ij}$, and g^{pq} being defined by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}), \quad \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}, \quad (g^{pq}) := (g_{ij})^{-1},$$

define the functions

$$\begin{aligned} R_{qijk} &:= \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}, \\ R^p_{\cdot ijk} &:= \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell}. \end{aligned}$$

Then

$$R^p_{\cdot ijk} = g^{pq} R_{qijk} \text{ and } R_{qijk} = g_{pq} R^p_{\cdot ijk}.$$

Using the relations

$$\Gamma_{jq\ell} + \Gamma_{\ell jq} = \partial_j g_{q\ell} \text{ and } \Gamma_{ikq} = g_{q\ell}\Gamma^\ell_{ik},$$

which themselves follow from the definitions of the functions Γ_{ijq} and Γ^p_{ij}, and noting that

$$(g^{pq}\partial_j g_{q\ell} + g_{q\ell}\partial_j g^{pq}) = \partial_j(g^{pq} g_{q\ell}) = 0,$$

we obtain

$$\begin{aligned} g^{pq}(\partial_j \Gamma_{ikq} - \Gamma^r_{ik}\Gamma_{jqr}) &= \partial_j \Gamma^p_{ik} - \Gamma_{ikq}\partial_j g^{pq} - \Gamma^\ell_{ik} g^{pq}(\partial_j g_{q\ell} - \Gamma_{\ell jq}) \\ &= \partial_j \Gamma^p_{ik} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ik}(g^{pq}\partial_j g_{q\ell} + g_{q\ell}\partial_j g^{pq}) \\ &= \partial_j \Gamma^p_{ik} + \Gamma^\ell_{ik}\Gamma^p_{j\ell}. \end{aligned}$$

Likewise,

$$g^{pq}(\partial_k \Gamma_{ijq} - \Gamma_{ij}^r \Gamma_{kqr}) = \partial_k \Gamma_{ij}^p - \Gamma_{ij}^\ell \Gamma_{k\ell}^p,$$

and thus the relations $R_{\cdot ijk}^p = g^{pq} R_{qijk}$ are established. The relations $R_{qijk} = g_{pq} R_{\cdot ijk}^p$ are clearly equivalent to these ones.

We next establish the existence of solutions to the system

$$\partial_i F_{\ell j} = \Gamma_{ij}^p F_{\ell p} \text{ in } \Omega.$$

(ii) *Let* Ω *be a connected and simply-connected open subset of* $\mathbb{R}^3$ *and let there be given functions* $\Gamma_{ij}^p = \Gamma_{ji}^p \in \mathcal{C}^1(\Omega)$ *satisfying the relations*

$$\partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p - \Gamma_{ij}^\ell \Gamma_{k\ell}^p = 0 \text{ in } \Omega,$$

which are equivalent to the relations

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0 \text{ in } \Omega,$$

by part (i).

Let a point $x^0 \in \Omega$ *and a matrix* $(F_{\ell j}^0) \in \mathbb{M}^3$ *be given. Then there exists one, and only one, field* $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ *that satisfies*

$$\begin{aligned} \partial_i F_{\ell j}(x) &= \Gamma_{ij}^p(x) F_{\ell p}(x), \ x \in \Omega, \\ F_{\ell j}(x^0) &= F_{\ell j}^0. \end{aligned}$$

Let x^1 be an arbitrary point in the set Ω, *distinct from* x^0. Since Ω is *connected*, there exists a *path* $\boldsymbol{\gamma} = (\gamma^i) \in \mathcal{C}^1([0,1]; \mathbb{R}^3)$ *joining* x^0 *to* x^1 *in* Ω; this means that

$$\boldsymbol{\gamma}(0) = x^0, \ \boldsymbol{\gamma}(1) = x^1, \text{ and } \boldsymbol{\gamma}(t) \in \Omega \text{ for all } 0 \le t \le 1.$$

Assume that a matrix field $(F_{\ell j}) \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ satisfies $\partial_i F_{\ell j}(x) = \Gamma_{ij}^p(x) F_{\ell p}(x)$, $x \in \Omega$. Then, *for each integer* $\ell \in \{1, 2, 3\}$, the three functions $\zeta_j \in \mathcal{C}^1([0,1])$ defined by (for simplicity, the dependence on ℓ is dropped)

$$\zeta_j(t) := F_{\ell j}(\boldsymbol{\gamma}(t)), \ 0 \le t \le 1,$$

satisfy the following *Cauchy problem for a linear system of three ordinary differential equations with respect to three unknowns*:

$$\begin{aligned} \frac{\mathrm{d}\zeta_j}{\mathrm{d}t}(t) &= \Gamma_{ij}^p(\boldsymbol{\gamma}(t)) \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t) \zeta_p(t), \ 0 \le t \le 1, \\ \zeta_j(0) &= \zeta_j^0, \end{aligned}$$

where the *initial values* ζ_j^0 are given by

$$\zeta_j^0 := F_{\ell j}^0.$$

Note in passing that the three Cauchy problems obtained by letting $\ell = 1, 2$, or 3 only differ by their initial values ζ_j^0.

It is well known that a Cauchy problem of the form (with self-explanatory notations)

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t}(t) &= \mathbf{A}(t)\boldsymbol{\zeta}(t),\ 0 \le t \le 1,\\ \boldsymbol{\zeta}(0) &= \boldsymbol{\zeta}^0,\end{aligned}$$

has *one* and *only one* solution $\boldsymbol{\zeta} \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ if $\mathbf{A} \in \mathcal{C}^0([0,1];\mathbb{M}^3)$ (see, e.g., Schwartz [1992, Theorem 4.3.1, p. 388]). Hence each one of the three Cauchy problems has one and only one solution.

Incidentally, this result already shows that, *if it exists, the unknown field* $(F_{\ell j})$ *is unique.*

In order that the three values $\zeta_j(1)$ found by solving the above Cauchy problem for a given integer $\ell \in \{1,2,3\}$ be acceptable candidates for the three unknown values $F_{\ell j}(x^1)$, they must be of course *independent of the path chosen for joining* x^0 *to* x^1.

So, let $\boldsymbol{\gamma}_0 \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ and $\boldsymbol{\gamma}_1 \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ be two paths joining x^0 to x^1 in Ω. The open set Ω being *simply-connected*, there exists a *homotopy* $\mathbf{G} = (G^i) : [0,1] \times [0,1] \to \mathbb{R}^3$ *joining* $\boldsymbol{\gamma}_0$ *to* $\boldsymbol{\gamma}_1$ *in* Ω, i.e., such that

$$\begin{gathered}\mathbf{G}(\cdot,0) = \boldsymbol{\gamma}_0,\ \mathbf{G}(\cdot,1) = \boldsymbol{\gamma}_1,\ \mathbf{G}(t,\lambda) \in \Omega \text{ for all } 0 \le t \le 1,\ 0 \le \lambda \le 1,\\ \mathbf{G}(0,\lambda) = x^0 \text{ and } \mathbf{G}(1,\lambda) = x^1 \text{ for all } 0 \le \lambda \le 1,\end{gathered}$$

and smooth enough in the sense that

$$\mathbf{G} \in \mathcal{C}^1([0,1]\times[0,1];\mathbb{R}^3) \text{ and } \frac{\partial}{\partial t}\Big(\frac{\partial \mathbf{G}}{\partial \lambda}\Big) = \frac{\partial}{\partial \lambda}\Big(\frac{\partial \mathbf{G}}{\partial t}\Big) \in \mathcal{C}^0([0,1]\times[0,1];\mathbb{R}^3).$$

Let $\boldsymbol{\zeta}(\cdot,\lambda) = (\zeta_j(\cdot,\lambda)) \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ denote for each $0 \le \lambda \le 1$ the solution of the Cauchy problem corresponding to the path $\mathbf{G}(\cdot,\lambda)$ joining x^0 to x^1. We thus have

$$\begin{aligned}\frac{\partial \zeta_j}{\partial t}(t,\lambda) &= \Gamma_{ij}^p(\mathbf{G}(t,\lambda))\frac{\partial G^i}{\partial t}(t,\lambda)\zeta_p(t,\lambda) \text{ for all } 0 \le t \le 1,\ 0 \le \lambda \le 1,\\ \zeta_j(0,\lambda) &= \zeta_j^0 \text{ for all } 0 \le \lambda \le 1.\end{aligned}$$

Our objective is to show that

$$\frac{\partial \zeta_j}{\partial \lambda}(1,\lambda) = 0 \text{ for all } 0 \le \lambda \le 1,$$

as this relation will imply that $\zeta_j(1,0) = \zeta_j(1,1)$, as desired. For this purpose, a direct differentiation shows that, for all $0 \le t \le 1$, $0 \le \lambda \le 1$,

$$\frac{\partial}{\partial \lambda}\Big(\frac{\partial \zeta_j}{\partial t}\Big) = \{\Gamma_{ij}^q \Gamma_{kq}^p + \partial_k \Gamma_{ij}^p\}\zeta_p \frac{\partial G^k}{\partial \lambda}\frac{\partial G^i}{\partial t} + \Gamma_{ij}^p \zeta_p \frac{\partial}{\partial \lambda}\Big(\frac{\partial G^i}{\partial t}\Big) + \sigma_q \Gamma_{ij}^q \frac{\partial G^i}{\partial t},$$

where
$$\sigma_j := \frac{\partial \zeta_j}{\partial \lambda} - \Gamma^p_{kj}\zeta_p \frac{\partial G^k}{\partial \lambda},$$
on the one hand (in the relations above and below, $\Gamma^q_{ij}, \partial_k \Gamma^p_{ij}$, etc., stand for $\Gamma^q_{ij}(\mathbf{G}(\cdot,\cdot)), \partial_k\Gamma^p_{ij}(\mathbf{G}(\cdot,\cdot))$, etc.).

On the other hand, a direct differentiation of the equation defining the functions σ_j shows that, for all $0 \le t \le 1, 0 \le \lambda \le 1$,
$$\frac{\partial}{\partial t}\Big(\frac{\partial \zeta_j}{\partial \lambda}\Big) = \frac{\partial \sigma_j}{\partial t} + \Big\{\partial_i \Gamma^p_{kj}\frac{\partial G^i}{\partial t}\zeta_p + \Gamma^q_{kj}\frac{\partial \zeta_q}{\partial t}\Big\}\frac{\partial G^k}{\partial \lambda} + \Gamma^p_{ij}\zeta_p \frac{\partial}{\partial t}\Big(\frac{\partial G^i}{\partial \lambda}\Big).$$
But $\dfrac{\partial \zeta_j}{\partial t} = \Gamma^p_{ij}\dfrac{\partial G^i}{\partial t}\zeta_p$, so that we also have
$$\frac{\partial}{\partial t}\Big(\frac{\partial \zeta_j}{\partial \lambda}\Big) = \frac{\partial \sigma_j}{\partial t} + \{\partial_i \Gamma^p_{kj} + \Gamma^q_{kj}\Gamma^p_{iq}\}\zeta_p \frac{\partial G^i}{\partial t}\frac{\partial G^k}{\partial \lambda} + \Gamma^p_{ij}\zeta_p \frac{\partial}{\partial t}\Big(\frac{\partial G^i}{\partial \lambda}\Big).$$

Hence, subtracting the above relations and noting that $\dfrac{\partial}{\partial \lambda}\Big(\dfrac{\partial \zeta_j}{\partial t}\Big) = \dfrac{\partial}{\partial t}\Big(\dfrac{\partial \zeta_j}{\partial \lambda}\Big)$ and $\dfrac{\partial}{\partial \lambda}\Big(\dfrac{\partial G^i}{\partial t}\Big) = \dfrac{\partial}{\partial t}\Big(\dfrac{\partial G^i}{\partial \lambda}\Big)$ by assumption, we infer that
$$\frac{\partial \sigma_j}{\partial t} + \{\partial_i \Gamma^p_{kj} - \partial_k \Gamma^p_{ij} + \Gamma^q_{kj}\Gamma^p_{iq} - \Gamma^q_{ij}\Gamma^p_{kq}\}\zeta_p \frac{\partial G^k}{\partial \lambda}\frac{\partial G^i}{\partial t} - \Gamma^q_{ij}\frac{\partial G^i}{\partial t}\sigma_q = 0.$$

The assumed symmetries $\Gamma^p_{ij} = \Gamma^p_{ji}$ combined with the assumed relations $\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell} = 0$ in Ω show that
$$\partial_i \Gamma^p_{kj} - \partial_k \Gamma^p_{ij} + \Gamma^q_{kj}\Gamma^p_{iq} - \Gamma^q_{ij}\Gamma^p_{kq} = 0,$$
on the one hand. On the other hand,
$$\sigma_j(0,\lambda) = \frac{\partial \zeta_j}{\partial \lambda}(0,\lambda) - \Gamma^p_{kj}(\mathbf{G}(0,\lambda))\zeta_p(0,\lambda)\frac{\partial G^k}{\partial \lambda}(0,\lambda) = 0,$$
since $\zeta^0_j(0,\lambda) = \zeta^0_j$ and $\mathbf{G}(0,\lambda) = x^0$ for all $0 \le \lambda \le 1$. Therefore, for any *fixed* value of the parameter $\lambda \in [0,1]$, each function $\sigma_j(\cdot,\lambda)$ satisfies a *Cauchy problem for an ordinary differential equation*, viz.,
$$\begin{aligned} &\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}(t,\lambda) = \Gamma^q_{ij}(\mathbf{G}(t,\lambda))\frac{\partial G^i}{\partial t}(t,\lambda)\sigma_q(t,\lambda),\ 0 \le t \le 1,\\ &\sigma_j(0,\lambda) = 0. \end{aligned}$$

But the solution of such a Cauchy problem is unique; hence $\sigma_j(t,\lambda) = 0$ for all $0 \le t \le 1$. In particular then,
$$\begin{aligned} \sigma_j(1,\lambda) &= \frac{\partial \zeta_j}{\partial \pi}(1,\lambda) - \Gamma^p_{kj}(\mathbf{G}(1,\lambda))\zeta_p(1,\lambda)\frac{\partial G^k}{\partial \pi}(1,\lambda)\\ &= 0 \text{ for all } 0 \le \lambda \le 1, \end{aligned}$$

and thus $\dfrac{\partial \zeta_j}{\partial \lambda}(1, \lambda) = 0$ for all $0 \le \lambda \le 1$, since $\mathbf{G}(1, \lambda) = x^1$ for all $0 \le \lambda \le 1$.

For each integer ℓ, we may thus unambiguously define a vector field $(F_{\ell j}) : \Omega \to \mathbb{R}^3$ by letting

$$F_{\ell j}(x^1) := \zeta_j(1) \text{ for any } x^1 \in \Omega,$$

where $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^3)$ is any path joining x^0 to x^1 in Ω and the vector field $(\zeta_j) \in \mathcal{C}^1([0,1])$ is the solution to the Cauchy problem

$$\begin{aligned} \frac{\mathrm{d}\zeta_j}{\mathrm{d}t}(t) &= \Gamma^p_{ij}(\boldsymbol{\gamma}(t)) \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t) \zeta_p(t),\ 0 \le t \le 1, \\ \zeta_j(0) &= \zeta^0_j, \end{aligned}$$

corresponding to such a path.

To establish that such a vector field is indeed the ℓ-th row-vector field of the unknown matrix field we are seeking, we need to show that $(F_{\ell j})^3_{j=1} \in \mathcal{C}^1(\Omega; \mathbb{R}^3)$ and that this field does satisfy the partial differential equations $\partial_i F_{\ell j} = \Gamma^p_{ij} F_{\ell p}$ in Ω corresponding to the fixed integer ℓ used in the above Cauchy problem.

Let x be an arbitrary point in Ω and let the integer $i \in \{1, 2, 3\}$ be *fixed* in what follows. Then there exist $x^1 \in \Omega$, a path $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^3)$ joining x^0 to x^1, $\tau \in]0, 1[$, and an open interval $I \subset [0, 1]$ containing τ such that

$$\boldsymbol{\gamma}(t) = x + (t - \tau)\boldsymbol{e}_i \text{ for } t \in I,$$

where $\boldsymbol{e}_i$ is the i-th basis vector in $\mathbb{R}^3$. Since each function ζ_j is continuously differentiable in $[0, 1]$ and satisfies $\dfrac{\mathrm{d}\zeta_j}{\mathrm{d}t}(t) = \Gamma^p_{ij}(\boldsymbol{\gamma}(t)) \dfrac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t) \zeta_p(t)$ for all $0 \le t \le 1$, we have

$$\begin{aligned} \zeta_j(t) &= \zeta_j(\tau) + (t - \tau) \frac{\mathrm{d}\zeta_j}{\mathrm{d}t}(\tau) + o(t - \tau) \\ &= \zeta_j(\tau) + (t - \tau)\Gamma^p_{ij}(\boldsymbol{\gamma}(\tau))\zeta_p(\tau) + o(t - \tau) \end{aligned}$$

for all $t \in I$. Equivalently,

$$F_{\ell j}(x + (t - \tau)\boldsymbol{e}_i) = F_{\ell j}(x) + (t - \tau)\Gamma^p_{ij}(x) F_{\ell p}(x) + o(t - x).$$

This relation shows that each function $F_{\ell j}$ possesses partial derivatives in the set Ω, given at each $x \in \Omega$ by

$$\partial_i F_{\ell p}(x) = \Gamma^p_{ij}(x) F_{\ell p}(x).$$

Consequently, the matrix field $(F_{\ell j})$ is of class $\mathcal{C}^1$ in Ω (its partial derivatives are continuous in Ω) and it satisfies the partial differential equations $\partial_i F_{\ell j} = \Gamma^p_{ij} F_{\ell p}$ in Ω, as desired. Differentiating these equations shows that the matrix field $(F_{\ell j})$ is in fact of class $\mathcal{C}^2$ in Ω.

In order to conclude the proof of the theorem, it remains to adequately choose the initial values $F^0_{\ell j}$ at x^0 in step (ii).

(iii) *Let* Ω *be a connected and simply-connected open subset of* $\mathbb{R}^3$ *and let* $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *be a matrix field satisfying*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega,$$

the functions $\Gamma_{ijq}, \Gamma^p_{ij}$, *and* g^{pq} *being defined by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}), \quad \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}, \quad (g^{pq}) := (g_{ij})^{-1}.$$

Given an arbitrary point $x^0 \in \Omega$, *let* $(F^0_{\ell j}) \in \mathbb{S}^3_>$ *denote the square root of the matrix* $(g^0_{ij}) := (g_{ij}(x^0)) \in \mathbb{S}^3_>$.

Let $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ *denote the solution to the corresponding system*

$$\begin{aligned} \partial_i F_{\ell j}(x) &= \Gamma^p_{ij}(x) F_{\ell p}(x),\ x \in \Omega, \\ F_{\ell j}(x^0) &= F^0_{\ell j}, \end{aligned}$$

which exists and is unique by parts (i) *and* (ii). *Then there exists an immersion* $\boldsymbol{\Theta} = (\Theta_\ell) \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *such that*

$$\partial_j \Theta_\ell = F_{\ell j} \text{ and } g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \text{ in } \Omega.$$

To begin with, we show that *the three vector fields defined by*

$$\boldsymbol{g}_j := (F_{\ell j})^3_{\ell=1} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$$

satisfy

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} \text{ in } \Omega.$$

To this end, we note that, by construction, these fields satisfy

$$\begin{aligned} \partial_i \boldsymbol{g}_j &= \Gamma^p_{ij} \boldsymbol{g}_p \text{ in } \Omega, \\ \boldsymbol{g}_j(x^0) &= \boldsymbol{g}^0_j, \end{aligned}$$

where $\boldsymbol{g}^0_j$ is the j-th column vector of the matrix $(F^0_{\ell j}) \in \mathbb{S}^3_>$. Hence the matrix field $(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfies

$$\begin{aligned} \partial_k(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) &= \Gamma^m_{kj}(\boldsymbol{g}_m \cdot \boldsymbol{g}_i) + \Gamma^m_{ki}(\boldsymbol{g}_m \cdot \boldsymbol{g}_j) \text{ in } \Omega, \\ (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)(x^0) &= g^0_{ij}. \end{aligned}$$

The definitions of the functions Γ_{ijq} and Γ^p_{ij} imply that

$$\partial_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki} \text{ and } \Gamma_{ijq} = g_{pq}\Gamma^p_{ij}.$$

Hence the matrix field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfies

$$\begin{aligned} \partial_k g_{ij} &= \Gamma^m_{kj} g_{mi} + \Gamma^m_{ki} g_{mj} \text{ in } \Omega, \\ g_{ij}(x^0) &= g^0_{ij}. \end{aligned}$$

Viewed as a system of partial differential equations, together with initial values at x^0, with respect to the matrix field $(g_{ij}) : \Omega \to \mathbb{M}^3$, the above system can have *at most one solution* in the space $\mathcal{C}^2(\Omega; \mathbb{M}^3)$.

To see this, let $x^1 \in \Omega$ be distinct from x^0 and let $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^3)$ be any path joining x^0 to x^1 in Ω, as in part (ii). Then the nine functions $g_{ij}(\boldsymbol{\gamma}(t))$, $0 \le t \le 1$, satisfy a Cauchy problem for a linear system of nine ordinary differential equations and this system has *at most one* solution.

An inspection of the two above systems therefore shows that their solutions are identical, i.e., that $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij}$.

It remains to show that *there exists an immersion* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *such that*

$$\partial_i \boldsymbol{\Theta} = \boldsymbol{g}_i \text{ in } \Omega,$$

where $\boldsymbol{g}_i := (F_{\ell j})_{\ell=1}^3$.

Since the functions Γ_{ij}^p satisfy $\Gamma_{ij}^p = \Gamma_{ji}^p$, any solution $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ of the system

$$\begin{aligned} \partial_i F_{\ell j}(x) &= \Gamma_{ij}^p(x) F_{\ell p}(x), \ x \in \Omega, \\ F_{\ell j}(x^0) &= F_{\ell j}^0 \end{aligned}$$

satisfies

$$\partial_i F_{\ell j} = \partial_j F_{\ell i} \text{ in } \Omega.$$

The open set Ω being *simply-connected, Poincaré's theorem* (for a proof, see, e.g., Schwartz [1992, Vol. 2, Theorem 59 and Corollary 1, p. 234–235]) shows that, *for each integer* ℓ, there exists a function $\Theta_\ell \in \mathcal{C}^3(\Omega)$ such that

$$\partial_i \Theta_\ell = F_{\ell i} \text{ in } \Omega,$$

or, equivalently, such that the mapping $\boldsymbol{\Theta} := (\Theta_\ell) \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfies

$$\partial_i \boldsymbol{\Theta} = \boldsymbol{g}_i \text{ in } \Omega.$$

That $\boldsymbol{\Theta}$ is an immersion follows from the assumed invertibility of the matrices (g_{ij}). The proof is thus complete. □

Remarks. (1) The assumptions

$$\partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p - \Gamma_{ij}^\ell \Gamma_{k\ell}^p = 0 \text{ in } \Omega,$$

made in part (ii) on the functions $\Gamma_{ij}^p = \Gamma_{ji}^p$ are thus *sufficient* conditions for the equations $\partial_i F_{\ell j} = \Gamma_{ij}^p F_{\ell p}$ in Ω to have solutions. Conversely, a simple computation shows that they are also *necessary* conditions, simply expressing that, if these equations have a solution, then necessarily $\partial_{ik} F_{\ell j} = \partial_{ki} F_{\ell j}$ in Ω.

It is no surprise that these necessary conditions are of the same nature as those of Theorem 1.5-1, viz., $\partial_{ik} \boldsymbol{g}_j = \partial_{ij} \boldsymbol{g}_k$ in Ω.

(2) The assumed positive definiteness of the matrices (g_{ij}) is used only in part (iii), for defining *ad hoc* initial vectors $\boldsymbol{g}_i^0$. □

The definitions of the functions Γ_{ij}^p and Γ_{ijq} imply that the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}$$

satisfy, for all i, j, k, p,

$$\begin{aligned} &R_{qijk} = R_{jkqi} = -R_{qikj}, \\ &R_{qijk} = 0 \text{ if } j = k \text{ or } q = i. \end{aligned}$$

These relations in turn imply that *the eighty-one sufficient conditions*

$$R_{qijk} = 0 \text{ in } \Omega \text{ for all } i, j, k, q \in \{1, 2, 3\},$$

are satisfied if and only if the six relations

$$R_{1212} = R_{1213} = R_{1223} = R_{1313} = R_{1323} = R_{2323} = 0 \text{ in } \Omega$$

are satisfied (as is immediately verified, there are other sets of six relations that will suffice as well, again owing to the relations satisfied by the functions R_{qijk} for all i, j, k, q).

To conclude, we briefly review various *extensions of the fundamental existence result* of Theorem 1.6-1. First, a quick look at its proof reveals that it holds *verbatim* in *any dimension* $d \geq 2$, i.e., with $\mathbb{R}^3$ replaced by $\mathbb{R}^d$ and $\mathbf{E}^3$ by a d-dimensional Euclidean space $\mathbf{E}^d$. This extension only demands that Latin indices and exponents now range in the set $\{1, 2, \ldots, d\}$ and that the sets of matrices $\mathbb{M}^3, \mathbb{S}^3$, and $\mathbb{S}^3_>$ be replaced by their d-dimensional counterparts $\mathbb{M}^d, \mathbb{S}^d$, and $\mathbb{S}^d_>$.

The *regularity assumptions* on the components g_{ij} of the symmetric positive definite matrix field $\mathbf{C} = (g_{ij})$ made in Theorem 1.6-1, viz., that $g_{ij} \in \mathcal{C}^2(\Omega)$, can be significantly weakened. More specifically, C. Mardare [2003a] has shown that the existence theorem still holds if $g_{ij} \in \mathcal{C}^1(\Omega)$, with a resulting mapping $\mathbf{\Theta}$ in the space $\mathcal{C}^2(\Omega; \mathbf{E}^d)$; likewise, S. Mardare [2004] has shown that the existence theorem still holds if $g_{ij} \in W^{2,\infty}_{\mathrm{loc}}(\Omega)$, with a resulting mapping $\mathbf{\Theta}$ in the space $W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbf{E}^d)$. As expected, the sufficient conditions $R_{qijk} = 0$ in Ω of Theorem 1.6-1 are then assumed to hold only in the sense of distributions, viz., as

$$\int_\Omega \{-\Gamma_{ikq} \partial_j \varphi + \Gamma_{ijq} \partial_k \varphi + \Gamma_{ij}^p \Gamma_{kqp} \varphi - \Gamma_{ik}^p \Gamma_{jqp} \varphi\} \, dx = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$.

The existence result has also been extended "*up to the boundary of the set* Ω" by Ciarlet & C. Mardare [2004a]. More specifically, assume that the set Ω satisfies the "*geodesic property*" (in effect, a mild smoothness assumption on the boundary $\partial\Omega$, satisfied in particular if $\partial\Omega$ is Lipschitz-continuous) and that the functions g_{ij} and their partial derivatives of order ≤ 2 can be extended by continuity to the closure $\overline{\Omega}$, the symmetric matrix field extended in this fashion remaining positive-definite over the set $\overline{\Omega}$. Then the immersion $\mathbf{\Theta}$ and its partial derivatives of order ≤ 3 can be also extended by continuity to $\overline{\Omega}$.

Ciarlet & C. Mardare [2004a] have also shown that, if in addition the geodesic distance is equivalent to the Euclidean distance on Ω (a property stronger than the "geodesic property", but again satisfied if the boundary $\partial\Omega$ is Lipschitz-continuous), then a matrix field $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ with a Riemann curvature tensor vanishing in Ω can be extended to a matrix field $(\widetilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ defined on a connected open set $\widetilde{\Omega}$ containing $\overline{\Omega}$ and whose Riemann curvature tensor still vanishes in $\widetilde{\Omega}$. This result relies on the existence of continuous extensions to $\overline{\Omega}$ of the immersion $\mathbf{\Theta}$ and its partial derivatives of order ≤ 3 and on a deep extension theorem of Whitney [1934].

1.7 UNIQUENESS UP TO ISOMETRIES OF IMMERSIONS WITH THE SAME METRIC TENSOR

In Section 1.6, we have established the *existence* of an immersion $\mathbf{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$ giving rise to a set $\mathbf{\Theta}(\Omega)$ with a prescribed metric tensor, provided the given metric tensor field satisfies *ad hoc* sufficient conditions. We now turn to the question of *uniqueness* of such immersions.

This uniqueness result is the object of the next theorem, aptly called a **rigidity theorem** in view of its geometrical interpretation: It asserts that, if two immersions $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ share the same metric tensor field, then the set $\widetilde{\mathbf{\Theta}}(\Omega)$ is obtained by subjecting the set $\mathbf{\Theta}(\Omega)$ either to a *rotation* (together represented by an orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = 1$), or to a *symmetry with respect to a plane followed by a rotation* (together represented by an orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = -1$), then by subjecting the rotated set to a *translation* (represented by a vector $\boldsymbol{c}$).

Such a geometric transformation is called a **rigid deformation** of the set $\mathbf{\Theta}(\Omega)$, to remind that it indeed corresponds to the idea of a *"rigid"* one in $\mathbf{E}^3$. It is also an **isometry**, i.e., a transformation that preserves the distances.

Let $\mathbb{O}^3$ denote the set of all orthogonal matrices of order three.

Theorem 1.7-1. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ be two immersions such that their associated metric tensors satisfy*

$$\boldsymbol{\nabla}\mathbf{\Theta}^T \boldsymbol{\nabla}\mathbf{\Theta} = \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^T \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} \text{ in } \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and an orthogonal matrix $\mathbf{Q} \in \mathbb{O}^3$ such that

$$\mathbf{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\widetilde{\mathbf{\Theta}}(x) \text{ for all } x \in \Omega.$$

Proof. For convenience, *the three-dimensional vector space $\mathbb{R}^3$ is identified throughout this proof with the Euclidean space $\mathbf{E}^3$*. In particular then, $\mathbb{R}^3$ inherits the inner product and norm of $\mathbf{E}^3$. The *spectral norm* of a matrix $\mathbf{A} \in \mathbb{M}^3$ is denoted

$$|\mathbf{A}| := \sup\{|\mathbf{A}\boldsymbol{b}|;\ \boldsymbol{b} \in \mathbb{R}^3,\ |\boldsymbol{b}| = 1\}.$$

To begin with, we consider the *special case* where $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbf{E}^3 = \mathbb{R}^3$ is the *identity mapping*. The issue of uniqueness reduces in this case to finding $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ such that

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{I} \text{ for all } x \in \Omega.$$

Parts (i) to (iii) are devoted to solving these equations.

(i) We first establish that *a mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *that satisfies*

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{I} \text{ for all } x \in \Omega$$

is locally an isometry: Given any point $x^0 \in \Omega$*, there exists an open neighborhood* V *of* x^0 *contained in* Ω *such that*

$$|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| = |y - x| \text{ for all } x, y \in V.$$

Let B be an open ball centered at x^0 and contained in Ω. Since the set B is convex, the *mean-value theorem* (for a proof, see, e.g., Schwartz [1992]) can be applied. It shows that

$$|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| \le \sup_{z \in]x,y[} |\boldsymbol{\nabla}\boldsymbol{\Theta}(z)||y - x| \text{ for all } x, y \in B.$$

Since the spectral norm of an orthogonal matrix is one, we thus have

$$|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| \le |y - x| \text{ for all } x, y \in B.$$

Since the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x^0)$ is invertible, the *local inversion theorem* (for a proof, see, e.g., Schwartz [1992]) shows that there exist an open neighborhood V of x^0 contained in Ω and an open neighborhood $\widehat{V}$ of $\boldsymbol{\Theta}(x^0)$ in $\mathbf{E}^3$ such that the restriction of $\boldsymbol{\Theta}$ to V is a $\mathcal{C}^1$-diffeomorphism from V onto $\widehat{V}$. Besides, there is no loss of generality in assuming that V is contained in B and that $\widehat{V}$ is convex (to see this, apply the local inversion theorem first to the restriction of $\boldsymbol{\Theta}$ to B, thus producing a "first" neighborhood V' of x^0 contained in B, then to the restriction of the inverse mapping obtained in this fashion to an open ball $\widehat{V}$ centered at $\boldsymbol{\Theta}(x^0)$ and contained in $\boldsymbol{\Theta}(V')$).

Let $\boldsymbol{\Theta}^{-1} : \widehat{V} \to V$ denote the inverse mapping of $\boldsymbol{\Theta} : V \to \widehat{V}$. The chain rule applied to the relation $\boldsymbol{\Theta}^{-1}(\boldsymbol{\Theta}(x)) = x$ for all $x \in V$ then shows that

$$\widehat{\boldsymbol{\nabla}}\boldsymbol{\Theta}^{-1}(\widehat{x}) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^{-1} \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x), x \in V.$$

The matrix $\widehat{\boldsymbol{\nabla}}\boldsymbol{\Theta}^{-1}(\widehat{x})$ being thus orthogonal for all $\widehat{x} \in \widehat{V}$, the mean-value theorem applied in the convex set $\widehat{V}$ shows that

$$|\boldsymbol{\Theta}^{-1}(\widehat{y}) - \boldsymbol{\Theta}^{-1}(\widehat{x})| \le |\widehat{y} - \widehat{x}| \text{ for all } \widehat{x}, \widehat{y} \in \widehat{V},$$

or equivalently, that

$$|y - x| \le |\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| \text{ for all } x, y \in V.$$

The restriction of the mapping $\boldsymbol{\Theta}$ to the open neighborhood V of x^0 is thus an isometry.

(ii) We next establish that, *if a mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *is locally an isometry*, in the sense that, given any $x^0 \in \Omega$, there exists an open neighborhood V of x^0 contained in Ω such that $|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| = |y - x|$ for all $x, y \in V$, *then its derivative is locally constant*, in the sense that

$$\boldsymbol{\nabla\Theta}(x) = \boldsymbol{\nabla\Theta}(x^0) \text{ for all } x \in V.$$

The set V being that found in (i), let the differentiable function $F : V \times V \to \mathbb{R}$ be defined for all $x = (x_p) \in V$ and all $y = (y_p) \in V$ by

$$F(x, y) := (\Theta_\ell(y) - \Theta_\ell(x))(\Theta_\ell(y) - \Theta_\ell(x)) - (y_\ell - x_\ell)(y_\ell - x_\ell).$$

Then $F(x, y) = 0$ for all $x, y \in V$ by (i). Hence

$$G_i(x, y) := \frac{1}{2}\frac{\partial F}{\partial y_i}(x, y) = \frac{\partial \Theta_\ell}{\partial y_i}(y)(\Theta_\ell(y) - \Theta_\ell(x)) - \delta_{i\ell}(y_\ell - x_\ell) = 0$$

for all $x, y \in V$. For a fixed $y \in V$, each function $G_i(\cdot, y) : V \to \mathbb{R}$ is differentiable and its derivative vanishes. Consequently,

$$\frac{\partial G_i}{\partial x_j}(x, y) = -\frac{\partial \Theta_\ell}{\partial y_i}(y)\frac{\partial \Theta_\ell}{\partial x_j}(x) + \delta_{ij} = 0 \text{ for all } x, y \in V,$$

or equivalently, in matrix form,

$$\boldsymbol{\nabla\Theta}(y)^T \boldsymbol{\nabla\Theta}(x) = \mathbf{I} \text{ for all } x, y \in V.$$

Letting $y = x^0$ in this relation shows that

$$\boldsymbol{\nabla\Theta}(x) = \boldsymbol{\nabla\Theta}(x^0) \text{ for all } x \in V.$$

(iii) By (ii), the mapping $\boldsymbol{\nabla\Theta} : \Omega \to \mathbb{M}^3$ is differentiable and its derivative vanishes in Ω. Therefore *the mapping* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *is twice differentiable and its second Fréchet derivative vanishes in* Ω. The open set Ω being *connected*, a classical result from differential calculus (see, e.g., Schwartz [1992, Theorem 3.7.10]) shows that the mapping $\boldsymbol{\Theta}$ is *affine* in Ω, i.e., that there exists a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{M}^3$ such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{ox} \text{ for all } x \in \Omega.$$

Since $\mathbf{Q} = \boldsymbol{\nabla\Theta}(x^0)$ and $\boldsymbol{\nabla\Theta}(x^0)^T \boldsymbol{\nabla\Theta}(x^0) = \mathbf{I}$ by assumption, the matrix $\mathbf{Q}$ is *orthogonal*.

(iv) We now consider the general equations $g_{ij} = \widetilde{g}_{ij}$ in Ω, noting that they also read

$$\boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x) = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

Given any point $x^0 \in \Omega$, let the neighborhoods V of x^0 and $\widehat{V}$ of $\mathbf{\Theta}(x^0)$ and the mapping $\mathbf{\Theta}^{-1} : \widehat{V} \to V$ be defined as in part (i) (by assumption, the mapping $\mathbf{\Theta}$ is an immersion; hence the matrix $\boldsymbol{\nabla}\mathbf{\Theta}(x^0)$ is invertible).

Consider the composite mapping

$$\widehat{\mathbf{\Phi}} := \widetilde{\mathbf{\Theta}} \circ \mathbf{\Theta}^{-1} : \widehat{V} \to \mathbf{E}^3.$$

Clearly, $\widehat{\mathbf{\Phi}} \in \mathcal{C}^1(\widehat{V}; \mathbf{E}^3)$ and

$$\begin{aligned}\widehat{\boldsymbol{\nabla}}\widehat{\mathbf{\Phi}}(\widehat{x}) &= \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)\widehat{\boldsymbol{\nabla}}\mathbf{\Theta}^{-1}(\widehat{x}) \\ &= \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)\boldsymbol{\nabla}\mathbf{\Theta}(x)^{-1} \text{ for all } \widehat{x} = \mathbf{\Theta}(x), x \in V.\end{aligned}$$

Hence the assumed relations

$$\boldsymbol{\nabla}\mathbf{\Theta}(x)^T\boldsymbol{\nabla}\mathbf{\Theta}(x) = \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)^T\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x) \text{ for all } x \in \Omega$$

imply that

$$\widehat{\boldsymbol{\nabla}}\widehat{\mathbf{\Phi}}(\widehat{x})^T\widehat{\boldsymbol{\nabla}}\widehat{\mathbf{\Phi}}(\widehat{x}) = \mathbf{I} \text{ for all } x \in V.$$

By parts (i) to (iii), there thus exist a vector $\boldsymbol{c} \in \mathbb{R}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that

$$\widehat{\mathbf{\Phi}}(\widehat{x}) = \widetilde{\mathbf{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\mathbf{\Theta}(x) \text{ for all } \widehat{x} = \mathbf{\Theta}(x), x \in V,$$

hence such that

$$\mathbf{\Xi}(x) := \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)\boldsymbol{\nabla}\mathbf{\Theta}(x)^{-1} = \mathbf{Q} \text{ for all } x \in V.$$

The *continuous* mapping $\mathbf{\Xi} : V \to \mathbb{M}^3$ defined in this fashion is thus *locally constant* in Ω. As in part (iii), we conclude from the assumed *connectedness* of Ω that the mapping $\mathbf{\Xi}$ is *constant in* Ω. Thus the proof is complete. □

The special case where $\mathbf{\Theta}$ is the identity mapping of $\mathbb{R}^3$ identified with $\mathbf{E}^3$ is the classical **Liouville theorem**. This theorem thus asserts that if a mapping $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ is such that $\boldsymbol{\nabla}\mathbf{\Theta}(x) \in \mathbb{O}^3$ for all $x \in \Omega$ where Ω is an open connected subset of $\mathbb{R}^3$, then there exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3$ such that

$$\mathbf{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{ox} \ \text{ for all } x \in \Omega.$$

Two mappings $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ are said to be **isometrically equivalent** if there exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3$ such that $\widetilde{\mathbf{\Theta}} = \boldsymbol{c} + \mathbf{Q}\mathbf{\Theta}$ in Ω, i.e., such that $\widetilde{\mathbf{\Theta}} = \mathbf{J} \circ \mathbf{\Theta}$, where $\mathbf{J} := \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{id}$ is thus an *isometry*. Theorem 1.7-1 thus asserts that *two mappings* $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *and* $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *share the same metric tensor field over an open connected subset* Ω *of* $\mathbb{R}^3$ *if and only if they are isometrically equivalent.*

Remarks. (1) In terms of covariant components g_{ij} of metric tensors, parts (i) to (iii) of the above proof provide the solution to the equations $g_{ij} = \delta_{ij}$ in Ω, while part (iv) provides the solution to the equations $g_{ij} = \partial_i \widetilde{\boldsymbol{\Theta}} \cdot \partial_j \widetilde{\boldsymbol{\Theta}}$ in Ω, where $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ is a given immersion.

(2) The classical *Mazur-Ulam theorem* asserts the following: Let Ω be a connected subset in $\mathbb{R}^d$, and let $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^d$ be a mapping that satisfies

$$|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| = |y - x| \text{ for all } x, y \in \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbb{R}^d$ and an orthogonal matrix $\mathbf{Q}$ of order d such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{ox} \text{ for all } x \in \Omega.$$

Parts (ii) and (iii) of the above proof thus provide a proof of this theorem under the additional assumption that the mapping $\boldsymbol{\Theta}$ is of class $\mathcal{C}^1$ (the extension from $\mathbb{R}^3$ to $\mathbb{R}^d$ is trivial). $\square$

While the immersions $\boldsymbol{\Theta}$ found in Theorem 1.6-1 are thus only defined up to isometries in $\mathbf{E}^3$, they become *uniquely determined* if they are required to satisfy *ad hoc* additional conditions, according to the following corollary to Theorems 1.6-1 and 1.7-1.

Theorem 1.7-2. *Let the assumptions on the set Ω and on the matrix field $\mathbf{C}$ be as in* Theorem 1.6-1, *let a point $x_0 \in \Omega$ be given, and let $\mathbf{F}_0 \in \mathbb{M}^3$ be any matrix that satisfies*

$$\mathbf{F}_0^T \mathbf{F}_0 = \mathbf{C}(x_0).$$

Then there exists one and only one immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ that satisfies

$$\begin{gathered} \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega, \\ \boldsymbol{\Theta}(x_0) = \mathbf{0} \text{ and } \boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \mathbf{F}_0. \end{gathered}$$

Proof. Given any immersion $\boldsymbol{\Phi} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ that satisfies $\boldsymbol{\nabla}\boldsymbol{\Phi}(x)^T \boldsymbol{\nabla}\boldsymbol{\Phi}(x) = \mathbf{C}(x)$ for all $x \in \Omega$ (such immersions exist by Theorem 1.6-1), let the mapping $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\Theta}(x) := \mathbf{F}_0 \boldsymbol{\nabla}\boldsymbol{\Phi}(x_0)^{-1}(\boldsymbol{\Phi}(x) - \boldsymbol{\Phi}(x_0)) \text{ for all } x \in \Omega.$$

Then it is immediately verified that this mapping $\boldsymbol{\Theta}$ satisfies the announced properties.

Besides, it is uniquely determined. To see this, let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\Phi} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be two immersions that satisfy

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \boldsymbol{\nabla}\boldsymbol{\Phi}(x)^T \boldsymbol{\nabla}\boldsymbol{\Phi}(x) \text{ for all } x \in \Omega.$$

Hence there exist (by Theorem 1.7-1) $\boldsymbol{c} \in \mathbb{R}^3$ and $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Phi}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$, so that $\boldsymbol{\nabla}\boldsymbol{\Phi}(x) = \mathbf{Q}\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$. The relation $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \boldsymbol{\nabla}\boldsymbol{\Phi}(x_0)$ then implies that $\mathbf{Q} = \mathbf{I}$ and the relation $\boldsymbol{\Theta}(x_0) = \boldsymbol{\Phi}(x_0)$ in turn implies that $\boldsymbol{c} = \mathbf{0}$. $\square$

Remark. One possible choice for the matrix $\mathbf{F}_0$ is the square root of the symmetric positive-definite matrix $\mathbf{C}(x_0)$. □

Theorem 1.7-1 constitutes the "classical" rigidity theorem, in that both immersions $\boldsymbol{\Theta}$ and $\widetilde{\boldsymbol{\Theta}}$ are assumed to be in the space $\mathcal{C}^1(\Omega; \mathbf{E}^3)$. The next theorem is an extension, due to Ciarlet & C. Mardare [2003], that covers the case where *one of the mappings belongs to the Sobolev space* $H^1(\Omega; \mathbf{E}^3)$.

The way the result in part (i) of the next proof is derived is due to Friesecke, James & Müller [2002]; the result of part (i) itself goes back to Reshetnyak [1967].

Let $\mathbb{O}^3_+$ denote the set of all *rotations*, i.e., of all orthogonal matrices $\mathbf{Q} \in \mathbb{O}^3$ with $\det \mathbf{Q} = 1$.

Theorem 1.7-3. *Let Ω be a connected open subset of $\mathbb{R}^3$, let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ be a mapping that satisfies*

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0 \text{ in } \Omega,$$

and let $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ be a mapping that satisfies

$$\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \text{ a.e. in } \Omega \text{ and } \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} \text{ a.e. in } \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a rotation $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

Proof. The Euclidean space $\mathbf{E}^3$ is identified with the space $\mathbb{R}^3$ throughout the proof.

(i) *To begin with, we consider the special case where $\boldsymbol{\Theta} = \boldsymbol{id}_\Omega$.* In other words, we are given a mapping $\widetilde{\boldsymbol{\Theta}} \in \mathbf{H}^1(\Omega)$ that satisfies $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$. Hence

$$\mathbf{Cof}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) = (\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x))\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^{-T} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^{-T} \text{ for almost all } x \in \Omega,$$

on the one hand. Since, on the other hand,

$$\operatorname{div} \mathbf{Cof}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \mathbf{0} \text{ in } (\mathcal{D}'(B))^3$$

in any open ball B such that $\overline{B} \subset \Omega$ (to see this, combine the density of $\mathcal{C}^2(\overline{B})$ in $H^1(B)$ with the classical Piola identity in the space $\mathcal{C}^2(\overline{B})$; for a proof of this identity, see, e.g., Ciarlet [1988, Theorem 1.7.1]), we conclude that

$$\Delta\widetilde{\boldsymbol{\Theta}} = \operatorname{div} \mathbf{Cof}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \mathbf{0} \text{ in } (\mathcal{D}'(B))^3.$$

Hence $\widetilde{\boldsymbol{\Theta}} = (\widetilde{\Theta}_j) \in (\mathcal{C}^\infty(\Omega))^3$. For such mappings, the identity

$$\Delta(\partial_i\widetilde{\Theta}_j\partial_i\widetilde{\Theta}_j) = 2\partial_i\widetilde{\Theta}_j\partial_i(\Delta\widetilde{\Theta}_j) + 2\partial_{ik}\widetilde{\Theta}_j\partial_{ik}\widetilde{\Theta}_j,$$

together with the relations $\Delta\widetilde{\Theta}_j = 0$ and $\partial_i\widetilde{\Theta}_j\partial_i\widetilde{\Theta}_j = 3$ in Ω, shows that $\partial_{ik}\widetilde{\Theta}_j = 0$ in Ω. The assumed connectedness of Ω then implies that there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ (by assumption, $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$) such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{ox} \text{ for almost all } x \in \Omega.$$

(ii) *We consider next the general case.* Let $x_0 \in \Omega$ be given. Since $\boldsymbol{\Theta}$ is an immersion, the local inversion theorem can be applied; there thus exist bounded open neighborhoods U of x_0 and $\widehat{U}$ of $\boldsymbol{\Theta}(x_0)$ satisfying $\overline{U} \subset \Omega$ and $\{\widehat{U}\}^- \subset \boldsymbol{\Theta}(\Omega)$, such that the restriction $\boldsymbol{\Theta}_U$ of $\boldsymbol{\Theta}$ to U can be extended to a $\boldsymbol{\mathcal{C}}^1$-diffeomorphism from $\overline{U}$ onto $\{\widehat{U}\}^-$.

Let $\boldsymbol{\Theta}_U^{-1} : \widehat{U} \to U$ denote the inverse mapping of $\boldsymbol{\Theta}_U$, which therefore satisfies $\widehat{\boldsymbol{\nabla}}\boldsymbol{\Theta}_U^{-1}(\widehat{x}) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^{-1}$ for all $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{U}$ (the notation $\widehat{\boldsymbol{\nabla}}$ indicates that differentiation is carried out with respect to the variable $\widehat{x} \in \widehat{U}$). Define the composite mapping

$$\widehat{\boldsymbol{\Phi}} := \widetilde{\boldsymbol{\Theta}} \cdot \boldsymbol{\Theta}_U^{-1} : \widehat{U} \to \mathbb{R}^3.$$

Since $\widetilde{\boldsymbol{\Theta}} \in \mathbf{H}^1(U)$ and $\boldsymbol{\Theta}_U^{-1}$ can be extended to a $\mathcal{C}^1$-diffeomorphism from $\{\widehat{U}\}^-$ onto $\overline{U}$, it follows that $\widehat{\boldsymbol{\Phi}} \in H^1(\widehat{U};\mathbb{R}^3)$ and that

$$\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\widehat{x}) = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)\widehat{\boldsymbol{\nabla}}\boldsymbol{\Theta}_U^{-1}(\widehat{x}) = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^{-1}$$

for almost all $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{U}$ (see, e.g., Adams [1975, Chapter 3]). Hence the assumptions $\det\boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ in Ω, $\det\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0$ a.e. in Ω, and $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}$ a.e. in Ω, together imply that $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\widehat{x}) \in \mathbb{O}^3_+$ for almost all $\widehat{x} \in \widehat{U}$. By (i), there thus exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widehat{\boldsymbol{\Phi}}(\widehat{x}) = \widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{o}\widehat{\boldsymbol{x}} \text{ for almost all } \widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{U},$$

or equivalently, such that

$$\boldsymbol{\Xi}(x) := \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^{-1} = \mathbf{Q} \text{ for almost all } x \in U.$$

Since the point $x_0 \in \Omega$ is arbitrary, this relation shows that $\boldsymbol{\Xi} \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$. By a classical result from distribution theory (cf. Schwartz [1966, Section 2.6]), we conclude from the assumed connectedness of Ω that $\boldsymbol{\Xi}(x) = \mathbf{Q}$ for almost all $x \in \Omega$, and consequently that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

□

Remarks. (1) The existence of $\widetilde{\mathbf{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ satisfying the assumptions of Theorem 1.7-3 thus implies that $\mathbf{\Theta} \in H^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$.

(2) If $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$, the assumptions $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω and $\det \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} > 0$ in Ω are no longer necessary; but then it can only be concluded that $\mathbf{Q} \in \mathbb{O}^3$: This is the *classical rigidity theorem* (Theorem 1.7-1), of which *Liouville's theorem* is the special case corresponding to $\mathbf{\Theta} = \boldsymbol{id}_\Omega$.

(3) The result established in part (i) of the above proof asserts that, given a connected open subset Ω of $\mathbb{R}^3$, if a mapping $\mathbf{\Theta} \in H^1(\Omega; \mathbf{E}^3)$ is such that $\boldsymbol{\nabla}\mathbf{\Theta}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$, then there exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3_+$ such that $\mathbf{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{ox}$ for almost all $x \in \Omega$. This result thus constitutes a *generalization of Liouville's theorem.*

(4) By contrast, if the mapping $\widetilde{\mathbf{\Theta}}$ is assumed to be instead in the space $H^1(\Omega; \mathbf{E}^3)$ (as in Theorem 1.7-3), an assumption about the sign of $\det \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}$ becomes necessary. To see this, let for instance Ω be an open ball centered at the origin in $\mathbb{R}^3$, let $\mathbf{\Theta}(x) = x$, and let $\widetilde{\mathbf{\Theta}}(x) = x$ if $x_1 \geq 0$ and $\widetilde{\mathbf{\Theta}}(x) = (-x_1, x_2, x_3)$ if $x_1 < 0$. Then $\widetilde{\mathbf{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} \in \mathbb{O}^3$ a.e. in Ω; yet there does not exist any orthogonal matrix such that $\widetilde{\mathbf{\Theta}}(x) = \mathbf{Q}\;\boldsymbol{ox}$ for all $x \in \Omega$, since $\widetilde{\mathbf{\Theta}}(\Omega) \subset \{x \in \mathbb{R}^3;\, x_1 \geq 0\}$ (this counter-example was kindly communicated to the author by Sorin Mardare).

(5) Surprisingly, the assumption $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω *cannot* be replaced by the weaker assumption $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ a.e. in Ω. To see this, let for instance Ω be an open ball centered at the origin in $\mathbb{R}^3$, let $\mathbf{\Theta}(x) = (x_1 x_2^2, x_2, x_3)$ and let $\widetilde{\mathbf{\Theta}}(x) = \mathbf{\Theta}(x)$ if $x_2 \geq 0$ and $\widetilde{\mathbf{\Theta}}(x) = (-x_1 x_2^2, -x_2, x_3)$ if $x_2 < 0$ (this counter-example was kindly communicated to the author by Hervé Le Dret).

(6) If a mapping $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ satisfies $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω, then $\mathbf{\Theta}$ is an immersion. Conversely, if Ω is a connected open set and $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ is an immersion, then either $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω or $\det \boldsymbol{\nabla}\mathbf{\Theta} < 0$ in Ω. The assumption that $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω made in Theorem 1.7-3 is simply intended to fix ideas (a similar result clearly holds under the other assumption).

(7) A little further ado shows that the conclusion of Theorem 1.7-3 is still valid if $\widetilde{\mathbf{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ is replaced by the weaker assumption $\widetilde{\mathbf{\Theta}} \in H^1_{\text{loc}}(\Omega; \mathbf{E}^3)$. $\square$

Like the existence results of Section 1.6, the uniqueness theorems of this section hold *verbatim* in any dimension $d \geq 2$, with $\mathbb{R}^3$ replaced by $\mathbb{R}^d$ and $\mathbf{E}^d$ by a d-dimensional Euclidean space.

1.8 CONTINUITY OF AN IMMERSION AS A FUNCTION OF ITS METRIC TENSOR

Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Together, Theorems 1.6-1 and 1.7-1 establish the existence of a mapping $\mathcal{F}$ that associates with any matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfying

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega,$$

where the functions Γ_{ijq} and Γ_{ij}^p are defined in terms of the functions g_{ij} as in Theorem 1.6-1, a well-defined element $\mathcal{F}(\mathbf{C})$ in the quotient set $\mathcal{C}^3(\Omega;\mathbf{E}^3)/\mathcal{R}$, where $(\mathbf{\Theta},\widetilde{\mathbf{\Theta}}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{a} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\mathbf{\Theta}(x) = \boldsymbol{a} + \mathbf{Q}\widetilde{\mathbf{\Theta}}(x)$ for all $x \in \Omega$.

A natural question thus arises as to whether there exist natural topologies on the space $\mathcal{C}^2(\Omega;\mathbb{S}^3)$ and on the quotient set $\mathcal{C}^3(\Omega;\mathbf{E}^3)/\mathcal{R}$ such that the mapping $\mathcal{F}$ defined in this fashion is *continuous*.

Equivalently, *is an immersion a continuous function of its metric tensor?*

The object of this section, which is based on Ciarlet & Laurent [2003], is to provide an affirmative answer to this question (see Theorem 1.8-5).

Note that such a question is not only clearly relevant to differential geometry *per se*, but it also naturally arises in *nonlinear three-dimensional elasticity*, where a smooth enough immersion $\mathbf{\Theta} : \Omega \to \mathbf{E}^3$ may be thought of as a *deformation* of the set Ω viewed as a *reference configuration* of a *nonlinearly elastic body* (although such an immersion should then be in addition injective and orientation-preserving in order to qualify for this definition; for details, see, e.g., Ciarlet [1988, Section 1.4] or Antman [1995, Section 12.1]). In this context, the associated matrix

$$\mathbf{C}(x) = (g_{ij}(x)) = \boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x),$$

is called the (right) *Cauchy-Green tensor at x* and the matrix

$$\boldsymbol{\nabla}\mathbf{\Theta}(x) = (\partial_j \Theta_i(x)) \in \mathbb{M}^3,$$

representing the Fréchet derivative of the mapping $\mathbf{\Theta}$ at x, is called the *deformation gradient at x*.

The Cauchy-Green tensor field $\mathbf{C} = \boldsymbol{\nabla}\mathbf{\Theta}^T\boldsymbol{\nabla}\mathbf{\Theta} : \Omega \to \mathbb{S}^3_>$ associated with a deformation $\mathbf{\Theta} : \Omega \to \mathbf{E}^3$ plays a major role in the theory of nonlinear three-dimensional elasticity, since the response function, or the stored energy function, of a frame-indifferent elastic, or hyperelastic, material necessarily depends on the deformation gradient through the Cauchy-Green tensor (see, e.g., Ciarlet [1988, Chapters 3 and 4]. As already suggested by Antman [1976], the Cauchy-Green tensor field of the *unknown deformed configuration* could thus also be regarded as the "*primary*" *unknown* rather than the deformation itself as is customary.

To begin with, we list some specific notations that will be used in this section for addressing the question raised above. Given a matrix $\mathbf{A} \in \mathbb{M}^3$, we let $\rho(\mathbf{A})$ denote its spectral radius and we let

$$|\mathbf{A}| := \sup_{\begin{cases} \boldsymbol{b} \in \mathbb{R}^3 \\ \boldsymbol{b} \neq \mathbf{0} \end{cases}} \frac{|\mathbf{A}\boldsymbol{b}|}{|\boldsymbol{b}|} = \{\rho(\mathbf{A}^T\mathbf{A})\}^{1/2}$$

denote its spectral norm.

Let Ω be an open subset of $\mathbb{R}^3$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If $g \in \mathcal{C}^\ell(\Omega;\mathbb{R}), \ell \geq 0$, and $K \Subset \Omega$, we define the *semi-norms*

$$|g|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|=\ell \end{cases}} |\partial^\alpha g(x)| \quad \text{and} \quad \|g\|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|\leq\ell \end{cases}} |\partial^\alpha g(x)|,$$

where ∂^α stands for the standard multi-index notation for partial derivatives. If $\boldsymbol{\Theta} \in \mathcal{C}^\ell(\Omega;\mathbf{E}^3)$ or $\mathbf{A} \in \mathcal{C}^\ell(\Omega;\mathbb{M}^3)$, $\ell \geq 0$, and $K \Subset \Omega$, we likewise set

$$|\boldsymbol{\Theta}|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|=\ell \end{cases}} |\partial^\alpha \boldsymbol{\Theta}(x)| \quad \text{and} \quad \|\boldsymbol{\Theta}\|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|\leq\ell \end{cases}} |\partial^\alpha \boldsymbol{\Theta}(x)|,$$

$$|\mathbf{A}|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|=\ell \end{cases}} |\partial^\alpha \mathbf{A}(x)| \quad \text{and} \quad \|\mathbf{A}\|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|\leq\ell \end{cases}} |\partial^\alpha \mathbf{A}(x)|,$$

where $|\cdot|$ denotes either the Euclidean vector norm or the matrix spectral norm.

The next sequential continuity results (Theorems 1.8-1, 1.8-2, and 1.8-3) constitute key steps toward establishing the continuity of the mapping $\mathcal{F}$ (see Theorem 1.8-5). Note that the functions R^n_{qijk} occurring in their statements are meant to be constructed from the functions g^n_{ij} in the same way that the functions R_{qijk} are constructed from the functions g_{ij}. To begin with, we establish the sequential continuity of the mapping $\mathcal{F}$ at $\mathbf{C} = \mathbf{I}$.

Theorem 1.8-1. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega;\mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{I}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega;\mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{id}\|_{3,K} = 0 \text{ for all } K \Subset \Omega$$

where $\boldsymbol{id}$ denotes the identity mapping of $\mathbb{R}^3$, identified here with $\mathbf{E}^3$.

Proof. The proof is broken into four parts, numbered (i) to (iv). The first part is a preliminary result about matrices (for convenience, it is stated here for matrices of order three, but it holds as well for matrices of arbitrary order).

(i) *Let matrices $\mathbf{A}^n \in \mathbb{M}^3$, $n \geq 0$, satisfy*

$$\lim_{n\to\infty} (\mathbf{A}^n)^T\mathbf{A}^n = \mathbf{I}.$$

Then there exist matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, that satisfy

$$\lim_{n\to\infty} \mathbf{Q}^n\mathbf{A}^n = \mathbf{I}.$$

Since the set $\mathbb{O}^3$ is compact, there exist matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$|\mathbf{Q}^n \mathbf{A}^n - \mathbf{I}| = \inf_{\mathbf{R} \in \mathbb{O}^3} |\mathbf{R}\mathbf{A}^n - \mathbf{I}|.$$

We assert that the matrices $\mathbf{Q}^n$ defined in this fashion satisfy $\lim_{n\to\infty} \mathbf{Q}^n \mathbf{A}^n = \mathbf{I}$. For otherwise, there would exist a subsequence $(\mathbf{Q}^p)_{p\geq 0}$ of the sequence $(\mathbf{Q}^n)_{n\geq 0}$ and $\delta > 0$ such that

$$|\mathbf{Q}^p \mathbf{A}^p - \mathbf{I}| = \inf_{\mathbf{R} \in \mathbb{O}^3} |\mathbf{R}\mathbf{A}^p - \mathbf{I}| \geq \delta \text{ for all } p \geq 0.$$

Since

$$\lim_{p\to\infty} |\mathbf{A}^p| = \lim_{p\to\infty} \sqrt{\rho((\mathbf{A}^p)^T \mathbf{A}^p)} = \sqrt{\rho(\mathbf{I})} = 1,$$

the sequence $(\mathbf{A}^p)_{p\geq 0}$ is bounded. Therefore there exists a further subsequence $(\mathbf{A}^q)_{q\geq 0}$ that converges to a matrix $\mathbf{S}$, which is orthogonal since

$$\mathbf{S}^T\mathbf{S} = \lim_{q\to\infty} (\mathbf{A}^q)^T \mathbf{A}^q = \mathbf{I}.$$

But then

$$\lim_{q\to\infty} \mathbf{S}^T \mathbf{A}^q = \mathbf{S}^T\mathbf{S} = \mathbf{I},$$

which contradicts $\inf_{\mathbf{R}\in\mathbb{O}^3} |\mathbf{R}\mathbf{A}^q - \mathbf{I}| \geq \delta$ for all $q \geq 0$. This proves (i).

In the remainder of this proof, the matrix fields $\mathbf{C}^n, n \geq 0$, are meant to be those appearing in the statement of Theorem 1.8-1.

(ii) *Let mappings* $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, *satisfy* $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ *in* Ω (such mappings exist by Theorem 1.6-1). *Then*

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{\ell,K} = 0 \textit{ for all } K \Subset \Omega \textit{ and for } \ell = 2, 3.$$

As usual, given any immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, let $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}$, let $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$, and let the vectors $\boldsymbol{g}^q$ be defined by the relations $\boldsymbol{g}_i \cdot \boldsymbol{g}^q = \delta_i^q$. It is then immediately verified that

$$\partial_{ij}\boldsymbol{\Theta} = \partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})\boldsymbol{g}^q.$$

Applying this relation to the mappings $\boldsymbol{\Theta}^n$ thus gives

$$\partial_{ij}\boldsymbol{\Theta}^n = \frac{1}{2}(\partial_j g_{iq}^n + \partial_i g_{jq}^n - \partial_q g_{ij}^n)(\boldsymbol{g}^q)^n, \; n \geq 0,$$

where the vectors $(\boldsymbol{g}^q)^n$ are defined by means of the relations $\partial_i \boldsymbol{\Theta}^n \cdot (\boldsymbol{g}^q)^n = \delta_i^q$. Let K denote an arbitrary compact subset of Ω. On the one hand,

$$\lim_{n\to\infty} |\partial_j g_{iq}^n + \partial_i g_{jq}^n - \partial_q g_{ij}^n|_{0,K} = 0,$$

since $\lim_{n\to\infty} |g_{ij}^n|_{1,K} = \lim_{n\to\infty} |g_{ij}^n - \delta_{ij}|_{1,K} = 0$ by assumption. On the other hand, the norms $|(\boldsymbol{g}^q)^n|_{0,K}$ are bounded independently of $n \geq 0$; to see this, observe that $(\boldsymbol{g}^q)^n$ is the q-th column vector of the matrix $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^{-1}$, then that

$$\begin{aligned}|(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^{-1}|_{0,K} &= |\{\rho((\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^{-T}(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^{-1})\}^{1/2}|_{0,K} \\ &= |\{\rho((g_{ij}^n)^{-1})\}^{1/2}|_{0,K} \leq \{|(g_{ij}^n)^{-1}|_{0,K}\}^{1/2},\end{aligned}$$

and, finally, that

$$\lim_{n\to\infty} |(g_{ij}^n) - \mathbf{I}|_{0,K} = 0 \Longrightarrow \lim_{n\to\infty} |(g_{ij}^n)^{-1} - \mathbf{I}|_{0,K} = 0.$$

Consequently,

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{2,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Differentiating the relations $\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})$ yields

$$\begin{aligned}\partial_{ijp}\boldsymbol{\Theta} &= \partial_{ip}\boldsymbol{g}_j = (\partial_{ip}\boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q \\ &= \Big(\frac{1}{2}(\partial_{jp} g_{iq} + \partial_{ip} g_{jq} - \partial_{pq} g_{ij}) - \partial_i \boldsymbol{g}_j \cdot \partial_p \boldsymbol{g}_q\Big)\boldsymbol{g}^q.\end{aligned}$$

Observing that $\lim_{n\to\infty} |g_{ij}^n|_{\ell,K} = \lim_{n\to\infty} |g_{ij}^n - \delta_{ij}|_{\ell,K} = 0$ for $\ell = 1, 2$ by assumption and recalling that the norms $|(\boldsymbol{g}^q)^n|_{0,K}$ are bounded independently of $n \geq 0$, we likewise conclude that

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{3,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

(iii) *There exist mappings* $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *that satisfy* $(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \mathbf{C}^n$ *in* Ω, $n \geq 0$, *and*

$$\lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{id}|_{1,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be mappings that satisfy $(\boldsymbol{\nabla}\boldsymbol{\psi}^n)^T \boldsymbol{\nabla}\boldsymbol{\psi}^n = \mathbf{C}^n$ in Ω, $n \geq 0$ (such mappings exist by Theorem 1.6-1), and let x_0 denote a point in the set Ω. Since $\lim_{n\to\infty} \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0)^T \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \mathbf{I}$ by assumption, part (i) implies that there exist orthogonal matrices $\mathbf{Q}^n(x_0)$, $n \geq 0$, such that

$$\lim_{n\to\infty} \mathbf{Q}^n(x_0)\boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \mathbf{I}.$$

Then the mappings $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, defined by

$$\widetilde{\boldsymbol{\Theta}}^n(x) := \mathbf{Q}^n(x_0)\boldsymbol{\psi}^n(x), \; x \in \Omega,$$

satisfy

$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \mathbf{C}^n \text{ in } \Omega,$$

so that their gradients $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfy

$$\lim_{n\to\infty} |\partial_i \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n|_{0,K} = \lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n|_{2,K} = 0 \text{ for all } K \Subset \Omega,$$

by part (ii). In addition,

$$\lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x_0) = \lim_{n\to\infty} \mathbf{Q}^n \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \mathbf{I}.$$

Hence a classical theorem about the differentiability of the limit of a sequence of mappings that are continuously differentiable on a connected open set and that take their values in a Banach space (see, e.g., Schwartz [1992, Theorem 3.5.12]) shows that *the mappings* $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n$ *uniformly converge on every compact subset of* Ω *toward a limit* $\mathbf{R} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ *that satisfies*

$$\partial_i \mathbf{R}(x) = \lim_{n\to\infty} \partial_i \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x) = \mathbf{0} \text{ for all } x \in \Omega.$$

This shows that $\mathbf{R}$ is a constant mapping since Ω is connected. Consequently, $\mathbf{R} = \mathbf{I}$ since in particular $\mathbf{R}(x_0) = \lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x_0) = \mathbf{I}$. We have therefore established that

$$\lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{id}|_{1,K} = \lim_{n\to\infty} |\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n - \mathbf{I}|_{0,K} = 0 \text{ for all } K \Subset \Omega.$$

(iv) *There exist mappings* $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *satisfying* $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ *in* Ω, $n \geq 0$, and

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \text{ and for } \ell = 0, 1.$$

The mappings

$$\boldsymbol{\Theta}^n := \Big(\widetilde{\boldsymbol{\Theta}}^n - \{\widetilde{\boldsymbol{\Theta}}^n(x_0) - x_0\}\Big) \in \mathcal{C}^3(\Omega; \mathbf{E}^3),\ n \geq 0,$$

clearly satisfy

$$\begin{gathered}
(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n \text{ in } \Omega,\ n \geq 0,\\
\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{1,K} = \lim_{n\to\infty} |\boldsymbol{\nabla}\boldsymbol{\Theta}^n - \mathbf{I}|_{0,K} = 0 \text{ for all } K \Subset \Omega,\\
\boldsymbol{\Theta}^n(x_0) = x_0,\ n \geq 0.
\end{gathered}$$

Again applying the theorem about the differentiability of the limit of a sequence of mappings used in part (iii), we conclude from the last two relations that the mappings $\boldsymbol{\Theta}^n$ uniformly converge on every compact subset of Ω toward a limit $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ that satisfies

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \lim_{n\to\infty} \boldsymbol{\nabla}\boldsymbol{\Theta}^n(x) = \mathbf{I} \text{ for all } x \in \Omega.$$

This shows that $(\boldsymbol{\Theta} - \boldsymbol{id})$ is a constant mapping since Ω is connected. Consequently, $\boldsymbol{\Theta} = \boldsymbol{id}$ since in particular $\boldsymbol{\Theta}(x_0) = \lim_{n\to\infty} \boldsymbol{\Theta}^n(x_0) = x_0$. We have thus established that

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{0,K} = 0 \text{ for all } K \Subset \Omega.$$

This completes the proof of Theorem 1.8-1. $\square$

We next establish the sequential continuity of the mapping $\mathcal{F}$ at those matrix fields $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ that can be written as $\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ with an *injective* mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$.

Theorem 1.8-2. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying respectively $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Assume that there exists an injective mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \mathbf{C}$ in Ω. Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T\boldsymbol{\nabla\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. The assumptions made on the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$ imply that the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$ is open, connected, and simply-connected, and that the inverse mapping $\widehat{\boldsymbol{\Theta}} : \widehat{\Omega} \subset \mathbf{E}^3 \to \mathbb{R}^3$ belongs to the space $\mathcal{C}^3(\widehat{\Omega}; \mathbb{R}^3)$. Define the matrix fields $(\widehat{g}^n_{ij}) \in \mathcal{C}^2(\widehat{\Omega}; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$(\widehat{g}^n_{ij}(\widehat{x})) := \boldsymbol{\nabla\Theta}(x)^{-T}(g^n_{ij}(x))\boldsymbol{\nabla\Theta}(x)^{-1} \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}.$$

Given any compact subset $\widehat{K}$ of $\widehat{\Omega}$, let $K := \widehat{\boldsymbol{\Theta}}(\widehat{K})$. Since $\lim_{n\to\infty} \|g^n_{ij} - g_{ij}\|_{2,K} = 0$ because K is a compact subset of Ω, the definition of the functions $\widehat{g}^n_{ij} : \widehat{\Omega} \to \mathbb{R}$ and the chain rule together imply that

$$\lim_{n\to\infty} \|\widehat{g}^n_{ij} - \delta_{ij}\|_{2,\widehat{K}} = 0.$$

Given $\widehat{x} = (\widehat{x}_i) \in \widehat{\Omega}$, let $\widehat{\partial}_i = \partial/\partial\widehat{x}_i$. Let $\widehat{R}^n_{qijk}$ denote the functions constructed from the functions $\widehat{g}^n_{ij}$ in the same way that the functions R_{qijk} are constructed from the functions g_{ij}. Since it is easily verified that these functions satisfy $\widehat{R}^n_{qijk} = 0$ in $\widehat{\Omega}$, Theorem 1.8-1 applied over the set $\widehat{\Omega}$ shows that there exist mappings $\widehat{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\widehat{\Omega}; \mathbf{E}^3)$ satisfying

$$\widehat{\partial}_i\widehat{\boldsymbol{\Theta}}^n \cdot \widehat{\partial}_j\widehat{\boldsymbol{\Theta}}^n = \widehat{g}^n_{ij} \text{ in } \widehat{\Omega},\ n \geq 0,$$

such that

$$\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0 \text{ for all } \widehat{K} \Subset \widehat{\Omega},$$

where $\widehat{\boldsymbol{id}}$ denotes the identity mapping of $\mathbf{E}^3$, identified here with $\mathbb{R}^3$. Define the mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$\boldsymbol{\Theta}^n(x) = \widehat{\boldsymbol{\Theta}}^n(\widehat{x}) \text{ for all } x = \widehat{\boldsymbol{\Theta}}(\widehat{x}) \in \Omega.$$

Given any compact subset K of Ω, let $\widehat{K} := \boldsymbol{\Theta}(K)$. Since $\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0$, the definition of the mappings $\boldsymbol{\Theta}^n$ and the chain rule together imply that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0,$$

on the one hand. Since, on the other hand, $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, the proof is complete. $\square$

We are now in a position to establish the sequential continuity of the mapping $\mathcal{F}$ at *any* matrix field $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ that can be written as $\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ with $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; E^3)$.

Theorem 1.8-3. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields respectively satisfying $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω (such mappings exist by Theorem 1.6-1). *Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. The proof is broken into four parts. In what follows, $\mathbf{C}$ and $\mathbf{C}^n$ designate matrix fields possessing the properties listed in the statement of the theorem.

(i) *Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω. Then there exist a countable number of open balls $B_r \subset \Omega$, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^\infty B_r$ and such that, for each $r \geq 1$, the set $\bigcup_{s=1}^r B_s$ is connected and the restriction of $\boldsymbol{\Theta}$ to B_r is injective.*

Given any $x \in \Omega$, there exists an open ball $V_x \subset \Omega$ such that the restriction of $\boldsymbol{\Theta}$ to V_x is injective. Since $\Omega = \bigcup_{x\in\Omega} V_x$ can also be written as a countable union of compact subsets of Ω, there already exist countably many such open balls, denoted V_r, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^\infty V_r$.

Let $r_1 := 1$, $B_1 := V_{r_1}$, and $r_2 := 2$. If the set $B_{r_1} \cup V_{r_2}$ is connected, let $B_2 := V_{r_2}$ and $r_3 := 3$. Otherwise, there exists a path γ_1 in Ω joining the centers of V_{r_1} and V_{r_2} since Ω is connected. Then there exists a finite set $I_1 = \{r_1(1), r_1(2), \cdots, r_1(N_1)\}$ of integers, with $N_1 \geq 1$ and $2 < r_1(1) < r_1(2) < \cdots < r_1(N_1)$, such that

$$\gamma_1 \subset V_{r_1} \cup V_{r_2} \cup \Big(\bigcup_{r\in I_1} V_r \Big).$$

Furthermore there exists a permutation σ_1 of $\{1, 2, \dots, N_1\}$ such that the sets $V_{r_1} \cup (\bigcup_{s=1}^{r} V_{\sigma_1(s)})$, $1 \le r \le N_1$, and $V_{r_1} \cup (\bigcup_{s=1}^{N_1} V_{\sigma_1(s)}) \cup V_{r_2}$ are connected. Let

$$B_r := V_{\sigma_1(r-1)},\ 2 \le r \le N_1 + 1, \quad B_{N_1+2} := V_{r_2},$$
$$r_3 := \min\bigl\{i \in \{\sigma_1(1), \dots, \sigma_1(N_1)\};\ i \ge 3\bigr\}.$$

If the set $(\bigcup_{r=1}^{N_1+2} B_r) \cup V_{r_3}$ is connected, let $B_{N_1+3} := V_{r_3}$. Otherwise, apply the same argument as above to a path γ_2 in Ω joining the centers of V_{r_2} and V_{r_3}, and so forth.

The iterative procedure thus produces a countable number of open balls B_r, $r \ge 1$, that possess the announced properties. In particular, $\Omega = \bigcup_{r=1}^{\infty} B_r$ since, by construction, the integer r_i appearing at the i-th stage satisfies $r_i \ge i$.

(ii) By Theorem 1.8-2, there exist mappings $\boldsymbol{\Theta}_1^n \in \mathcal{C}^3(B_1; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\Theta}}_2^n \in \mathcal{C}^3(B_2; \mathbf{E}^3)$, $n \ge 0$, that satisfy

$$\begin{aligned} (\boldsymbol{\nabla}\boldsymbol{\Theta}_1^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_1^n &= \mathbf{C}^n \text{ in } B_1 \quad \text{and} \quad \lim_{n\to\infty} \|\boldsymbol{\Theta}_1^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1, \\ (\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_2^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_2^n &= \mathbf{C}^n \text{ in } B_2 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_2, \end{aligned}$$

and by Theorem 1.7-1, there exist vectors $\boldsymbol{c}^n \in \mathbf{E}^3$ and matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \ge 0$, such that

$$\widetilde{\boldsymbol{\Theta}}_2^n(x) = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\Theta}_1^n(x) \text{ for all } x \in B_1 \cap B_2.$$

Then we assert that

$$\lim_{n\to\infty} \boldsymbol{c}^n = \mathbf{0} \text{ and } \lim_{n\to\infty} \mathbf{Q}^n = \mathbf{I}.$$

Let $(\mathbf{Q}^p)_{p\ge0}$ be a subsequence of the sequence $(\mathbf{Q}^n)_{n\ge0}$ that converges to a (necessarily orthogonal) matrix $\mathbf{Q}$ and let x_1 denote a point in the set $B_1 \cap B_2$. Since $\boldsymbol{c}^p = \widetilde{\boldsymbol{\Theta}}_2^p(x_1) - \mathbf{Q}^p \boldsymbol{\Theta}_1(x_1)$ and $\lim_{n\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x_1) = \lim_{n\to\infty} \boldsymbol{\Theta}_1^p(x_1) = \boldsymbol{\Theta}(x_1)$, the subsequence $(\boldsymbol{c}^p)_{p\ge0}$ also converges. Let $\boldsymbol{c} := \lim_{p\to\infty} \boldsymbol{c}^p$. Thus

$$\begin{aligned} \boldsymbol{\Theta}(x) &= \lim_{p\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x) \\ &= \lim_{p\to\infty} (\boldsymbol{c}^p + \mathbf{Q}^p \boldsymbol{\Theta}_1^p(x)) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for all } x \in B_1 \cap B_2, \end{aligned}$$

on the one hand. On the other hand, the differentiability of the mapping $\boldsymbol{\Theta}$ implies that

$$\boldsymbol{\Theta}(x) = \boldsymbol{\Theta}(x_1) + \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)(x - x_1) + o(|x - x_1|) \text{ for all } x \in B_1 \cap B_2.$$

Note that $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)$ is an invertible matrix, since $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1) = (g_{ij}(x_1))$.

Let $\boldsymbol{b} := \boldsymbol{\Theta}(x_1)$ and $\mathbf{A} := \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)$. Together, the last two relations imply that

$$\boldsymbol{b} + \mathbf{A}(x - x_1) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{b} + \mathbf{Q}\mathbf{A}(x - x_1) + o(|x - x_1|),$$

and hence (letting $x = x_1$ shows that $\boldsymbol{b} = \boldsymbol{c} + \mathbf{Q}\boldsymbol{b}$) that

$$\mathbf{A}(x - x_1) = \mathbf{Q}\mathbf{A}(x - x_1) + o(|x - x_1|) \text{ for all } x \in B_1 \cap B_2.$$

The invertibility of $\mathbf{A}$ thus implies that $\mathbf{Q} = \mathbf{I}$ and therefore that $\boldsymbol{c} = \boldsymbol{b} - \mathbf{Q}\boldsymbol{b} = \mathbf{0}$. The uniqueness of these limits shows that the whole sequences $(\mathbf{Q}^n)_{n\geq 0}$ and $(\boldsymbol{c}^n)_{n\geq 0}$ converge.

(iii) *Let the mappings* $\boldsymbol{\Theta}_2^n \in \mathcal{C}^3(B_1 \cup B_2; \mathbf{E}^3)$, $n \geq 0$, *be defined by*

$$\boldsymbol{\Theta}_2^n(x) := \boldsymbol{\Theta}_1^n(x) \text{ for all } x \in B_1,$$
$$\boldsymbol{\Theta}_2^n(x) := (\mathbf{Q}^n)^T(\widetilde{\boldsymbol{\Theta}}_2^n(x) - \boldsymbol{c}^n) \text{ for all } x \in B_2.$$

Then

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_2^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_2^n = \mathbf{C}^n \text{ in } B_1 \cup B_2$$

(as is clear), *and*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1 \cup B_2.$$

The plane containing the intersection of the boundaries of the open balls B_1 and B_2 is the common boundary of two closed half-spaces in $\mathbb{R}^3$, H_1 containing the center of B_1, and H_2 containing that of B_2 (by construction, the set $B_1 \cup B_2$ is connected; see part (i)). Any compact subset K of $B_1 \cup B_2$ may thus be written as $K = K_1 \cup K_2$, where $K_1 := (K \cap H_1) \subset B_1$ and $K_2 := (K \cap H_2) \subset B_2$ (that the open sets found in part (i) may be chosen as *balls* thus play an essential rôle here). Hence

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_1} = 0 \text{ and } \lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0,$$

the second relation following from the definition of the mapping $\boldsymbol{\Theta}_2^n$ on $B_2 \supset K_2$ and on the relations $\lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$ (part (ii)) and $\lim_{n\to\infty} \mathbf{Q}^n = \mathbf{I}$ and $\lim_{n\to\infty} \boldsymbol{c}^n = \mathbf{0}$ (part (iii)).

(iv) It remains to iterate the procedure described in parts (ii) and (iii). For some $r \geq 2$, assume that mappings $\boldsymbol{\Theta}_r^n \in \mathcal{C}^3(\bigcup_{s=1}^r B_s; \mathbf{E}^3)$, $n \geq 0$, have been found that satisfy

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_r^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_r^n = \mathbf{C}^n \text{ in } \bigcup_{s=1}^r B_s,$$
$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_r^n - \boldsymbol{\Theta}\|_{2,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^r B_s.$$

Since the restriction of $\boldsymbol{\Theta}$ to B_{r+1} is injective (part (i)), Theorem 1.8-2 shows that there exist mappings $\widetilde{\boldsymbol{\Theta}}_{r+1}^n \in \mathcal{C}^3(B_{r+1}; \mathbf{E}^3)$, $n \geq 0$, that satisfy

$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{r+1}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{r+1}^n = \mathbf{C}^n \text{ in } B_{r+1},$$
$$\lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_{r+1}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_{r+1},$$

and since the set $\bigcup_{s=1}^{r+1} B_s$ is connected (part (i)), Theorem 1.7-1 shows that there exist vectors $\boldsymbol{c}^n \in \mathbf{E}^3$ and matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$\widetilde{\mathbf{\Theta}}_{r+1}^n(x) = \boldsymbol{c}^n + \mathbf{Q}^n \mathbf{\Theta}_r^n(x) \text{ for all } x \in \Big(\bigcup_{s=1}^{r} B_s\Big) \cap B_{r+1}.$$

Then an argument similar to that used in part (ii) shows that $\lim_{n\to\infty} \mathbf{Q}^n = \mathbf{I}$ and $\lim_{n\to\infty} \boldsymbol{c}^n = \mathbf{0}$, and an argument similar to that used in part (iii) (note that the ball B_{r+1} may intersect more than one of the balls B_s, $1 \leq s \leq r$) shows that the mappings $\mathbf{\Theta}_{r+1}^n \in \mathcal{C}^3(\bigcup_{s=1}^{r} B_s; \mathbf{E}^3)$, $n \geq 0$, defined by

$$\mathbf{\Theta}_{r+1}^n(x) := \mathbf{\Theta}_r^n(x) \text{ for all } x \in \bigcup_{s=1}^{r} B_s,$$
$$\mathbf{\Theta}_{r+1}^n(x) := (\mathbf{Q}^n)^T(\widetilde{\mathbf{\Theta}}_r^n(x) - \boldsymbol{c}^n) \text{ for all } x \in B_{r+1},$$

satisfy

$$\lim_{n\to\infty} \|\mathbf{\Theta}_{r+1}^n - \mathbf{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^{r} B_s.$$

Then the mappings $\mathbf{\Theta}^n : \Omega \to \mathbf{E}^3$, $n \geq 0$, defined by

$$\mathbf{\Theta}^n(x) := \mathbf{\Theta}_r^n(x) \text{ for all } x \in \bigcup_{s=1}^{r} B_s,\ r \geq 1,$$

possess all the required properties: They are unambiguously defined since for all $s > r$, $\mathbf{\Theta}_s^n(x) = \mathbf{\Theta}_r^n(x)$ for all $x \in \bigcup_{s=1}^{r} B_s$ by construction; they are of class $\mathcal{C}^3$ since the mappings $\mathbf{\Theta}_r^n : \bigcup_{s=1}^{r} B_s \to \mathbf{E}^3$ are themselves of class $\mathcal{C}^3$; they satisfy $(\boldsymbol{\nabla}\mathbf{\Theta}^n)^T \boldsymbol{\nabla}\mathbf{\Theta}^n = \mathbf{C}^n$ in Ω since the mappings $\mathbf{\Theta}_r^n$ satisfy the same relations in $\bigcup_{s=1}^{r} B_s$; and finally, they satisfy $\lim_{n\to\infty} \|\mathbf{\Theta}^n - \mathbf{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$ since any compact subset of Ω is contained in $\bigcup_{s=1}^{r} B_s$ for r large enough. This completes the proof. $\square$

It is easily seen that the assumptions $R_{qijk} = 0$ in Ω are in fact superfluous in Theorem 1.8-3 (as shown in the next proof, these relations are consequences of the assumptions $R_{qijk}^n = 0$ in $\Omega, n \geq 0$, and $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$ for all $K \Subset \Omega$). This observation gives rise to the following corollary to Theorem 1.8-3, in the form of another *sequential continuity result*, of interest by itself. The novelties are that the assumptions are now made on the immersions $\mathbf{\Theta}^n$, $n \geq 0$, and that this result also provides the existence of a "limit" immersion $\mathbf{\Theta}$.

Theorem 1.8-4. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let there be given immersions $\mathbf{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, and a matrix field $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ such that*

$$\lim_{n\to\infty} \|(\boldsymbol{\nabla}\mathbf{\Theta}^n)^T \boldsymbol{\nabla}\mathbf{\Theta}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Then there exist mappings $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, *of the form*

$$\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\Theta}^n, \ \boldsymbol{c}^n \in \mathbf{E}^3, \ \mathbf{Q}^n \in \mathbb{O}^3,$$

which thus satisfy $(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = (\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n$ *in* Ω *for all* $n \geq 0$, *and there exists a mapping* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *such that*

$$\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C} \text{ in } \Omega \text{ and } \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. Let the functions R^n_{qijk}, $n \geq 0$, and R_{qijk} be constructed from the components g^n_{ij} and g_{ij} of the matrix fields $\mathbf{C}^n := (\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n$ and $\mathbf{C}$ in the usual way (see, e.g., Theorem 1.6-1). Then $R^n_{qijk} = 0$ in Ω for all $n \geq 0$, since these relations are simply the necessary conditions of Theorem 1.5-1.

We now show that $R_{qijk} = 0$ in Ω. To this end, let K be any compact subset of Ω. The relations

$$\mathbf{C}^n = \mathbf{C}(\mathbf{I} + \mathbf{C}^{-1}(\mathbf{C}^n - \mathbf{C})), \ n \geq 0,$$

together with the inequalities $\|\mathbf{AB}\|_{2,K} \leq 4\|\mathbf{A}\|_{2,K}\|\mathbf{B}\|_{2,K}$ valid for any matrix fields $\mathbf{A}, \mathbf{B} \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$, show that there exists $n_0 = n_0(K)$ such that the matrix fields $(\mathbf{I} + \mathbf{C}^{-1}(\mathbf{C}^n - \mathbf{C}))(x)$ are invertible at all $x \in K$ for all $n \geq n_0$. The same relations also show that there exists a constant M such that $\|(\mathbf{C}^n)^{-1}\|_{2,K} \leq M$ for all $n \geq n_0$. Hence the relations

$$(\mathbf{C}^n)^{-1} - \mathbf{C}^{-1} = \mathbf{C}^{-1}(\mathbf{C} - \mathbf{C}^n)(\mathbf{C}^n)^{-1}, \ n \geq n_0,$$

together with the assumptions $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$, in turn imply that the components $g^{ij,n}$, $n \geq n_0$, and g^{ij} of the matrix fields $(\mathbf{C}^n)^{-1}$ and $\mathbf{C}^{-1}$ satisfy

$$\lim_{n\to\infty} \|g^{ij,n} - g^{ij}\|_{2,K} = 0.$$

With self-explanatory notations, it thus follows that

$$\lim_{n\to\infty} \|\Gamma^n_{ijq} - \Gamma_{ijq}\|_{1,K} = 0 \text{ and } \lim_{n\to\infty} \|\Gamma^{p,n}_{ij} - \Gamma^p_{ij}\|_{1,K} = 0,$$

hence that $\lim_{n\to\infty} \|R^n_{qijk} - R_{qijk}\|_{0,K} = 0$. This shows that $R_{qijk} = 0$ in K, hence that $R_{qijk} = 0$ in Ω since K is an arbitrary compact subset of Ω.

By the fundamental existence theorem (Theorem 1.6-1), there thus exists a mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω. Theorem 1.8-3 can now be applied, showing that there exist mappings $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that

$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \mathbf{C}^n \text{ in } \Omega, \ n \geq 0, \text{ and } \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{\Theta}\|_{3,K} \text{ for all } K \Subset \Omega.$$

Finally, the rigidity theorem (Theorem 1.7-1) shows that, for each $n \geq 0$, there exist $\boldsymbol{c}^n \in \mathbf{E}^3$ and $\mathbf{Q}^n \in \mathbb{O}^3$ such that $\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n\boldsymbol{\Theta}^n$ in Ω because the mappings $\widetilde{\boldsymbol{\Theta}}^n$ and $\boldsymbol{\Theta}^n$ share the same metric tensor field and the set Ω is connected. $\square$

It remains to show how the *sequential continuity* established in Theorem 1.8-3 implies the *continuity of a deformation as a function of its metric tensor* for *ad hoc* topologies.

Let Ω be an open subset of $\mathbb{R}^3$. For any integers $\ell \geq 0$ and $d \geq 1$, the space $\mathcal{C}^\ell(\Omega; \mathbb{R}^d)$ becomes a *locally convex topological space* when its topology is defined by the family of semi-norms $\|\cdot\|_{\ell,K}$, $K \Subset \Omega$, defined earlier. Then a sequence $(\boldsymbol{\Theta}^n)_{n\geq 0}$ converges to $\boldsymbol{\Theta}$ with respect to this topology if and only if

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega.$$

Furthermore, this topology is *metrizable*: Let $(K_i)_{i\geq 0}$ be any sequence of subsets of Ω that satisfy

$$K_i \Subset \Omega \text{ and } K_i \subset \operatorname{int} K_{i+1} \text{ for all } i \geq 0, \text{ and } \Omega = \bigcup_{i=0}^{\infty} K_i.$$

Then

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \Longleftrightarrow \lim_{n\to\infty} d_\ell(\boldsymbol{\Theta}^n, \boldsymbol{\Theta}) = 0,$$

where

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\Theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}.$$

For details, see, e.g., Yosida [1966, Chapter 1].

Let $\dot{\mathcal{C}}^3(\Omega; \mathbf{E}^3) := \mathcal{C}^3(\Omega; \mathbf{E}^3)/\mathcal{R}$ denote the quotient set of $\mathcal{C}^3(\Omega; \mathbf{E}^3)$ by the equivalence relation $\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$. Then it is easily verified that the set $\dot{\mathcal{C}}^3(\Omega; \mathbf{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\Theta}}, \dot{\boldsymbol{\psi}}) = \inf_{\begin{cases} \boldsymbol{\kappa} \in \dot{\boldsymbol{\Theta}} \\ \boldsymbol{\chi} \in \dot{\boldsymbol{\psi}} \end{cases}} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}) = \inf_{\begin{cases} \boldsymbol{c} \in \mathbf{E}^3 \\ \mathbf{Q} \in \mathbb{O}^3 \end{cases}} d_3(\boldsymbol{\Theta}, \boldsymbol{c} + \mathbf{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\Theta}}$ denotes the equivalence class of $\boldsymbol{\Theta}$ *modulo* $\mathcal{R}$.

We now show that the announced continuity of an immersion as a function of its metric tensor is a corollary to Theorem 1.8-1. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 1.8-5. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let*

$$\mathcal{C}^2_0(\Omega; \mathbb{S}^3_>) := \{(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>);\ R_{qijk} = 0 \text{ in } \Omega\},$$

and, given any matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2_0(\Omega; \mathbb{S}^3_>)$, let $\mathcal{F}(\mathbf{C}) \in \dot{\mathcal{C}}^3(\Omega; \mathbf{E}^3)$ denote the equivalence class modulo $\mathcal{R}$ *of any $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω. Then the mapping*

$$\mathcal{F} : \{\mathcal{C}^2_0(\Omega; \mathbb{S}^3_>); d_2\} \longrightarrow \{\dot{\mathcal{C}}^3(\Omega; \mathbf{E}^3); \dot{d}_3\}$$

defined in this fashion is continuous.

Proof. Since $\{\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>); d_2\}$ and $\{\dot{\mathcal{C}}^3(\Omega;\mathbf{E}^3); \dot{d}_3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through $\mathcal{F}$ into convergent sequences.

Let then $\mathbf{C} \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$ and $\mathbf{C}^n \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$, $n \geq 0$, be such that

$$\lim_{n\to\infty} d_2(\mathbf{C}^n, \mathbf{C}) = 0,$$

i.e., such that $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$ for all $K \Subset \Omega$. Given any $\boldsymbol{\Theta} \in \mathcal{F}(\mathbf{C})$, Theorem 1.8-3 shows that there exist $\boldsymbol{\Theta}^n \in \mathcal{F}(\mathbf{C}^n)$, $n \geq 0$, such that $\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$, i.e., such that

$$\lim_{n\to\infty} d_3(\boldsymbol{\Theta}^n, \boldsymbol{\Theta}) = 0.$$

Consequently,

$$\lim_{n\to\infty} \dot{d}_3(\mathcal{F}(\mathbf{C}^n), \mathcal{F}(\mathbf{C})) = 0.$$

$\square$

As shown by Ciarlet & C. Mardare [2004b], the above continuity result can be extended "*up to the boundary of the set* Ω", as follows. If Ω is bounded and has a Lipschitz-continuous boundary, the mapping $\mathcal{F}$ of Theorem 1.8-5 can be extended to a mapping that is locally Lipschitz-continuous with respect to the topologies of the Banach spaces $\mathcal{C}^2(\overline{\Omega};\mathbb{S}^3)$ for the continuous extensions of the symmetric matrix fields $\mathbf{C}$, and $\mathcal{C}^3(\overline{\Omega};\mathbf{E}^3)$ for the continuous extensions of the immersions $\boldsymbol{\Theta}$ (the existence of such continuous extensions is briefly commented upon at the end of Section 1.6).

Another extension, motivated by three-dimensional nonlinear elasticity, is the following: Let Ω be a bounded and connected subset of $\mathbb{R}^3$, and let $\mathcal{B}$ be an elastic body with Ω as its *reference configuration*. Thanks mostly to the landmark existence theory of Ball [1977], it is now customary in nonlinear three-dimensional elasticity to view any mapping $\boldsymbol{\Theta} \in H^1(\Omega;\mathbf{E}^3)$ that is almost-everywhere injective and satisfies $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ a.e. in Ω as a possible *deformation* of $\mathcal{B}$ when $\mathcal{B}$ is subjected to *ad hoc* applied forces and boundary conditions. The almost-everywhere injectivity of $\boldsymbol{\Theta}$ (understood in the sense of Ciarlet & Nečas [1987]) and the restriction on the sign of $\det \boldsymbol{\nabla}\boldsymbol{\Theta}$ mathematically express (in an arguably weak way) the *non-interpenetrability* and *orientation-preserving conditions* that any physically realistic deformation should satisfy.

As mentioned earlier, the *Cauchy-Green tensor field* $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} \in L^1(\Omega;\mathbb{S}^3)$ associated with a deformation $\boldsymbol{\Theta} \in H^1(\Omega;\mathbf{E}^3)$ pervades the mathematical modeling of three-dimensional nonlinear elasticity. Conceivably, an alternative approach to the existence theory in three-dimensional elasticity could thus regard *the Cauchy-Green tensor as the primary unknown*, instead of the deformation itself as is usually the case.

Clearly, *the Cauchy-Green tensors depend continuously on the deformations*, since the Cauchy-Schwarz inequality immediately shows that the mapping

$$\boldsymbol{\Theta} \in H^1(\Omega;\mathbf{E}^3) \to \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} \in L^1(\Omega;\mathbb{S}^3)$$

is continuous (irrespectively of whether the mappings $\boldsymbol{\Theta}$ are almost-everywhere injective and orientation-preserving).

Then Ciarlet & C. Mardare [2005a] have shown that, *under appropriate smoothness and orientation-preserving assumptions, the converse holds, i.e., the deformations depend continuously on their Cauchy-Green tensors, the topologies being those of the same spaces* $H^1(\Omega;\mathbf{E}^3)$ *and* $L^1(\Omega;\mathbb{S}^3)$ (by contrast with the orientation-preserving condition, the issue of non-interpenetrability turns out to be irrelevant to this issue). In fact, this continuity result holds in an arbitrary dimension d, at no extra cost in its proof; so it will be stated below in this more general setting. The notation $\mathbf{E}^d$ then denotes a d-dimensional Euclidean space and $\mathbb{S}^d$ denotes the space of all symmetric matrices of order d.

This continuity result is itself a simple consequence of a *nonlinear Korn inequality*, which constitutes the main result of *ibid.*: Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ with a Lipschitz-continuous boundary and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbf{E}^d)$ be a mapping satisfying $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$. Then there exists a constant $C(\boldsymbol{\Theta})$ with the following property: For each orientation-preserving mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbf{E}^d)$, there exist a $d \times d$ *rotation* $\mathbf{R} = \mathbf{R}(\boldsymbol{\Phi},\boldsymbol{\Theta})$ (i.e., an orthogonal matrix of order d with a determinant equal to one) and a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi},\boldsymbol{\Theta})$ in $\mathbf{E}^d$ such that

$$\|\boldsymbol{\Phi} - (\boldsymbol{b} + \mathbf{R}\boldsymbol{\Theta})\|_{H^1(\Omega;\mathbf{E}^d)} \leq C(\boldsymbol{\Theta})\|\boldsymbol{\nabla}\boldsymbol{\Phi}^T\boldsymbol{\nabla}\boldsymbol{\Phi} - \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^d)}.$$

That a vector $\boldsymbol{b}$ and a rotation $\mathbf{R}$ should appear in the left-hand side of such an inequality is of course reminiscent of the classical *rigidity theorem* (Theorem 1.7-1), which asserts that, if two mappings $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega;\mathbf{E}^d)$ and $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega;\mathbf{E}^d)$ satisfying $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0$ and $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ in an open connected subset Ω of $\mathbb{R}^d$ have the same Cauchy-Green tensor field, then the two mappings are *isometrically equivalent*, i.e., there exist a vector $\boldsymbol{b}$ in $\mathbf{E}^d$ and a $d \times d$ orthogonal matrix $\mathbf{R}$ (a rotation in this case) such that $\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \mathbf{R}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$.

More generally, we shall say that two orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega;\mathbf{E}^d)$ and $\boldsymbol{\Theta} \in H^1(\Omega;\mathbf{E}^d)$ are *isometrically equivalent* if there exist a vector $\boldsymbol{b}$ in $\mathbf{E}^d$ and a $d \times d$ orthogonal matrix $\mathbf{R}$ (again a rotation in this case) such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \mathbf{R}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

One application of the above key inequality is the following *sequential continuity property*: Let $\boldsymbol{\Theta}^k \in H^1(\Omega;\mathbf{E}^d)$, $k \geq 1$, and $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbf{E}^d)$ be orientation-preserving mappings. Then there exist a constant $C(\boldsymbol{\Theta})$ and orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}}^k \in H^1(\Omega;\mathbf{E}^d)$, $k \geq 1$, that are isometrically equivalent to $\boldsymbol{\Theta}^k$ such that

$$\|\widetilde{\boldsymbol{\Theta}}^k - \boldsymbol{\Theta}\|_{H^1(\Omega;\mathbf{E}^d)} \leq C(\boldsymbol{\Theta})\|(\boldsymbol{\nabla}\boldsymbol{\Theta}^k)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^k - \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^d)}.$$

Hence *the sequence* $(\widetilde{\boldsymbol{\Theta}}^k)_{k=1}^{\infty}$ *converges to* $\boldsymbol{\Theta}$ *in* $H^1(\Omega;\mathbf{E}^d)$ *as* $k \to \infty$ *if the sequence* $((\boldsymbol{\nabla}\boldsymbol{\Theta}^k)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^k)_{k=1}^{\infty}$ *converges to* $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta}$ *in* $L^1(\Omega;\mathbb{S}^d)$ *as* $k \to \infty$.

Should the Cauchy-Green strain tensor be viewed as the primary unknown (as suggested above), such a sequential continuity could thus prove to be useful when considering *infimizing sequences* of the total energy, in particular for handling the part of the energy that takes into account the applied forces and the boundary conditions, which are both naturally expressed in terms of the deformation itself.

They key inequality is first established in the special case where $\boldsymbol{\Theta}$ is the identity mapping of the set Ω, by making use in particular of a fundamental "*geometric rigidity lemma*" recently proved by Friesecke, James & Müller [2002]. It is then extended to an arbitrary mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$, thanks in particular to a methodology that bears some similarity with that used in Theorems 1.8-2 and 1.8-3.

Such results are to be compared with the earlier, pioneering estimates of John [1961], John [1972] and Kohn [1982], which implied *continuity at rigid body deformations*, i.e., at a mapping $\boldsymbol{\Theta}$ that is isometrically equivalent to the identity mapping of Ω. The recent and noteworthy continuity result of Reshetnyak [2003] for *quasi-isometric mappings* is in a sense complementary to the above one (it also deals with Sobolev type norms).

Chapter 2

DIFFERENTIAL GEOMETRY OF SURFACES

2.1 CURVILINEAR COORDINATES ON A SURFACE

In addition to the rules governing Latin indices that we set in Section 1.1, we henceforth require that *Greek* indices and exponents vary in the set $\{1, 2\}$ and that the summation convention be systematically used in conjunction with these rules. For instance, the relation

$$\partial_\alpha(\eta_i \boldsymbol{a}^i) = (\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + (\eta_{3|\alpha} + b_\alpha^\beta \eta_\beta)\boldsymbol{a}^3$$

means that, for $\alpha = 1, 2$,

$$\partial_\alpha\Big(\sum_{i=1}^{3}\eta_i \boldsymbol{a}^i\Big) = \sum_{\beta=1}^{2}(\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + \Big(\eta_{3|\alpha} + \sum_{\beta=1}^{2} b_\alpha^\beta \eta_\beta\Big)\boldsymbol{a}^3.$$

Kronecker's symbols are designated by $\delta_\alpha^\beta, \delta_{\alpha\beta}$, or $\delta^{\alpha\beta}$ according to the context.

Let there be given as in Section 1.1 a *three-dimensional Euclidean space* $\mathbf{E}^3$, equipped with an orthonormal basis consisting of three vectors $\widehat{\boldsymbol{e}}^i = \widehat{\boldsymbol{e}}_i$, and let $\boldsymbol{a} \cdot \boldsymbol{b}, |\boldsymbol{a}|$, and $\boldsymbol{a} \wedge \boldsymbol{b}$ denote the Euclidean inner product, the Euclidean norm, and the vector product of vectors $\boldsymbol{a}, \boldsymbol{b}$ in the space $\mathbf{E}^3$.

In addition, let there be given a *two-dimensional vector space*, in which two vectors $\boldsymbol{e}^\alpha = \boldsymbol{e}_\alpha$ form a basis. *This space will be identified with* $\mathbb{R}^2$. Let y_α denote the coordinates of a point $y \in \mathbb{R}^2$ and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

Finally, let there be given an *open* subset ω of $\mathbb{R}^2$ and a smooth enough mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ (specific smoothness assumptions on $\boldsymbol{\theta}$ will be made later, according to each context). The set

$$\widehat{\omega} := \boldsymbol{\theta}(\omega)$$

is called a **surface** in $\mathbf{E}^3$.

If the mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ is injective, each point $\widehat{y} \in \widehat{\omega}$ can be unambiguously written as 1

$$\widehat{y} = \boldsymbol{\theta}(y), \quad y \in \omega,$$

and the two coordinates y_α of y are called the **curvilinear coordinates** of $\widehat{y}$ (Figure 2.1-1). Well-known *examples* of surfaces and of curvilinear coordinates and their corresponding coordinate lines (defined in Section 2.2) are given in Figures 2.1-2 and 2.1-3.

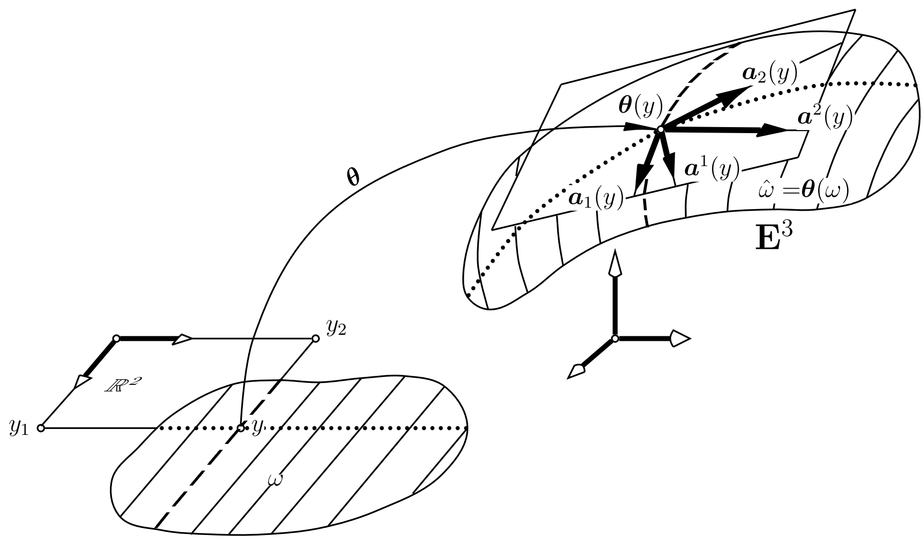

Figure 2.1-1: *Curvilinear coordinates on a surface and covariant and contravariant bases of the tangent plane.* Let $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ be a surface in $\mathbf{E}^3$. The two coordinates y_1, y_2 of $y \in \omega$ are the curvilinear coordinates of $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$. If the two vectors $\boldsymbol{a}_\alpha(y) = \partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent, they are tangent to the coordinate lines passing through $\widehat{y}$ and they form the covariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$. The two vectors $\boldsymbol{a}^\alpha(y)$ from this tangent plane defined by $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$ form its contravariant basis.

Naturally, once a surface $\widehat{\omega}$ is defined as $\widehat{\omega} = \boldsymbol{\theta}(\omega)$, *there are infinitely many other ways of defining curvilinear coordinates on* $\widehat{\omega}$, depending on how the domain ω and the mapping $\boldsymbol{\theta}$ are chosen. For instance, a portion $\widehat{\omega}$ of a sphere may be represented by means of *Cartesian coordinates, spherical coordinates*, or *stereographic coordinates* (Figure 2.1-3). Incidentally, this example illustrates the variety of restrictions that have to be imposed on $\widehat{\omega}$ according to which kind of curvilinear coordinates it is equipped with!

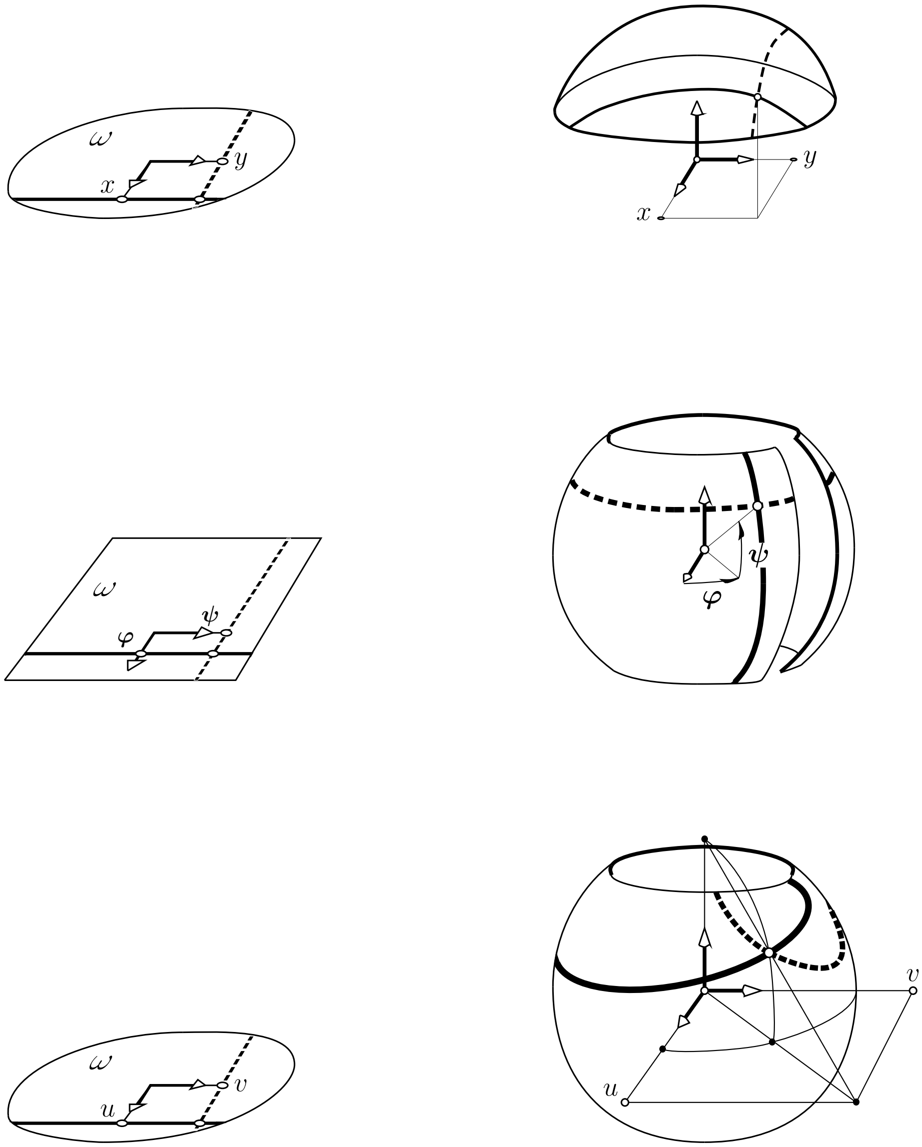

Figure 2.1-2: *Several systems of curvilinear coordinates on a sphere.* Let Σ be a sphere of radius R. A portion of Σ contained "in the northern hemisphere" can be represented by means of Cartesian coordinates, with a mapping $\boldsymbol{\theta}$ of the form:

$$\boldsymbol{\theta} : (x, y) \in \omega \to (x, y, \{R^2 - (x^2 + y^2)\}^{1/2}) \in \mathbf{E}^3.$$

A portion of Σ that excludes a neighborhood of both "poles" and of a "meridian" (to fix ideas) can be represented by means of spherical coordinates, with a mapping $\boldsymbol{\theta}$ of the form:

$$\boldsymbol{\theta} : (\varphi, \psi) \in \omega \to (R \cos\psi \cos\varphi, R\cos\psi \sin\varphi, R \sin\psi) \in \mathbf{E}^3.$$

A portion of Σ that excludes a neighborhood of the "North pole" can be represented by means of stereographic coordinates, with a mapping $\boldsymbol{\theta}$ of the form:

$$\boldsymbol{\theta} : (u, v) \in \omega \to \Big(\frac{2R^2 u}{u^2 + v^2 + R^2}, \frac{2R^2 v}{u^2 + v^2 + R^2}, R\frac{u^2 + v^2 - R^2}{u^2 + v^2 + R^2}\Big) \in \mathbf{E}^3.$$

The corresponding coordinate lines are represented in each case, with self-explanatory graphical conventions.

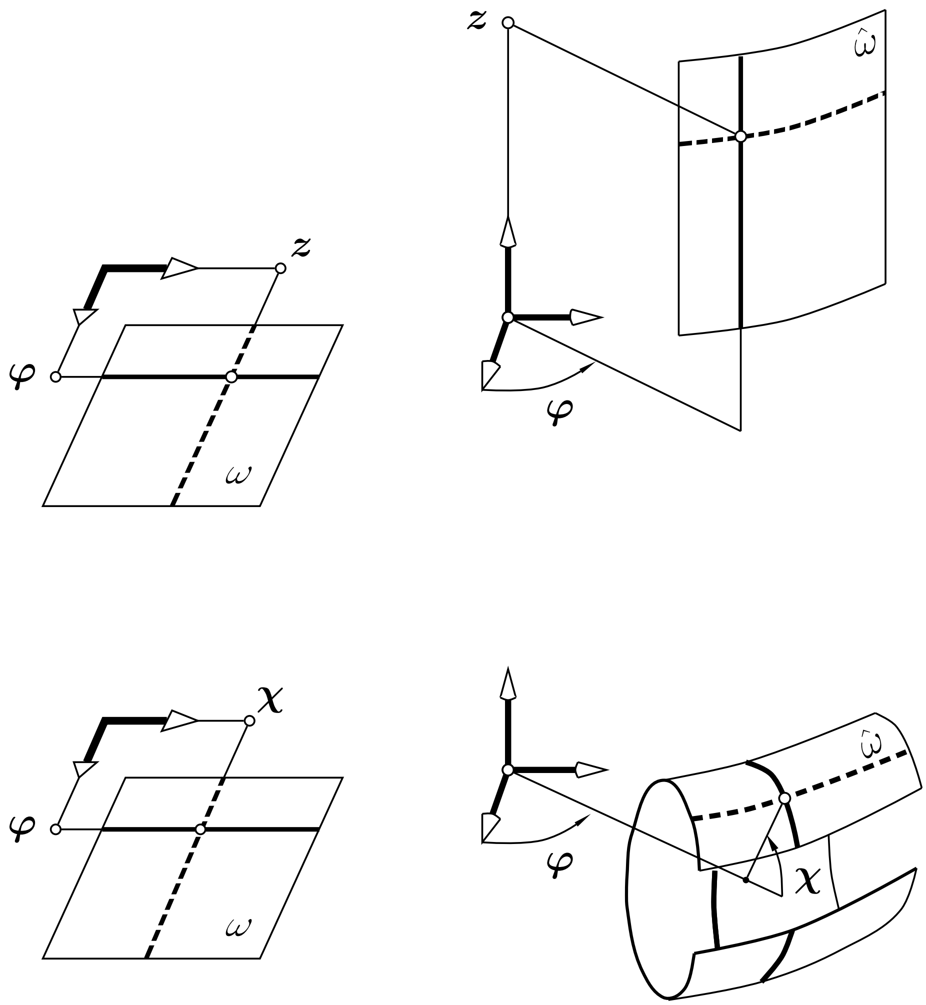

Figure 2.1-3: *Two familiar examples of surfaces and curvilinear coordinates.* A portion $\widehat{\omega}$ of a circular cylinder of radius R can be represented by a mapping $\boldsymbol{\theta}$ of the form

$$\boldsymbol{\theta} : (\varphi, z) \in \omega \to (R\cos\varphi, R\sin\varphi, z) \in \mathbf{E}^3.$$

A portion $\widehat{\omega}$ of a torus can be represented by a mapping $\boldsymbol{\theta}$ of the form

$$\boldsymbol{\theta} : (\varphi, \chi) \in \omega \to ((R + r\cos\chi)\cos\varphi, (R + r\cos\chi)\sin\varphi, r\sin\chi) \in \mathbf{E}^3,$$

with $R > r$.

The corresponding coordinate lines are represented in each case, with self-explanatory graphical conventions.

2.2 FIRST FUNDAMENTAL FORM

Let ω be an open subset of $\mathbb{R}^2$ and let

$$\boldsymbol{\theta} = \theta_i \widehat{\boldsymbol{e}}^i : \omega \subset \mathbb{R}^2 \to \boldsymbol{\theta}(\omega) = \widehat{\omega} \subset \mathbf{E}^3$$

be a mapping that is *differentiable at a point* $y \in \omega$. If $\boldsymbol{\delta y}$ is such that $(y+\boldsymbol{\delta y}) \in \omega$, then

$$\boldsymbol{\theta}(y + \boldsymbol{\delta y}) = \boldsymbol{\theta}(y) + \boldsymbol{\nabla\theta}(y)\boldsymbol{\delta y} + o(\boldsymbol{\delta y}),$$

where the 3×2 matrix $\boldsymbol{\nabla\theta}(y)$ is defined by

$$\boldsymbol{\nabla\theta}(y) := \begin{pmatrix} \partial_1\theta_1 & \partial_2\theta_1 \\ \partial_1\theta_2 & \partial_2\theta_2 \\ \partial_1\theta_3 & \partial_2\theta_3 \end{pmatrix}(y).$$

Let the two vectors $\boldsymbol{a}_\alpha(y) \in \mathbb{R}^3$ be defined by

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha\boldsymbol{\theta}(y) = \begin{pmatrix} \partial_\alpha\theta_1 \\ \partial_\alpha\theta_2 \\ \partial_\alpha\theta_3 \end{pmatrix}(y),$$

i.e., $\boldsymbol{a}_\alpha(y)$ *is the* α-th *column vector of the matrix* $\boldsymbol{\nabla\theta}(y)$ and let $\boldsymbol{\delta y} = \delta y^\alpha \boldsymbol{e}_\alpha$. Then the expansion of $\boldsymbol{\theta}$ about y may be also written as

$$\boldsymbol{\theta}(y + \boldsymbol{\delta y}) = \boldsymbol{\theta}(y) + \delta y^\alpha \boldsymbol{a}_\alpha(y) + o(\boldsymbol{\delta y}).$$

If in particular $\boldsymbol{\delta y}$ is of the form $\boldsymbol{\delta y} = \delta t \boldsymbol{e}_\alpha$, where $\delta t \in \mathbb{R}$ and $\boldsymbol{e}_\alpha$ is one of the basis vectors in $\mathbb{R}^2$, this relation reduces to

$$\boldsymbol{\theta}(y + \delta t \boldsymbol{e}_\alpha) = \boldsymbol{\theta}(y) + \delta t \boldsymbol{a}_\alpha(y) + o(\delta t).$$

A mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ is an **immersion at** $y \in \omega$ if it is differentiable at y and the 3×2 matrix $\boldsymbol{\nabla\theta}(y)$ is of rank two, or equivalently if the two vectors $\boldsymbol{a}_\alpha(y) = \partial_\alpha\boldsymbol{\theta}(y)$ are linearly independent.

Assume from now on in this section that *the mapping* $\boldsymbol{\theta}$ *is an immersion at* y. In this case, the last relation shows that *each vector* $\boldsymbol{a}_\alpha(y)$ *is tangent to the* α-th **coordinate line** *passing through* $\widehat{y} = \boldsymbol{\theta}(y)$, defined as the image by $\boldsymbol{\theta}$ of the points of ω that lie on a line parallel to $\boldsymbol{e}_\alpha$ passing through y (there exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the α-*th* coordinate line is given by $t \in \,]t_0, t_1[\, \to \boldsymbol{f}_\alpha(t) := \boldsymbol{\theta}(y + t\boldsymbol{e}_\alpha)$ in a neighborhood of $\widehat{y}$; hence $\boldsymbol{f}'_\alpha(0) = \partial_\alpha\boldsymbol{\theta}(y) = \boldsymbol{a}_\alpha(y)$); see Figures 2.1-1, 2.1-2, and 2.1-3.

The vectors $\boldsymbol{a}_\alpha(y)$, which thus span the *tangent plane* to the surface $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$, form the **covariant basis of the tangent plane** to $\widehat{\omega}$ at $\widehat{y}$; see Figure 2.1-1.

Returning to a general increment $\boldsymbol{\delta y} = \delta y^\alpha \boldsymbol{e}_\alpha$, we also infer from the expansion of $\boldsymbol{\theta}$ about y that

$$\begin{aligned} |\boldsymbol{\theta}(y + \boldsymbol{\delta y}) - \boldsymbol{\theta}(y)|^2 &= \boldsymbol{\delta y}^T \boldsymbol{\nabla\theta}(y)^T \boldsymbol{\nabla\theta}(y) \boldsymbol{\delta y} + o(|\boldsymbol{\delta y}|^2) \\ &= \delta y^\alpha \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) \delta y^\beta + o(|\boldsymbol{\delta y}|^2). \end{aligned}$$

In other words, the principal part with respect to $\boldsymbol{\delta y}$ of the length between the points $\boldsymbol{\theta}(y + \boldsymbol{\delta y})$ and $\boldsymbol{\theta}(y)$ is $\{\delta y^\alpha \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) \delta y^\beta\}^{1/2}$. This observation suggests to define a matrix $(a_{\alpha\beta}(y))$ of order two by letting

$$a_{\alpha\beta}(y) := \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \big(\boldsymbol{\nabla}\boldsymbol{\theta}(y)^T \boldsymbol{\nabla}\boldsymbol{\theta}(y)\big)_{\alpha\beta}.$$

The elements $a_{\alpha\beta}(y)$ of this symmetric matrix are called the **covariant components** of the **first fundamental form**, also called the **metric tensor**, *of the surface* $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$.

Note that *the matrix* $(a_{\alpha\beta}(y))$ *is positive definite* since the vectors $\boldsymbol{a}_\alpha(y)$ are assumed to be linearly independent.

The two vectors $\boldsymbol{a}_\alpha(y)$ being thus defined, *the four relations*

$$\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$$

unambiguously define two linearly independent vectors $\boldsymbol{a}^\alpha(y)$ *in the tangent plane*. To see this, let *a priori* $\boldsymbol{a}^\alpha(y) = Y^{\alpha\sigma}(y)\boldsymbol{a}_\sigma(y)$ in the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$. This gives $Y^{\alpha\sigma}(y)a_{\sigma\beta}(y) = \delta^\alpha_\beta$; hence $Y^{\alpha\sigma}(y) = a^{\alpha\sigma}(y)$, where

$$(a^{\alpha\beta}(y)) := (a_{\alpha\beta}(y))^{-1}.$$

Hence $\boldsymbol{a}^\alpha(y) = a^{\alpha\sigma}(y)\boldsymbol{a}_\sigma(y)$. These relations in turn imply that

$$\begin{aligned}\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}^\beta(y) &= a^{\alpha\sigma}(y)a^{\beta\tau}(y)\boldsymbol{a}_\sigma(y) \cdot \boldsymbol{a}_\tau(y) \\ &= a^{\alpha\sigma}(y)a^{\beta\tau}(y)a_{\sigma\tau}(y) \;=\; a^{\alpha\sigma}(y)\delta^\beta_\sigma \;=\; a^{\alpha\beta}(y),\end{aligned}$$

and thus the vectors $\boldsymbol{a}^\alpha(y)$ are linearly independent since the matrix $(a^{\alpha\beta}(y))$ is positive definite. We would likewise establish that $\boldsymbol{a}_\alpha(y) = a_{\alpha\beta}(y)\boldsymbol{a}^\beta(y)$.

The two vectors $\boldsymbol{a}^\alpha(y)$ form the **contravariant basis of the tangent plane** to the surface $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$ (Figure 2.1-1) and the elements $a^{\alpha\beta}(y)$ of the symmetric matrix $(a^{\alpha\beta}(y))$ are called the **contravariant components** of the **first fundamental form**, or **metric tensor**, of the surface $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$.

Let us record for convenience the fundamental relations that exist between the vectors of the covariant and contravariant bases of the tangent plane and the covariant and contravariant components of the first fundamental tensor:

$$\begin{aligned} a_{\alpha\beta}(y) = \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) \quad &\text{and} \quad a^{\alpha\beta}(y) = \boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}^\beta(y), \\ \boldsymbol{a}_\alpha(y) = a_{\alpha\beta}(y)\boldsymbol{a}^\beta(y) \quad &\text{and} \quad \boldsymbol{a}^\alpha(y) = a^{\alpha\beta}(y)\boldsymbol{a}_\beta(y). \end{aligned}$$

A word of caution. The presentation in this section closely follows that of Section 1.2, the mapping $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ "replacing" the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$. There are indeed strong *similarities* between the two presentations, such as the way the metric tensor is defined in both cases, but there are also sharp *differences*. In particular, the matrix $\boldsymbol{\nabla}\boldsymbol{\theta}(y)$ is *not* a square matrix, while the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ *is* square! $\square$

2.3 AREAS AND LENGTHS ON A SURFACE

We now review fundamental formulas expressing *area* and *length elements* at a point $\widehat{y} = \boldsymbol{\theta}(y)$ of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in terms of the matrix $(a_{\alpha\beta}(y))$; see Figure 2.3-1.

These formulas highlight in particular the crucial rôle played by the matrix $(a_{\alpha\beta}(y))$ for computing "metric" notions at $\widehat{y} = \boldsymbol{\theta}(y)$. Indeed, the first fundamental form well deserves "metric tensor" as its *alias*!

A mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ is an **immersion** if it is an immersion at each $y \in \omega$, i.e., if $\boldsymbol{\theta}$ is differentiable in ω and the two vectors $\partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at each $y \in \omega$.

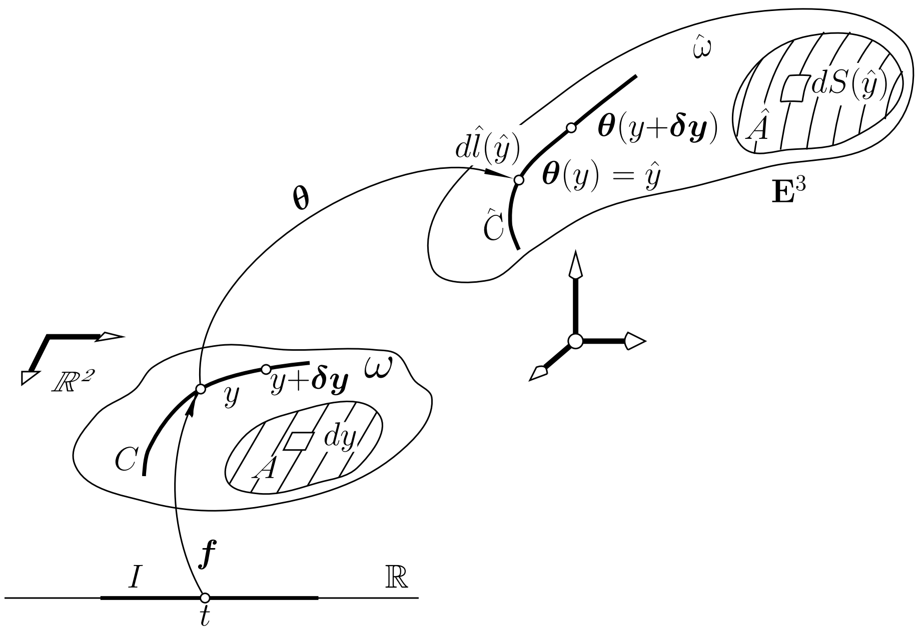

Figure 2.3-1: *Area and length elements on a surface.* The elements $\mathrm{d}\widehat{a}(\widehat{y})$ and $\mathrm{d}\widehat{\ell}(\widehat{y})$ at $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ are related to $\mathrm{d}y$ and $\boldsymbol{\delta y}$ by means of the covariant components of the metric tensor of the surface $\widehat{\omega}$; cf. Theorem 2.3-1. The corresponding relations are used for computing the area of a surface $\widehat{A} = \boldsymbol{\theta}(A) \subset \widehat{\omega}$ and the length of a curve $\widehat{C} = \boldsymbol{\theta}(C) \subset \widehat{\omega}$, where $C = \boldsymbol{f}(I)$ and I is a compact interval of $\mathbb{R}$.

Theorem 2.3-1. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ be an injective and smooth enough immersion, and let $\widehat{\omega} = \boldsymbol{\theta}(\omega)$.*

(a) *The area element $\mathrm{d}\widehat{a}(\widehat{y})$ at $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ is given in terms of the area element $\mathrm{d}y$ at $y \in \omega$ by*

$$\mathrm{d}\widehat{a}(\widehat{y}) = \sqrt{a(y)}\,\mathrm{d}y, \text{ where } a(y) := \det(a_{\alpha\beta}(y)).$$

(b) *The length element* $\mathrm{d}\widehat{\ell}(\widehat{y})$ *at* $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ *is given by*

$$\mathrm{d}\widehat{\ell}(\widehat{y}) = \left\{\delta y^\alpha a_{\alpha\beta}(y)\delta y^\beta\right\}^{1/2}.$$

Proof. The relation (a) between the area elements is well known. It can also be deduced directly from the relation between the area elements $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ and $\mathrm{d}\Gamma(x)$ given in Theorem 1.3-1 (b) by means of an *ad hoc* "three-dimensional extension" of the mapping $\boldsymbol{\theta}$.

The expression of the length element in (b) recalls that $\mathrm{d}\widehat{\ell}(\widehat{y})$ is by definition the principal part with respect to $\boldsymbol{\delta y} = \delta y^\alpha \boldsymbol{e}_\alpha$ of the length $|\boldsymbol{\theta}(y + \boldsymbol{\delta y}) - \boldsymbol{\theta}(y)|$, whose expression precisely led to the introduction of the matrix $(a_{\alpha\beta}(y))$. $\square$

The relations found in Theorem 2.3-1 are used for computing *surface integrals* and *lengths* on the surface $\widehat{\omega}$ by means of integrals inside ω, i.e., in terms of the *curvilinear coordinates* used for defining the surface $\widehat{\omega}$ (see again Figure 2.3-1).

Let A be a domain in $\mathbb{R}^2$ such that $\overline{A} \subset \omega$ (a domain in $\mathbb{R}^2$ is a bounded, open, and connected subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary; cf. Section 1.3), let $\widehat{A} := \boldsymbol{\theta}(A)$, and let $\widehat{f} \in L^1(\widehat{A})$ be given. Then

$$\int_{\widehat{A}} \widehat{f}(\widehat{y})\,\mathrm{d}\widehat{a}(\widehat{y}) = \int_A (\widehat{f} \circ \boldsymbol{\theta})(y)\sqrt{a(y)}\,\mathrm{d}y.$$

In particular, the *area* of $\widehat{A}$ is given by

$$area\,\widehat{A} := \int_{\widehat{A}} \mathrm{d}\widehat{a}(\widehat{y}) = \int_A \sqrt{a(y)}\,\mathrm{d}y.$$

Consider next a *curve* $C = \boldsymbol{f}(I)$ in ω, where I is a compact interval of $\mathbb{R}$ and $\boldsymbol{f} = f^\alpha \boldsymbol{e}_\alpha : I \to \omega$ is a smooth enough injective mapping. Then the *length* of the *curve* $\widehat{C} := \boldsymbol{\theta}(C) \subset \widehat{\omega}$ is given by

$$length\,\widehat{C} := \int_I \left|\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\theta} \circ \boldsymbol{f})(t)\right| \mathrm{d}t = \int_I \sqrt{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}\,\mathrm{d}t.$$

The last relation shows in particular that the lengths of curves inside the surface $\boldsymbol{\theta}(\omega)$ are precisely those induced by the Euclidean metric of the space $\mathbf{E}^3$. For this reason, the surface $\boldsymbol{\theta}(\omega)$ is said to be **isometrically imbedded** in $\mathbf{E}^3$.

2.4 SECOND FUNDAMENTAL FORM; CURVATURE ON A SURFACE

While the image $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$ of a *three-dimensional* open set $\Omega \subset \mathbb{R}^3$ by a smooth enough immersion $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$ is well defined by its "metric", uniquely up to isometries in $\mathbf{E}^3$ (provided *ad hoc* compatibility conditions are satisfied by the covariant components $g_{ij} : \Omega \to \mathbb{R}$ of its *metric tensor*; cf. Theorems 1.6-1 and 1.7-1), a surface given as the image $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$ of a *two-dimensional* open set $\omega \subset \mathbb{R}^2$ by a smooth enough immersion $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ *cannot* be defined by its metric alone.

As intuitively suggested by Figure 2.4-1, the missing information is provided by the "curvature" of a surface. A natural way to give substance to this otherwise vague notion consists in specifying how the *curvature of a curve on a surface* can be computed. As shown in this section, solving this question relies on the knowledge of the *second fundamental form* of a surface, which naturally appears for this purpose through its covariant components (Theorem 2.4-1).

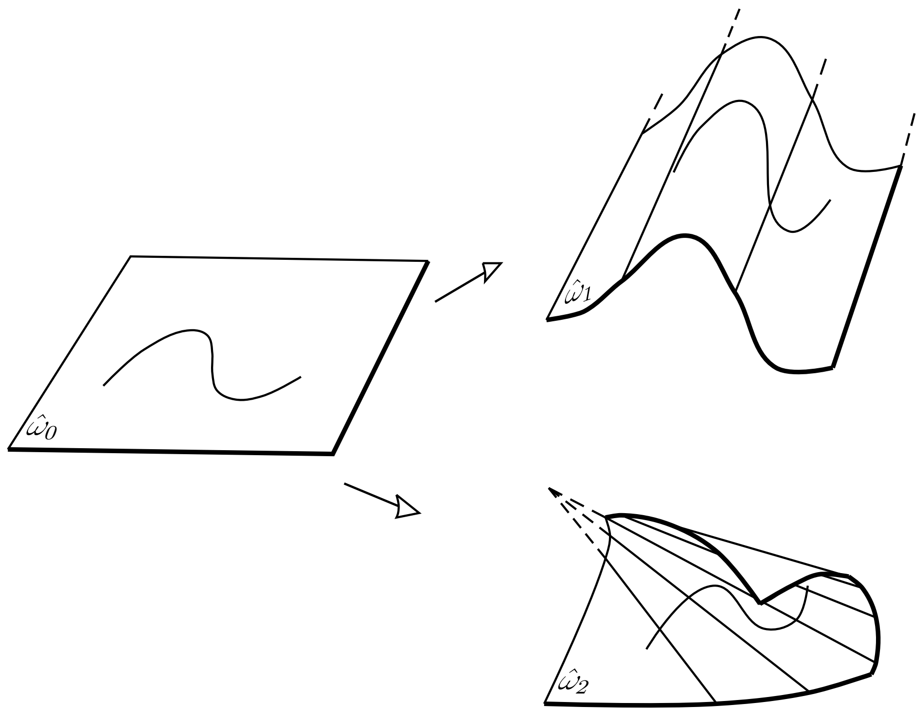

Figure 2.4-1: *A metric alone does not define a surface in* $\mathbf{E}^3$. A flat surface $\widehat{\omega}_0$ may be deformed into a portion $\widehat{\omega}_1$ of a cylinder or a portion $\widehat{\omega}_2$ of a cone without altering the length of any curve drawn on it (cylinders and cones are instances of "developable surfaces"; cf. Section 2.5). Yet it should be clear that in general $\widehat{\omega}_0$ and $\widehat{\omega}_1$, or $\widehat{\omega}_0$ and $\widehat{\omega}_2$, or $\widehat{\omega}_1$ and $\widehat{\omega}_2$, are not identical surfaces *modulo* an isometry of $\mathbf{E}^3$!

Consider as in Section 2.1 a *surface* $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in $\mathbf{E}^3$, where ω is an open subset of $\mathbb{R}^2$ and $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ is a smooth enough immersion. For each $y \in \omega$, the vector

$$\boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$$

is thus well defined, has Euclidean norm one, and is normal to the surface $\widehat{\omega}$ at the point $\widehat{y} = \boldsymbol{\theta}(y)$.

Remark. The denominator in the definition of $\boldsymbol{a}_3(y)$ may be also written as

$$|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)| = \sqrt{a(y)},$$

where $a(y) := \det(a_{\alpha\beta}(y))$. □

Fix $y \in \omega$ and consider a *plane P normal to $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$*, i.e., a plane that contains the vector $\boldsymbol{a}_3(y)$. The intersection $\widehat{C} = P \cap \widehat{\omega}$ is thus a *planar curve* on $\widehat{\omega}$.

As shown in Theorem 2.4-1, it is remarkable that the *curvature* of $\widehat{C}$ at $\widehat{y}$ can be computed by means of the covariant components $a_{\alpha\beta}(y)$ of the first fundamental form of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ introduced in Section 2.2, together with the covariant components $b_{\alpha\beta}(y)$ of the *"second" fundamental form* of $\widehat{\omega}$. The definition of the **curvature** *of a planar curve* is recalled in Figure 2.4-2.

If the algebraic curvature of $\widehat{C}$ at $\widehat{y}$ is $\neq 0$, it can be written as $\dfrac{1}{R}$, and R is then called the **algebraic radius of curvature** of the curve $\widehat{C}$ at $\widehat{y}$. This means that the **center of curvature** of the curve $\widehat{C}$ at $\widehat{y}$ is the point $(\widehat{y} + R\boldsymbol{a}_3(y))$; see Figure 2.4-3. While R is *not* intrinsically defined, as its sign changes in any system of curvilinear coordinates where the normal vector $\boldsymbol{a}_3(y)$ is replaced by its opposite, *the center of curvature is intrinsically defined.*

If the curvature of $\widehat{C}$ at $\widehat{y}$ is 0, the radius of curvature of the curve $\widehat{C}$ at $\widehat{y}$ is said to be *infinite*; for this reason, it is customary to still write the curvature as $\dfrac{1}{R}$ in this case.

Note that *the real number $\dfrac{1}{R}$ is always well defined* by the formula given in the next theorem, since the symmetric matrix $(a_{\alpha\beta}(y))$ is positive definite. This implies in particular that *the radius of curvature never vanishes along a curve on a surface $\boldsymbol{\theta}(\omega)$ defined by a mapping $\boldsymbol{\theta}$ satisfying the assumptions of the next theorem, hence in particular of class $\mathcal{C}^2$ on ω.*

It is intuitively clear that if $R = 0$, the mapping $\boldsymbol{\theta}$ "cannot be too smooth". Think of a surface made of two portions of planes intersecting along a segment, which thus constitutes a *fold* on the surface. Or think of a surface $\boldsymbol{\theta}(\omega)$ with $0 \in \omega$ and $\boldsymbol{\theta}(y_1, y_2) = |y_1|^{1+\alpha}$ for some $0 < \alpha < 1$, so that $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ but $\boldsymbol{\theta} \notin \mathcal{C}^2(\omega; \mathbf{E}^3)$: The radius of curvature of a curve corresponding to a constant y_2 vanishes at $y_1 = 0$.

Theorem 2.4-1. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ be an injective immersion, and let $y \in \omega$ be fixed.*

Consider a plane P normal to $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at the point $\widehat{y} = \boldsymbol{\theta}(y)$. The intersection $P \cap \widehat{\omega}$ is a curve $\widehat{C}$ on $\widehat{\omega}$, which is the image $\widehat{C} = \boldsymbol{\theta}(C)$ of a curve C in the set $\overline{\omega}$. Assume that, in a sufficiently small neighborhood of y, the restriction of C to this neighborhood is the image $\boldsymbol{f}(I)$ of an open interval $I \subset \mathbb{R}$, where $\boldsymbol{f} = f^\alpha \boldsymbol{e}_\alpha : I \to \mathbb{R}$ is a smooth enough injective mapping that satisfies $\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\, \boldsymbol{e}_\alpha \neq \mathbf{0}$, where $t \in I$ is such that $y = \boldsymbol{f}(t)$ (Figure 2.4-3).

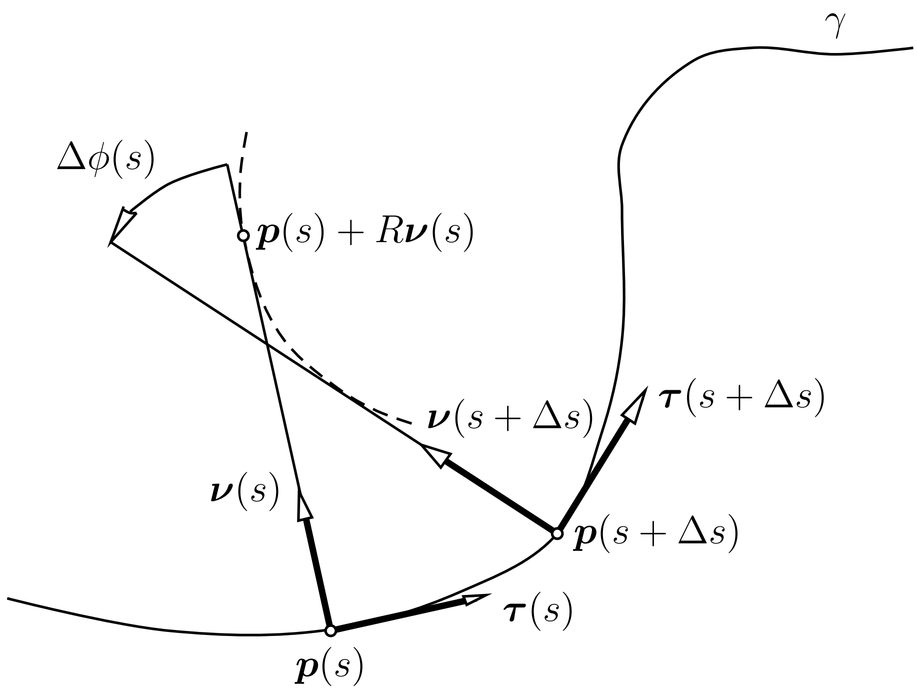

Figure 2.4-2: *Curvature of a planar curve.* Let γ be a smooth enough planar curve, parametrized by its curvilinear abscissa s. Consider two points $\boldsymbol{p}(s)$ and $\boldsymbol{p}(s+\Delta s)$ with curvilinear abscissae s and $s+\Delta s$ and let $\Delta\phi(s)$ be the algebraic angle between the two normals $\boldsymbol{\nu}(s)$ and $\boldsymbol{\nu}(s+\Delta s)$ (oriented in the usual way) to γ at those points. When $\Delta s \to 0$, the ratio $\dfrac{\Delta\phi(s)}{\Delta s}$ has a limit, called the "curvature" of γ at $\boldsymbol{p}(s)$. If this limit is non-zero, its inverse R is called the "algebraic radius of curvature" of γ at $\boldsymbol{p}(s)$ (the sign of R depends on the orientation chosen on γ).

The point $\boldsymbol{p}(s) + R\boldsymbol{\nu}(s)$, which is intrinsically defined, is called the "center of curvature" of γ at $\boldsymbol{p}(s)$: It is the center of the "osculating circle" at $\boldsymbol{p}(s)$, i.e., the limit as $\Delta s \to 0$ of the circle tangent to γ at $\boldsymbol{p}(s)$ that passes through the point $\boldsymbol{p}(s+\Delta s)$. The center of curvature is also the limit as $\Delta s \to 0$ of the intersection of the normals $\boldsymbol{\nu}(s)$ and $\boldsymbol{\nu}(s+\Delta s)$. Consequently, the centers of curvature of γ lie on a curve (dashed on the figure), called "la développée" in French, that is tangent to the normals to γ.

Then the curvature $\dfrac{1}{R}$ *of the planar curve* $\widehat{C}$ *at* $\widehat{y}$ *is given by the ratio*

$$\frac{1}{R} = \frac{b_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}{a_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)},$$

where $a_{\alpha\beta}(y)$ *are the covariant components of the first fundamental form of* $\widehat{\omega}$ *at* y (Section 2.1) *and*

$$b_{\alpha\beta}(y) := \boldsymbol{a}_3(y)\cdot\partial_\alpha\boldsymbol{a}_\beta(y) = -\partial_\alpha\boldsymbol{a}_3(y)\cdot\boldsymbol{a}_\beta(y) = b_{\beta\alpha}(y).$$

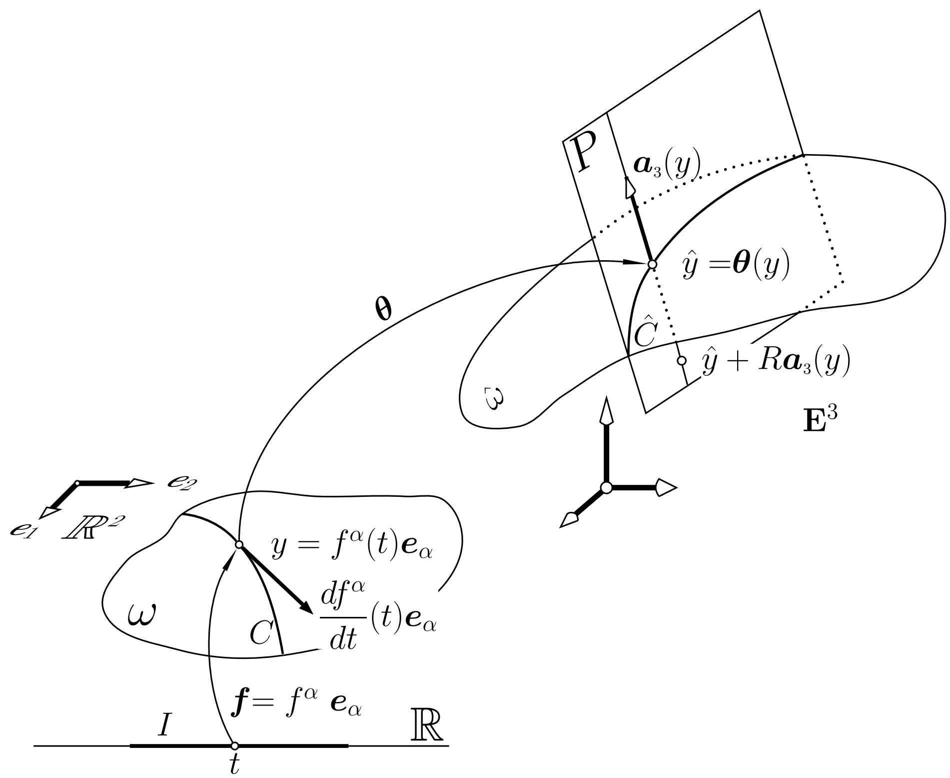

Figure 2.4-3: *Curvature on a surface.* Let P be a plane containing the vector $\boldsymbol{a}_3(y) = \dfrac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$, which is normal to the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$. The algebraic curvature $\dfrac{1}{R}$ of the planar curve $\widehat{C} = P \cap \widehat{\omega} = \boldsymbol{\theta}(C)$ at $\widehat{y} = \boldsymbol{\theta}(y)$ is given by the ratio

$$\frac{1}{R} = \frac{b_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}{a_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)},$$

where $a_{\alpha\beta}(y)$ and $b_{\alpha\beta}(y)$ are the covariant components of the first and second fundamental forms of the surface $\widehat{\omega}$ at $\widehat{y}$ and $\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)$ are the components of the vector tangent to the curve $C = \boldsymbol{f}(I)$ at $y = \boldsymbol{f}(t) = f^\alpha(t)\boldsymbol{e}_\alpha$. If $\dfrac{1}{R} \neq 0$, the center of curvature of the curve $\widehat{C}$ at $\widehat{y}$ is the point $(\widehat{y} + R\boldsymbol{a}_3(y))$, which is intrinsically defined in the Euclidean space $\mathbf{E}^3$.

Proof. (i) We first establish a well-known formula giving the curvature $\dfrac{1}{R}$ of a planar curve. Using the notations of Figure 2.4-2, we note that

$$\sin\Delta\phi(s) = \boldsymbol{\nu}(s)\cdot\boldsymbol{\tau}(s+\Delta s) = -\{\boldsymbol{\nu}(s+\Delta s) - \boldsymbol{\nu}(s)\}\cdot\boldsymbol{\tau}(s+\Delta s),$$

so that

$$\frac{1}{R} := \lim_{\Delta s \to 0}\frac{\Delta\phi(s)}{\Delta s} = \lim_{\Delta s\to 0}\frac{\sin\Delta\phi(s)}{\Delta s} = -\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s}(s)\cdot\boldsymbol{\tau}(s).$$

(ii) The curve $(\boldsymbol{\theta} \circ \boldsymbol{f})(I)$, which is *a priori* parametrized by $t \in I$, can be also parametrized by its curvilinear abscissa s in a neighborhood of the point $\widehat{y}$. There thus exist an interval $\tilde{I} \subset I$ and a mapping $\boldsymbol{p} : J \to P$, where $J \subset \mathbb{R}$ is an interval, such that

$$(\boldsymbol{\theta} \circ \boldsymbol{f})(t) = \boldsymbol{p}(s) \text{ and } (\boldsymbol{a}_3 \circ \boldsymbol{f})(t) = \boldsymbol{\nu}(s) \text{ for all } t \in \tilde{I},\ s \in J.$$

By (i), the curvature $\dfrac{1}{R}$ of $\widehat{C}$ is given by

$$\frac{1}{R} = -\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s}(s) \cdot \boldsymbol{\tau}(s),$$

where

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s}(s) &= \frac{\mathrm{d}(\boldsymbol{a}_3 \circ \boldsymbol{f})}{\mathrm{d}t}(t)\frac{\mathrm{d}t}{\mathrm{d}s} = \partial_\alpha \boldsymbol{a}_3(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\frac{\mathrm{d}t}{\mathrm{d}s},\\
\boldsymbol{\tau}(s) &= \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}s}(s) = \frac{\mathrm{d}(\boldsymbol{\theta} \circ \boldsymbol{f})}{\mathrm{d}t}(t)\frac{\mathrm{d}t}{\mathrm{d}s}\\
&= \partial_\beta \boldsymbol{\theta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\,\frac{\mathrm{d}t}{\mathrm{d}s} = \boldsymbol{a}_\beta(\boldsymbol{f}(t))\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\,\frac{\mathrm{d}t}{\mathrm{d}s}.
\end{aligned}$$

Hence

$$\frac{1}{R} = -\partial_\alpha \boldsymbol{a}_3(\boldsymbol{f}(t)) \cdot \boldsymbol{a}_\beta(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2.$$

To obtain the announced expression for $\dfrac{1}{R}$, it suffices to note that

$$-\partial_\alpha \boldsymbol{a}_3(\boldsymbol{f}(t)) \cdot \boldsymbol{a}_\beta(\boldsymbol{f}(t)) = b_{\alpha\beta}(\boldsymbol{f}(t)),$$

by definition of the functions $b_{\alpha\beta}$ and that (Theorem 2.3-1 (b))

$$\mathrm{d}s = \left\{\delta y^\alpha a_{\alpha\beta}(y)\delta y^\beta\right\}^{1/2} = \left\{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\right\}^{1/2}\mathrm{d}t.$$

$\square$

The knowledge of the curvatures of curves contained in planes *normal to* $\widehat{\omega}$ suffices for computing the curvature of *any* curve on $\widehat{\omega}$. More specifically, the radius of curvature $\widetilde{R}$ at $\widehat{y}$ of any smooth enough curve $\widetilde{C}$ (planar or not) on the surface $\widehat{\omega}$ is given by $\dfrac{\cos\varphi}{\widetilde{R}} = \dfrac{1}{R}$, where φ is the angle between the "principal normal" to $\widetilde{C}$ at $\widehat{y}$ and $\boldsymbol{a}_3(y)$ and $\dfrac{1}{R}$ is given in Theorem 2.4-1; see, e.g., Stoker [1969, Chapter 4, Section 12].

The elements $b_{\alpha\beta}(y)$ of the symmetric matrix $(b_{\alpha\beta}(y))$ defined in Theorem 2.4-1 are called the **covariant components** of the **second fundamental form** of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at $\widehat{y} = \boldsymbol{\theta}(y)$.

2.5 PRINCIPAL CURVATURES; GAUSSIAN CURVATURE

The analysis of the previous section suggests that precise information about the shape of a surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in a neighborhood of one of its points $\widehat{y} = \boldsymbol{\theta}(y)$ can be gathered by letting the plane P turn around the normal vector $\boldsymbol{a}_3(y)$ and by following in this process the variations of the curvatures at $\widehat{y}$ of the corresponding planar curves $P \cap \widehat{\omega}$, as given in Theorem 2.4-1.

As a first step in this direction, we show that *these curvatures span a compact interval of* $\mathbb{R}$. In particular then, they "stay away from infinity".

Note that this compact interval contains 0 if, and only if, the radius of curvature of the curve $P \cap \widehat{\omega}$ is infinite for at least one such plane P.

Theorem 2.5-1. (a) *Let the assumptions and notations be as in* Theorem 2.4-1. *For a fixed $y \in \omega$, consider the set $\mathcal{P}$ of all planes P normal to the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at $\widehat{y} = \boldsymbol{\theta}(y)$. Then the set of curvatures of the associated planar curves $P \cap \widehat{\omega}$, $P \in \mathcal{P}$, is a compact interval of* $\mathbb{R}$, *denoted* $\left[\dfrac{1}{R_1(y)}, \dfrac{1}{R_2(y)}\right]$.

(b) *Let the matrix $(b_\alpha^\beta(y))$, α being the row index, be defined by*

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y),$$

where $(a^{\alpha\beta}(y)) = (a_{\alpha\beta}(y))^{-1}$ (Section 2.2) *and the matrix $(b_{\alpha\beta}(y))$ is defined as in* Theorem 2.4-1. *Then*

$$\frac{1}{R_1(y)} + \frac{1}{R_2(y)} = b_1^1(y) + b_2^2(y),$$

$$\frac{1}{R_1(y)R_2(y)} = b_1^1(y)b_2^2(y) - b_1^2(y)b_2^1(y) = \frac{\det(b_{\alpha\beta}(y))}{\det(a_{\alpha\beta}(y))}.$$

(c) *If* $\dfrac{1}{R_1(y)} \neq \dfrac{1}{R_2(y)}$, *there is a unique pair of orthogonal planes $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that the curvatures of the associated planar curves $P_1 \cap \widehat{\omega}$ and $P_2 \cap \widehat{\omega}$ are precisely* $\dfrac{1}{R_1(y)}$ *and* $\dfrac{1}{R_2(y)}$.

Proof. (i) Let $\Delta(P)$ denote the intersection of $P \in \mathcal{P}$ with the tangent plane T to the surface $\widehat{\omega}$ at $\widehat{y}$, and let $\widehat{C}(P)$ denote the intersection of P with $\widehat{\omega}$. Hence $\Delta(P)$ is tangent to $\widehat{C}(P)$ at $\widehat{y} \in \widehat{\omega}$.

In a sufficiently small neighborhood of $\widehat{y}$ the restriction of the curve $\widehat{C}(P)$ to this neighborhood is given by $\widehat{C}(P) = (\boldsymbol{\theta} \circ \boldsymbol{f}(P))(I(P))$, where $I(P) \subset \mathbb{R}$ is an open interval and $\boldsymbol{f}(P) = f^\alpha(P)\boldsymbol{e}_\alpha : I(P) \to \mathbb{R}^2$ is a smooth enough injective mapping that satisfies $\dfrac{\mathrm{d}f^\alpha(P)}{\mathrm{d}t}(t)\boldsymbol{e}_\alpha \neq \boldsymbol{0}$, where $t \in I(P)$ is such that $y = \boldsymbol{f}(P)(t)$. Hence the line $\Delta(P)$ is given by

$$\Delta(P) = \left\{\widehat{y} + \lambda \frac{\mathrm{d}(\boldsymbol{\theta} \circ \boldsymbol{f}(P))}{\mathrm{d}t}(t); \lambda \in \mathbb{R}\right\} = \{\widehat{y} + \lambda \xi^\alpha \boldsymbol{a}_\alpha(y); \lambda \in \mathbb{R}\},$$

where $\xi^\alpha := \dfrac{\mathrm{d}f^\alpha(P)}{\mathrm{d}t}(t)$ and $\xi^\alpha \boldsymbol{e}_\alpha \neq \mathbf{0}$ by assumption.

Since the line $\{y + \mu\xi^\alpha \boldsymbol{e}_\alpha; \mu \in \mathbb{R}\}$ is tangent to the curve $C(P) := \boldsymbol{\theta}^{-1}(\widehat{C}(P))$ at $y \in \omega$ (the mapping $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ is injective by assumption) for each such parametrizing function $\boldsymbol{f}(P) : I(P) \to \mathbb{R}^2$ and since the vectors $\boldsymbol{a}_\alpha(y)$ are linearly independent, there exists a bijection between the set of all lines $\Delta(P) \subset T$, $P \in \mathcal{P}$, and the set of all lines supporting the nonzero tangent vectors to the curve $C(P)$.

Hence Theorem 2.4-1 shows that when P varies in $\mathcal{P}$, the curvature of the corresponding curves $\widehat{C} = \widehat{C}(P)$ at $\widehat{y}$ takes the same values as does the ratio $\dfrac{b_{\alpha\beta}(y)\xi^\alpha\xi^\beta}{a_{\alpha\beta}(y)\xi^\alpha\xi^\beta}$ when $\boldsymbol{\xi} := (\zeta_\alpha)$ varies in $\mathbb{R}^2 - \{\mathbf{0}\}$.

(ii) Let the symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ of order two be defined by

$$\mathbf{A} := (a_{\alpha\beta}(y)) \text{ and } \mathbf{B} := (b_{\alpha\beta}(y)).$$

Since $\mathbf{A}$ is positive definite, it has a (unique) *square root* $\mathbf{C}$, i.e., a symmetric positive definite matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{C}^2$. Hence the ratio

$$\frac{b_{\alpha\beta}(y)\xi^\alpha\xi^\beta}{a_{\alpha\beta}(y)\xi^\alpha\xi^\beta} = \frac{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}} = \frac{\boldsymbol{\eta}^T\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}\boldsymbol{\eta}}{\boldsymbol{\eta}^T\boldsymbol{\eta}}, \text{ where } \boldsymbol{\eta} = \mathbf{C}\boldsymbol{\xi},$$

is nothing but the *Rayleigh quotient* associated with the symmetric matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$. When $\boldsymbol{\eta}$ varies in $\mathbb{R}^2 - \{\mathbf{0}\}$, this Rayleigh quotient thus spans the compact interval of $\mathbb{R}$ whose end-points are the smallest and largest eigenvalue, respectively denoted $\dfrac{1}{R_1(y)}$ and $\dfrac{1}{R_2(y)}$, of the matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$ (for a proof, see, e.g., Ciarlet [1982, Theorem 1.3-1]). This proves (a).

Furthermore, the relation

$$\mathbf{C}(\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1})\mathbf{C}^{-1} = \mathbf{B}\mathbf{C}^{-2} = \mathbf{B}\mathbf{A}^{-1}$$

shows that the eigenvalues of the symmetric matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$ coincide with those of the (in general non-symmetric) matrix $\mathbf{B}\mathbf{A}^{-1}$. Note that $\mathbf{B}\mathbf{A}^{-1} = (b_\alpha^\beta(y))$ with $b_\alpha^\beta(y) := a^{\beta\sigma}(y)b_{\alpha\sigma}(y)$, α being the row index, since $\mathbf{A}^{-1} = (a^{\alpha\beta}(y))$.

Hence the relations in (b) simply express that the sum and the product of the eigenvalues of the matrix $\mathbf{B}\mathbf{A}^{-1}$ are respectively equal to its trace and to its determinant, which may be also written as $\dfrac{\det(b_{\alpha\beta}(y))}{\det(a_{\alpha\beta}(y))}$ since $\mathbf{B}\mathbf{A}^{-1} = (b_\alpha^\beta(y))$. This proves (b).

(iii) Let $\boldsymbol{\eta}_1 = (\eta_1^\alpha) = \mathbf{C}\boldsymbol{\xi}_1$ and $\boldsymbol{\eta}_2 = (\eta_2^\beta) = \mathbf{C}\boldsymbol{\xi}_2$, with $\boldsymbol{\xi}_1 = (\xi_1^\alpha)$ and $\boldsymbol{\xi}_2 = (\xi_2^\beta)$, be two orthogonal ($\boldsymbol{\eta}_1^T\boldsymbol{\eta}_2 = 0$) eigenvectors of the symmetric matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$, corresponding to the eigenvalues $\dfrac{1}{R_1(y)}$ and $\dfrac{1}{R_2(y)}$, respectively. Hence

$$0 = \boldsymbol{\eta}_1^T\boldsymbol{\eta}_2 = \boldsymbol{\xi}_1^T\mathbf{C}^T\mathbf{C}\boldsymbol{\xi}_2 = \boldsymbol{\xi}_1^T\mathbf{A}\boldsymbol{\xi}_2 = 0,$$

since $\mathbf{C}^T = \mathbf{C}$. By (i), the corresponding lines $\Delta(P_1)$ and $\Delta(P_2)$ of the tangent plane are parallel to the vectors $\xi_1^\alpha \boldsymbol{a}_\alpha(y)$ and $\xi_2^\beta \boldsymbol{a}_\beta(y)$, which are orthogonal since

$$\left\{\xi_1^\alpha \boldsymbol{a}_\alpha(y)\right\} \cdot \left\{\xi_2^\beta \boldsymbol{a}_\beta(y)\right\} = a_{\alpha\beta}(y)\xi_1^\alpha \xi_2^\beta = \boldsymbol{\xi}_1^T \mathbf{A} \boldsymbol{\xi}_2.$$

If $\dfrac{1}{R_1(y)} \neq \dfrac{1}{R_2(y)}$, the directions of the vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are uniquely determined and the lines $\Delta(P_1)$ and $\Delta(P_2)$ are likewise uniquely determined. This proves (c). □

We are now in a position to state several *fundamental definitions*:

The elements $b_\alpha^\beta(y)$ of the (in general non-symmetric) matrix $(b_\alpha^\beta(y))$ defined in Theorem 2.5-1 are called the **mixed components** of the **second fundamental form** of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at $\widehat{y} = \boldsymbol{\theta}(y)$.

The real numbers $\dfrac{1}{R_1(y)}$ and $\dfrac{1}{R_2(y)}$ (one or both possibly equal to 0) found in Theorem 2.5-1 are called the **principal curvatures** of $\widehat{\omega}$ at $\widehat{y}$.

If $\dfrac{1}{R_1(y)} = \dfrac{1}{R_2(y)}$, the curvatures of the planar curves $P \cap \widehat{\omega}$ are the same in all directions, i.e., for all $P \in \mathcal{P}$. If $\dfrac{1}{R_1(y)} = \dfrac{1}{R_2(y)} = 0$, the point $\widehat{y} = \boldsymbol{\theta}(y)$ is called a **planar point**. If $\dfrac{1}{R_1(y)} = \dfrac{1}{R_2(y)} \neq 0, \widehat{y}$ is called an **umbilical point**.

It is remarkable that, *if all the points of $\widehat{\omega}$ are planar, then $\widehat{\omega}$ is a portion of a plane*. Likewise, *if all the points of $\widehat{\omega}$ are umbilical, then $\widehat{\omega}$ is a portion of a sphere*. For proofs, see, e.g., Stoker [1969, p. 87 and p. 99].

Let $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ be a point that is neither planar nor umbilical; in other words, the principal curvatures at $\widehat{y}$ are not equal. Then the two orthogonal lines tangent to the planar curves $P_1 \cap \widehat{\omega}$ and $P_2 \cap \widehat{\omega}$ (Theorem 2.5-1 (c)) are called the **principal directions** at $\widehat{y}$.

A **line of curvature** is a curve on $\widehat{\omega}$ that is tangent to a principal direction at each one of its points. It can be shown that *a point that is neither planar nor umbilical possesses a neighborhood where two orthogonal families of lines of curvature can be chosen as coordinate lines*. See, e.g., Klingenberg [1973, Lemma 3.6.6].

If $\dfrac{1}{R_1(y)} \neq 0$ *and* $\dfrac{1}{R_2(y)} \neq 0$, the real numbers $R_1(y)$ and $R_2(y)$ are called the **algebraic principal radii of curvature** of $\widehat{\omega}$ at $\widehat{y}$. If, e.g., $\dfrac{1}{R_1(y)} = 0$, the corresponding radius of curvature $R_1(y)$ is said to be *infinite*, according to the convention made in Section 2.4. While the algebraic principal radii of curvature may simultaneously change their signs in another system of curvilinear coordinates, the associated *centers of curvature* are intrinsically defined.

The numbers $\left(\dfrac{1}{R_1(y)} + \dfrac{1}{R_2(y)}\right)$ and $\dfrac{1}{R_1(y)R_2(y)}$, which are the principal invariants of the matrix $(b_\alpha^\beta(y))$ (Theorem 2.5-1), are respectively called the **mean curvature** and the **Gaussian**, or **total, curvature** of the surface $\widehat{\omega}$ at $\widehat{y}$.

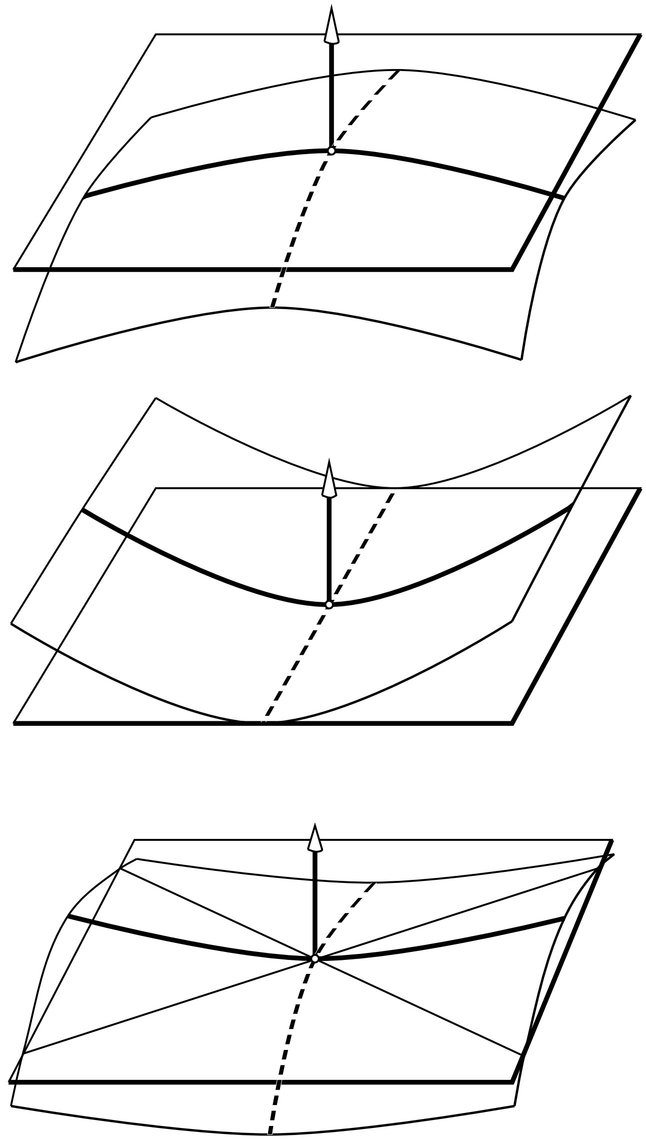

Figure 2.5-1: *Different kinds of points on a surface.* A point is elliptic if the Gaussian curvature is > 0 or equivalently, if the two principal radii of curvature are of the same sign; the surface is then locally on one side of its tangent plane. A point is parabolic if exactly one of the two principal radii of curvature is infinite; the surface is again locally on one side of its tangent plane. A point is hyperbolic if the Gaussian curvature is < 0 or equivalently, if the two principal radii of curvature are of different signs; the surface then intersects its tangent plane along two curves.

A point on a surface is **elliptic, parabolic**, or **hyperbolic**, according as its *Gaussian curvature* is $> 0, = 0$ but it is not a planar point, or < 0; see Figure 2.5-1.

An **asymptotic line** is a curve on a surface that is everywhere tangent to a direction along which the radius of curvature is infinite; any point along an asymptotic line is thus either parabolic or hyperbolic. It can be shown that, *if all the points of a surface are hyperbolic, any point possesses a neighborhood where two intersecting families of asymptotic lines can be chosen as coordinate lines.* See, e.g., Klingenberg [1973, Lemma 3.6.12].

As intuitively suggested by Figure 2.4-1, a surface in $\mathbb{R}^3$ cannot be defined by its *metric* alone, i.e., through its first fundamental form alone, since its *curvature* must be in addition specified through its second fundamental form. But quite surprisingly, the *Gaussian curvature* at a point can also be expressed solely in terms of the functions $a_{\alpha\beta}$ and their derivatives! This is the celebrated *Theorema egregium* ("astonishing theorem") of Gauß [1828]; see Theorem 2.6.2 in the next section.

Another striking result involving the *Gaussian curvature* is the equally celebrated **Gauß-Bonnet theorem**, so named after Gauß [1828] and Bonnet [1848] (for a "modern" proof, see, e.g., Klingenberg [1973, Theorem 6.3-5] or do Carmo [1994, Chapter 6, Theorem 1]): *Let S be a smooth enough, "closed", "orientable", and compact surface in* $\mathbb{R}^3$ (a "closed" surface is one "without boundary", such as a sphere or a torus; "orientable" surfaces, which exclude for instance Klein bottles, are defined in, e.g., Klingenberg [1973, Section 5.5]) *and let $K : S \to \mathbb{R}$ denote its Gaussian curvature. Then*

$$\int_S K(\widehat{y})\,\mathrm{d}\widehat{a}(\widehat{y}) = 2\pi(2 - 2g(S)),$$

where the **genus** *$g(S)$ is the number of "holes" of S* (for instance, a sphere has genus zero, while a torus has genus one). The integer $\chi(S)$ defined by $\chi(S) := (2 - 2g(S))$ is the **Euler characteristic** of $\widehat{\omega}$.

According to the definition of Stoker [1969, Chapter 5, Section 2], a **developable surface** is one whose *Gaussian curvature* vanishes everywhere. Developable surfaces are otherwise often defined as "ruled" surfaces whose Gaussian curvature vanishes everywhere, as in, e.g., Klingenberg [1973, Section 3.7]). A portion of a plane provides a first example, the only one of a developable surface *all* points of which are planar. Any developable surface *all* points of which are *parabolic* can be likewise fully described: It is *either* a portion of a cylinder, *or* a portion of a cone, *or* a portion of a surface spanned by the tangents to a skewed curve. The description of a developable surface comprising both planar and parabolic points is more subtle (although the above examples are in a sense the only ones possible, at least locally; see Stoker [1969, Chapter 5, Sections 2 to 6]).

The interest of developable surfaces is that they can be, at least locally, continuously "rolled out", or "developed" (hence their name), onto a *plane, without changing the metric* of the intermediary surfaces in the process.

2.6 COVARIANT DERIVATIVES OF A VECTOR FIELD AND CHRISTOFFEL SYMBOLS ON A SURFACE; THE GAUSS AND WEINGARTEN FORMULAS

As in Sections 2.2 and 2.4, consider a surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in $\mathbf{E}^3$, where $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ is a smooth enough injective immersion, and let

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}, \quad y \in \omega.$$

Then the vectors $\boldsymbol{a}_\alpha(y)$ (which form the covariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$; see Figure 2.1-1) together with the vector $\boldsymbol{a}_3(y)$ (which is normal to $\widehat{\omega}$ and has Euclidean norm one) form the **covariant basis** *at* $\widehat{y}$.

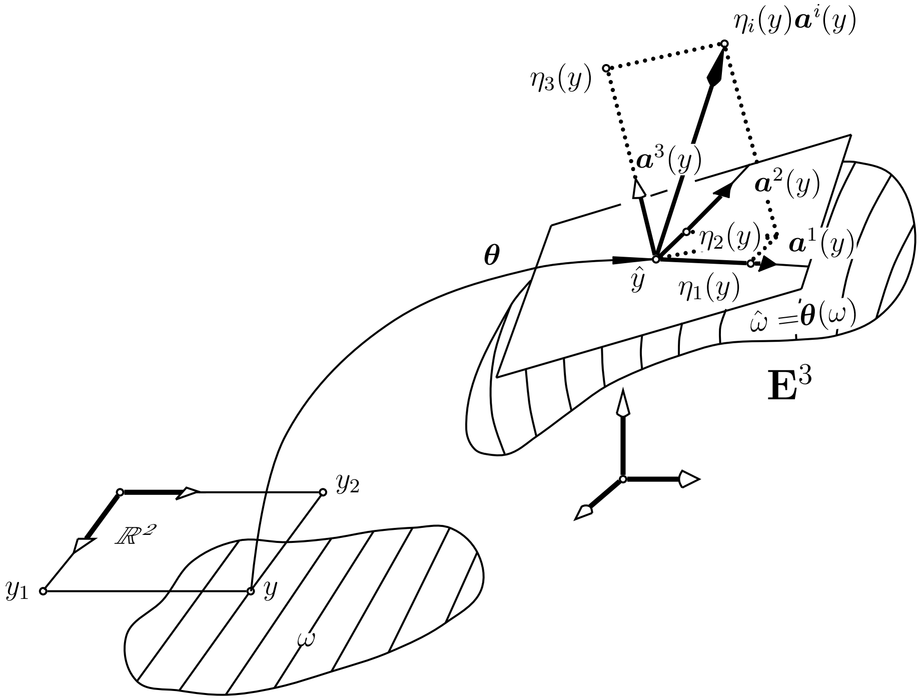

Figure 2.6-1: *Contravariant bases and vector fields along a surface.* At each point $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega} = \boldsymbol{\theta}(\omega)$, the three vectors $\boldsymbol{a}^i(y)$, where $\boldsymbol{a}^\alpha(y)$ form the contravariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y}$ (Figure 2.1-1) and $\boldsymbol{a}^3(y) = \dfrac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$, form the contravariant basis at $\widehat{y}$. An arbitrary vector field defined on $\widehat{\omega}$ may then be defined by its covariant components $\eta_i : \omega \to \mathbb{R}$. This means that $\eta_i(y)\boldsymbol{a}^i(y)$ is the vector at the point $\widehat{y}$.

Let the vectors $\boldsymbol{a}^\alpha(y)$ of the tangent plane to $\widehat{\omega}$ at $\widehat{y}$ be defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$. Then the vectors $\boldsymbol{a}^\alpha(y)$ (which form the contravariant basis of the tangent plane at $\widehat{y}$; see again Figure 2.1-1) together with the vector $\boldsymbol{a}^3(y)$ form the **contravariant basis** *at* $\widehat{y}$; see Figure 2.6-1. Note that the vectors of the covariant and contravariant bases at $\widehat{y}$ satisfy

$$\boldsymbol{a}^i(y) \cdot \boldsymbol{a}_j(y) = \delta^i_j.$$

Suppose that a *vector field* is defined on the surface $\widehat{\omega}$. One way to define such a field in terms of the *curvilinear coordinates* used for defining the surface $\widehat{\omega}$ consists in writing it as $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$, i.e., in specifying its **covariant components** $\eta_i : \omega \to \mathbb{R}$ over the vectors $\boldsymbol{a}^i$ of the *contravariant bases*. This means that $\eta_i(y)\boldsymbol{a}^i(y)$ is the vector at each point $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ (Figure 2.6-1).

Our objective in this section is to compute the *partial derivatives* $\partial_\alpha(\eta_i \boldsymbol{a}^i)$ of such a vector field. These are found in the next theorem, as immediate consequences of two basic formulas, those of *Gauß* and *Weingarten*. The *Christoffel symbols on a surface* and the *covariant derivatives on a surface* are also naturally introduced in this process.

Theorem 2.6-1. *Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ be an immersion.*

(a) *The derivatives of the vectors of the covariant and contravariant bases are given by*

$$\partial_\alpha \boldsymbol{a}_\beta = C^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3 \text{ and } \partial_\alpha \boldsymbol{a}^\beta = -C^\beta_{\alpha\sigma} \boldsymbol{a}^\sigma + b^\beta_\alpha \boldsymbol{a}^3,$$
$$\partial_\alpha \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{a}^3 = -b_{\alpha\beta} \boldsymbol{a}^\beta = -b^\sigma_\alpha \boldsymbol{a}_\sigma,$$

where the covariant and mixed components $b_{\alpha\beta}$ and b^β_α of the second fundamental form of $\widehat{\omega}$ are defined in Theorems 2.4-1 *and* 2.5-1 *and*

$$C^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta.$$

(b) *Let there be given a vector field $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ with covariant components $\eta_i \in \mathcal{C}^1(\omega)$. Then $\eta_i \boldsymbol{a}^i \in \boldsymbol{\mathcal{C}}^1(\omega)$ and the partial derivatives $\partial_\alpha(\eta_i \boldsymbol{a}^i) \in \boldsymbol{\mathcal{C}}^0(\omega)$ are given by*

$$\begin{aligned}\partial_\alpha(\eta_i \boldsymbol{a}^i) &= (\partial_\alpha \eta_\beta - C^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + (\partial_\alpha \eta_3 + b^\beta_\alpha \eta_\beta)\boldsymbol{a}^3 \\ &= (\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + (\eta_{3|\alpha} + b^\beta_\alpha \eta_\beta)\boldsymbol{a}^3,\end{aligned}$$

where

$$\eta_{\beta|\alpha} := \partial_\alpha \eta_\beta - C^\sigma_{\alpha\beta}\eta_\sigma \text{ and } \eta_{3|\alpha} := \partial_\alpha \eta_3.$$

Proof. Since any vector $\boldsymbol{c}$ in the tangent plane can be expanded as $\boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{a}_\beta)\boldsymbol{a}^\beta = (\boldsymbol{c} \cdot \boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma$, since $\partial_\alpha \boldsymbol{a}^3$ is in the tangent plane ($\partial_\alpha \boldsymbol{a}^3 \cdot \boldsymbol{a}^3 = \frac{1}{2}\partial_\alpha(\boldsymbol{a}^3 \cdot \boldsymbol{a}^3) = 0$), and since $\partial_\alpha \boldsymbol{a}^3 \cdot \boldsymbol{a}_\beta = -b_{\alpha\beta}$ (Theorem 2.4-1), it follows that

$$\partial_\alpha \boldsymbol{a}^3 = (\partial_\alpha \boldsymbol{a}^3 \cdot \boldsymbol{a}_\beta)\boldsymbol{a}^\beta = -b_{\alpha\beta}\boldsymbol{a}^\beta.$$

This formula, together with the definition of the functions b^β_α (Theorem 2.5-1), implies in turn that

$$\partial_\alpha \boldsymbol{a}_3 = (\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma = -b_{\alpha\beta}(\boldsymbol{a}^\beta \cdot \boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma = -b_{\alpha\beta}a^{\beta\sigma}\boldsymbol{a}_\sigma = -b^\sigma_\alpha \boldsymbol{a}_\sigma.$$

Any vector $\boldsymbol{c}$ can be expanded as $\boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{a}^i)\boldsymbol{a}_i = (\boldsymbol{c} \cdot \boldsymbol{a}_j)\boldsymbol{a}^j$. In particular,

$$\partial_\alpha \boldsymbol{a}_\beta = (\partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma + (\partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^3)\boldsymbol{a}_3 = C^\sigma_{\alpha\beta}\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3,$$

by definition of $C^\sigma_{\alpha\beta}$ and $b_{\alpha\beta}$. Finally,

$$\partial_\alpha \boldsymbol{a}^\beta = (\partial_\alpha \boldsymbol{a}^\beta \cdot \boldsymbol{a}_\sigma)\boldsymbol{a}^\sigma + (\partial_\alpha \boldsymbol{a}^\beta \cdot \boldsymbol{a}_3)\boldsymbol{a}^3 = -C^\beta_{\alpha\sigma}\boldsymbol{a}^\sigma + b^\beta_\alpha \boldsymbol{a}^3,$$

since

$$\partial_\alpha \boldsymbol{a}^\beta \cdot \boldsymbol{a}_3 = -\boldsymbol{a}^\beta \cdot \partial_\alpha \boldsymbol{a}_3 = b^\sigma_\alpha \boldsymbol{a}_\sigma \cdot \boldsymbol{a}^\beta = b^\beta_\alpha.$$

That $\eta_i \boldsymbol{a}^i \in \boldsymbol{\mathcal{C}}^1(\omega)$ if $\eta_i \in \mathcal{C}^1(\omega)$ is clear since $\boldsymbol{a}^i \in \boldsymbol{C}^1(\omega)$ if $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$. The formulas established *supra* immediately lead to the announced expression of $\partial_\alpha(\eta_i \boldsymbol{a}^i)$. □

The relations (found in Theorem 2.6-1)

$$\partial_\alpha \boldsymbol{a}_\beta = C^\sigma_{\alpha\beta}\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3 \text{ and } \partial_\alpha \boldsymbol{a}^\beta = -C^\beta_{\alpha\sigma}\boldsymbol{a}^\sigma + b^\beta_\alpha \boldsymbol{a}^3$$

and

$$\partial_\alpha \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{a}^3 = -b_{\alpha\beta}\boldsymbol{a}^\beta = -b^\sigma_\alpha \boldsymbol{a}_\sigma,$$

respectively constitute the **formulas of Gauß** and **Weingarten**. The functions (also found in Theorem 2.6-1)

$$\eta_{\beta|\alpha} = \partial_\alpha \eta_\beta - C^\sigma_{\alpha\beta}\eta_\sigma \text{ and } \eta_{3|\alpha} = \partial_\alpha \eta_3$$

are the **first-order covariant derivatives** of the *surface vector field* $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$, and the functions

$$C^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta = -\partial_\alpha \boldsymbol{a}^\sigma \cdot \boldsymbol{a}_\beta$$

are the **Christoffel symbols of the first kind**.

Remarks. (1) The Christoffel symbols $C^\sigma_{\alpha\beta}$ can be also defined solely in terms of the covariant components of the first fundamental form; see the proof of Theorem 2.7-1

(2) The notation $C^\sigma_{\alpha\beta}$ is preferred here instead of the customary notation $\Gamma^\sigma_{\alpha\beta}$, so as to avoid confusion with the "three-dimensional" Christoffel symbols Γ^p_{ij} introduced in Section 1.4. □

The definition of the covariant derivatives $\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ of a vector field defined on a surface $\boldsymbol{\theta}(\omega)$ given in Theorem 2.6-1 is highly reminiscent of the definition of the covariant derivatives $v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$ of a vector field defined on an open set $\boldsymbol{\Theta}(\Omega)$ given in Section 1.4. However, the former are more subtle to apprehend than the latter. To see this, recall that the covariant derivatives $v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$ may be also defined by the relations (Theorem 1.4-2)

$$v_{i\|j}\boldsymbol{g}^j = \partial_j(v_i \boldsymbol{g}^i).$$

By contrast, *even if only tangential vector fields $\eta_\alpha \boldsymbol{a}^\alpha$ on the surface $\boldsymbol{\theta}(\omega)$ are considered* (i.e., vector fields $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ for which $\eta_3 = 0$), *their covariant derivatives $\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ satisfy only the relations*

$$\eta_{\alpha|\beta}\boldsymbol{a}^\alpha = \mathbf{P}\left\{\partial_\beta(\eta_\alpha \boldsymbol{a}^\alpha)\right\},$$

where $\mathbf{P}$ denotes the projection operator on the tangent plane in the direction of the normal vector (i.e., $\mathbf{P}(c_i \boldsymbol{a}^i) := c_\alpha \boldsymbol{a}^\alpha$), since

$$\partial_\beta(\eta_\alpha \boldsymbol{a}^\alpha) = \eta_{\alpha|\beta}\boldsymbol{a}^\alpha + b^\alpha_\beta \eta_\alpha \boldsymbol{a}^3$$

for such tangential fields by Theorem 2.6-1. The reason is that a surface has in general a *nonzero curvature*, manifesting itself here by the "extra term" $b^\alpha_\beta \eta_\alpha \boldsymbol{a}^3$. This term vanishes in ω if $\widehat{\omega}$ is a *portion of a plane*, since in this case $b^\alpha_\beta = b_{\alpha\beta} = 0$. Note that, *again in this case*, the formula giving the partial derivatives in Theorem 2.9-1 (b) reduces to

$$\partial_\alpha(\eta_i \boldsymbol{a}^i) = (\eta_{i|\alpha})\boldsymbol{a}^i.$$

2.7 NECESSARY CONDITIONS SATISFIED BY THE FIRST AND SECOND FUNDAMENTAL FORMS: THE GAUSS AND CODAZZI-MAINARDI EQUATIONS; GAUSS' THEOREMA EGREGIUM

It is remarkable that *the components $a_{\alpha\beta} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} : \omega \to \mathbb{R}$ of the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega)$*, defined by a smooth enough immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$, *cannot be arbitrary functions.*

As shown in the next theorem, they must satisfy relations that take the form:

$$\begin{aligned}
\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} &= b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,\\
\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} &= 0 \text{ in } \omega,
\end{aligned}$$

where the functions $C_{\alpha\beta\tau}$ and $C^\sigma_{\alpha\beta}$ have simple expressions in terms of the functions $a_{\alpha\beta}$ and of some of their partial derivatives (as shown in the next proof, it so happens that the functions $C^\sigma_{\alpha\beta}$ as defined in Theorem 2.7-1 coincide with the Christoffel symbols introduced in the previous section; this explains why they are denoted by the same symbol).

These relations, which are meant to hold for all $\alpha, \beta, \sigma, \tau \in \{1, 2\}$, respectively constitute the **Gauß**, and **Codazzi-Mainardi, equations**.

Theorem 2.7-1. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ be an immersion, and let*

$$a_{\alpha\beta} := \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{\frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}\right\}$$

denote the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$. Let the functions $C_{\alpha\beta\tau} \in \mathcal{C}^1(\omega)$ and $C^\sigma_{\alpha\beta} \in \mathcal{C}^1(\omega)$ be defined by

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$
$$C^\sigma_{\alpha\beta} := a^{\sigma\tau} C_{\alpha\beta\tau} \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then, necessarily,

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,$$
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega.$$

Proof. Let $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$. It is then immediately verified that the functions $C_{\alpha\beta\tau}$ are also given by

$$C_{\alpha\beta\tau} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau.$$

Let $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$ and, for each $y \in \omega$, let the three vectors $\boldsymbol{a}^j(y)$ be defined by the relations $\boldsymbol{a}^j(y) \cdot \boldsymbol{a}_i(y) = \delta^j_i$. Since we also have $\boldsymbol{a}^\beta = a^{\alpha\beta} \boldsymbol{a}_\alpha$ and $\boldsymbol{a}^3 = \boldsymbol{a}_3$, the last relations imply that $C^\sigma_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma$, hence that

$$\partial_\alpha \boldsymbol{a}_\beta = C^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3,$$

since $\partial_\alpha \boldsymbol{a}_\beta = (\partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma) \boldsymbol{a}_\sigma + (\partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^3) \boldsymbol{a}_3$. Differentiating the same relations yields

$$\partial_\sigma C_{\alpha\beta\tau} = \partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau + \partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_\tau,$$

so that the above relations together give

$$\partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_\tau = C^\mu_{\alpha\beta} \boldsymbol{a}_\mu \cdot \partial_\sigma \boldsymbol{a}_\tau + b_{\alpha\beta} \boldsymbol{a}_3 \cdot \partial_\sigma \boldsymbol{a}_\tau = C^\mu_{\alpha\beta} C_{\sigma\tau\mu} + b_{\alpha\beta} b_{\sigma\tau}.$$

Consequently,

$$\partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_\sigma C_{\alpha\beta\tau} - C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - b_{\alpha\beta} b_{\sigma\tau}.$$

Since $\partial_{\alpha\sigma} \boldsymbol{a}_\beta = \partial_{\alpha\beta} \boldsymbol{a}_\sigma$, we also have

$$\partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_\beta C_{\alpha\sigma\tau} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} - b_{\alpha\sigma} b_{\beta\tau}.$$

Hence the Gauß equations immediately follow.

Since $\partial_\alpha \boldsymbol{a}_3 = (\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\sigma) \boldsymbol{a}^\sigma + (\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_3) \boldsymbol{a}^3$ and $\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\sigma = -b_{\alpha\sigma} = -\partial_\alpha \boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3$, we have

$$\partial_\alpha \boldsymbol{a}_3 = -b_{\alpha\sigma} \boldsymbol{a}^\sigma.$$

Differentiating the relations $b_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3$, we obtain

$$\partial_\sigma b_{\alpha\beta} = \partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 + \partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_3.$$

This relation and the relations $\partial_\alpha \boldsymbol{a}_\beta = C^\sigma_{\alpha\beta}\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3$ and $\partial_\alpha \boldsymbol{a}_3 = -b_{\alpha\sigma}\boldsymbol{a}^\sigma$ together imply that

$$\partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_3 = -C^\mu_{\alpha\beta} b_{\sigma\mu}.$$

Consequently,

$$\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\beta} b_{\sigma\mu}.$$

Since $\partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma$, we also have

$$\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_\beta b_{\alpha\sigma} + C^\mu_{\alpha\sigma} b_{\beta\mu}.$$

Hence the Codazzi-Mainardi equations immediately follow. □

Remark. The vectors $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}^\beta$ introduced above respectively form the covariant and contravariant bases of the tangent plane to the surface $\boldsymbol{\theta}(\omega)$, the unit vector $\boldsymbol{a}_3 = \boldsymbol{a}^3$ is normal to the surface, and the functions $a^{\alpha\beta}$ are the contravariant components of the first fundamental form (Sections 2.2 and 2.3). □

As shown in the above proof, the Gauß and Codazzi-Mainardi equations thus simply constitute a re-writing of the relations $\partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma$ in the form of the equivalent relations $\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_\tau$ and $\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3$.

The functions

$$C_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = C_{\beta\alpha\tau}$$

and

$$C^\sigma_{\alpha\beta} = a^{\sigma\tau} C_{\alpha\beta\tau} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma = C^\sigma_{\beta\alpha}$$

are the **Christoffel symbols of the first**, and **second, kind**. We recall that the same Christoffel symbols $C^\sigma_{\alpha\beta}$ also naturally appeared in a different context (that of covariant differentiation; cf. Section 2.6).

Finally, the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu}$$

are the **covariant components** of the **Riemann curvature tensor** of the surface $\boldsymbol{\theta}(\omega)$.

Remark. Like the notation $C^\sigma_{\alpha\beta}$ vs. Γ^p_{ij}, the notation $C_{\alpha\beta\tau}$ is intended to avoid confusions with the "three-dimensional" Christoffel symbols Γ_{ijq} introduced in Section 1.4. □

Letting $\alpha = 2$, $\beta = 1$, $\sigma = 2$, $\tau = 1$ in the Gauß equations gives in particular

$$S_{1212} = \det(b_{\alpha\beta}).$$

Consequently, the *Gaussian curvature* at each point $\boldsymbol{\Theta}(y)$ of the surface $\boldsymbol{\theta}(\omega)$ can be written as

$$\frac{1}{R_1(y)R_2(y)} = \frac{S_{1212}(y)}{\det(a_{\alpha\beta}(y))},\ y \in \omega,$$

since $\dfrac{1}{R_1(y)R_2(y)} = \dfrac{\det(b_{\alpha\beta}(y)}{\det(a_{\alpha\beta}(y))}$ (Theorem 2.5-1). By inspection of the function S_{1212}, we thus reach the astonishing conclusion that, *at each point of the surface, a notion involving the "curvature" of the surface, viz., the Gaussian curvature, is entirely determined by the knowledge of the "metric" of the surface at the same point*, viz., the components of the first fundamental forms and their partial derivatives of order ≤ 2 at the same point! This startling conclusion naturally deserves a theorem:

Theorem 2.7-2. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ be an immersion, let $a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta}$ denote the covariant components of the first fundamental form of the surface $\boldsymbol{\theta}(\omega)$, and let the functions $C_{\alpha\beta\tau}$ and S_{1212} be defined by*

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$
$$S_{1212} := \frac{1}{2}(2\partial_{12} a_{12} - \partial_{11} a_{22} - \partial_{22} a_{11}) + a^{\alpha\beta}(C_{12\alpha} C_{12\beta} - C_{11\alpha} C_{22\beta}).$$

Then, at each point $\boldsymbol{\theta}(y)$ of the surface $\boldsymbol{\theta}(\omega)$, the principal curvatures $\frac{1}{R_1(y)}$ and $\frac{1}{R_2(y)}$ satisfy

$$\frac{1}{R_1(y)R_2(y)} = \frac{S_{1212}(y)}{\det(a_{\alpha\beta}(y))},\ y \in \omega.$$

Theorem 2.7-2 constitutes the famed **Theorema Egregium** of Gauß [1828], so named by Gauß who had been himself astounded by his discovery.

2.8 EXISTENCE OF A SURFACE WITH PRESCRIBED FIRST AND SECOND FUNDAMENTAL FORMS

Let $\mathbb{M}^2, \mathbb{S}^2$, and $\mathbb{S}^2_>$ denote the sets of all square matrices of order two, of all symmetric matrices of order two, and of all symmetric, positive definite matrices of order two.

So far, we have considered that we are given an open set $\omega \subset \mathbb{R}^2$ and a smooth enough immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$, thus allowing us to define the fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$, where $a_{\alpha\beta} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} : \omega \to \mathbb{R}$ are the covariant components of the *first* and *second fundamental forms* of the surface $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$.

Note that *the immersion $\boldsymbol{\theta}$ need not be injective* in order that these matrix fields be well defined.

We now turn to the reciprocal questions:

Given an open subset ω of $\mathbb{R}^2$ and two smooth enough matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$, when are they the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$, i.e., when does there exist an immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ such that

$$a_{\alpha\beta} := \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} := \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega?$$

If such an immersion exists, to what extent is it unique?

The answers to these questions turn out to be remarkably simple: *If ω is simply-connected, the necessary conditions of* Theorem 2.7-1, i.e., *the Gauß and Codazzi-Mainardi equations, are also sufficient for the existence of such an immersion. If ω is connected, this immersion is unique up to isometries in* $\mathbf{E}^3$.

Whether an immersion found in this fashion is *injective* is a different issue, which accordingly should be resolved by different means.

Following Ciarlet & Larsonneur [2001], we now give a self-contained, complete, and essentially elementary, proof of this well-known result. This proof amounts to showing that it can be established as a simple *corollary* to the *fundamental theorem of three-dimensional differential geometry* (Theorems 1.6-1 and 1.7-1).

This proof has also the merit to shed light on the analogies (which cannot remain unnoticed!) between the assumptions and conclusions of both *existence* results (compare Theorems 1.6-1 and 2.8-1) and both *uniqueness* results (compare Theorems 1.7-1 and 2.9-1).

A direct proof of the fundamental theorem of surface theory is given in Klingenberg [1973, Theorem 3.8.8], where the global existence of the mapping $\boldsymbol{\theta}$ is based on an existence theorem for ordinary differential equations, analogous to that used in part (ii) of the proof of Theorem 1.6-1. A proof of the "local" version of this theorem, which constitutes *Bonnet's theorem*, is found in, e.g., do Carmo [1976].

This result is another special case of the *fundamental theorem of Riemannian geometry* alluded to in Section 1.6. We recall that this theorem asserts that a simply-connected Riemannian manifold of dimension p can be isometrically immersed into a Euclidean space of dimension $(p+q)$ if and only if there exist tensors satisfying together generalized *Gauß*, and *Codazzi-Mainardi, equations* and that the corresponding isometric immersions are unique up to isometries in the Euclidean space. A substantial literature has been devoted to this theorem and its various proofs, which usually rely on basic notions of Riemannian geometry, such as connections or normal bundles, and on the theory of differential forms. See in particular the earlier papers of Janet [1926] and Cartan [1927] and the more recent references of Szczarba [1970], Tenenblat [1971], Jacobowitz [1982], and Szopos [2005].

Like the fundamental theorem of three-dimensional differential geometry, this theorem comprises two essentially distinct parts, a *global existence result* (Theorem 2.8-1) and a *uniqueness result* (Theorem 2.9-1), the latter being also called *rigidity theorem.* Note that these two results are established under *different assumptions* on the set ω and on the smoothness of the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$.

These existence and uniqueness results together constitute the **fundamental theorem of surface theory**.

Theorem 2.8-1. *Let ω be a connected and simply-connected open subset of* $\mathbb{R}^2$ *and let* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ *be two matrix fields that satisfy* *the Gauß and Codazzi-Mainardi equations, viz.,*

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,$$
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega,$$

where

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$
$$C^\sigma_{\alpha\beta} := a^{\sigma\tau} C_{\alpha\beta\tau} \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ *such that*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Proof. The proof of this theorem as a corollary to Theorem 1.6-1 relies on the following elementary observation: Given a smooth enough immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ and $\varepsilon > 0$, let the mapping $\boldsymbol{\Theta} : \omega \times]-\varepsilon, \varepsilon[\to \mathbf{E}^3$ be defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \omega \times]-\varepsilon, \varepsilon[\,,$$

where $\boldsymbol{a}_3 := \dfrac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}$, and let

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \omega \times]-\varepsilon, \varepsilon[\,,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $c_{\alpha\beta} := a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

Assume that the matrices (g_{ij}) constructed in this fashion are *invertible*, hence positive definite, over the set $\omega \times]-\varepsilon, \varepsilon[$ (they need not be, of course; but the resulting difficulty is easily circumvented; see parts (i) and (viii) below). Then the field $(g_{ij}) : \omega \times]-\varepsilon, \varepsilon[\to \mathbb{S}^3_>$ becomes a natural candidate for applying the "three-dimensional" existence result of Theorem 1.6-1, provided of course that *the "three-dimensional" sufficient conditions* of this theorem, viz.,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

can be shown to hold, as consequences of the "two-dimensional" Gauß and Codazzi-Mainardi equations. That this is indeed the case is the essence of the present proof (see parts (i) to (vii)).

By Theorem 1.6-1, there then exists an immersion $\boldsymbol{\Theta} : \omega \times]-\varepsilon, \varepsilon[\to \mathbf{E}^3$ that satisfies $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in $\omega \times]-\varepsilon, \varepsilon[$. It thus remains to check that $\boldsymbol{\theta} := \boldsymbol{\Theta}(\cdot, 0)$ indeed satisfies (see part (ix))

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

The actual implementation of this program essentially involves elementary, but sometimes lengthy, computations, which accordingly will be omitted for the most part; only the main intermediate results will be recorded.

For clarity, the proof is broken into nine parts, numbered (i) to (ix).

(i) *Given two matrix fields* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$, *let the matrix field* $(g_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3)$ *be defined by*

$$g_{\alpha\beta} := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} := \delta_{i3} \text{ in } \omega \times \mathbb{R}$$

(the variable $y \in \omega$ is omitted; x_3 designates the variable in $\mathbb{R}$), *where*

$$c_{\alpha\beta} := b_\alpha^\tau b_{\beta\tau} \text{ and } b_\alpha^\tau := a^{\sigma\tau} b_{\alpha\sigma} \text{ in } \omega.$$

Let ω_0 *be an open subset of* $\mathbb{R}^2$ *such that* $\overline{\omega}_0$ *is a compact subset of* ω. *Then there exists* $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ *such that the symmetric matrices* (g_{ij}) *are positive definite at all points in* $\overline{\Omega}_0$, *where*

$$\Omega_0 := \omega_0 \times \left]-\varepsilon_0, \varepsilon_0\right[.$$

Besides, the elements of the inverse matrix (g^{pq}) *are given in* $\overline{\Omega}_0$ *by*

$$g^{\alpha\beta} = \sum_{n \geq 0} (n+1) x_3^n a^{\alpha\sigma} (\mathbf{B}^n)_\sigma^\beta \text{ and } g^{i3} = \delta^{i3},$$

where

$$(\mathbf{B})_\sigma^\beta := b_\sigma^\beta \text{ and } (\mathbf{B}^n)_\sigma^\beta := b_\sigma^{\sigma_1} \cdots b_{\sigma_{n-1}}^\beta \text{ for } n \geq 2,$$

i.e., $(\mathbf{B}^n)_\sigma^\beta$ *designates for any* $n \geq 0$ *the element at the* σ-th *row and* β-th *column of the matrix* $\mathbf{B}^n$. *The above series are absolutely convergent in the space* $\mathcal{C}^2(\overline{\Omega}_0)$.

Let *a priori* $g^{\alpha\beta} = \sum_{n\geq 0} x_3^n h_n^{\alpha\beta}$ where $h_n^{\alpha\beta}$ are functions of $y \in \overline{\omega}_0$ only, so that the relations $g^{\alpha\beta} g_{\beta\tau} = \delta_\tau^\alpha$ read

$$h_0^{\alpha\beta} a_{\beta\tau} + x_3 (h_1^{\alpha\beta} a_{\beta\tau} - 2h_0^{\alpha\beta} b_{\beta\tau}) + \sum_{n \geq 2} x_3^n (h_n^{\alpha\beta} a_{\beta\tau} - 2h_{n-1}^{\alpha\beta} b_{\beta\tau} + h_{n-2}^{\alpha\beta} c_{\beta\tau}) = \delta_\tau^\alpha.$$

It is then easily verified that the functions $h_n^{\alpha\beta}$ are given by

$$h_n^{\alpha\beta} = (n+1) a^{\alpha\sigma} (\mathbf{B}^n)_\sigma^\beta, \; n \geq 0,$$

so that

$$g^{\alpha\beta} = \sum_{n \geq 0} (n+1) x_3^n a^{\alpha\sigma} b_\sigma^{\sigma_1} \cdots b_{\sigma_{n-1}}^\beta.$$

It is clear that such a series is absolutely convergent in the space $\mathcal{C}^2(\overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0])$ if $\varepsilon_0 > 0$ is small enough.

(ii) *The functions* $C^\sigma_{\alpha\beta}$ *being defined by*

$$C^\sigma_{\alpha\beta} := a^{\sigma\tau} C_{\alpha\beta\tau},$$

where

$$(a^{\sigma\tau}) := (a_{\alpha\beta})^{-1} \text{ and } C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$

define the functions

$$b^\tau_\alpha|_\beta := \partial_\beta b^\tau_\alpha + C^\tau_{\beta\mu} b^\mu_\alpha - C^\mu_{\alpha\beta} b^\tau_\mu,$$
$$b_{\alpha\beta|\sigma} := \partial_\sigma b_{\alpha\beta} - C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\beta\sigma} b_{\alpha\mu} = b_{\beta\alpha|\sigma}.$$

Then

$$b^\tau_\alpha|_\beta = a^{\sigma\tau} b_{\alpha\sigma|\beta} \text{ and } b_{\alpha\sigma|\beta} = a_{\sigma\tau} b^\tau_\alpha|_\beta.$$

Furthermore, the assumed Codazzi-Mainardi equations imply that

$$b^\tau_\alpha|_\beta = b^\tau_\beta|_\alpha \text{ and } b_{\alpha\sigma|\beta} = b_{\alpha\beta|\sigma}.$$

The above relations follow from straightforward computations based on the definitions of the functions $b^\tau_\alpha|_\beta$ and $b_{\alpha\beta|\sigma}$. They are recorded here because they play a pervading rôle in the subsequent computations.

(iii) *The functions* $g_{ij} \in \mathcal{C}^2(\overline{\Omega}_0)$ *and* $g^{ij} \in \mathcal{C}^2(\overline{\Omega}_0)$ *being defined as in* part (i), *define the functions* $\Gamma_{ijq} \in \mathcal{C}^1(\overline{\Omega}_0)$ *and* $\Gamma^p_{ij} \in \mathcal{C}^1(\overline{\Omega}_0)$ *by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}.$$

Then the functions $\Gamma_{ijq} = \Gamma_{jiq}$ *and* $\Gamma^p_{ij} = \Gamma^p_{ji}$ *have the following expressions*:

$$\begin{aligned}
\Gamma_{\alpha\beta\sigma} &= C_{\alpha\beta\sigma} - x_3(b^\tau_\alpha|_\beta a_{\tau\sigma} + 2C^\tau_{\alpha\beta} b_{\tau\sigma}) + x_3^2(b^\tau_\alpha|_\beta b_{\tau\sigma} + C^\tau_{\alpha\beta} c_{\tau\sigma}),\\
\Gamma_{\alpha\beta 3} &= -\Gamma_{\alpha 3\beta} = b_{\alpha\beta} - x_3 c_{\alpha\beta},\\
\Gamma_{\alpha 33} &= \Gamma_{3\beta 3} = \Gamma_{33q} = 0,\\
\Gamma^\sigma_{\alpha\beta} &= C^\sigma_{\alpha\beta} - \sum\nolimits_{n\geq 0} x_3^{n+1} b^\tau_\alpha|_\beta (\mathbf{B}^n)^\sigma_\tau,\\
\Gamma^3_{\alpha\beta} &= b_{\alpha\beta} - x_3 c_{\alpha\beta},\\
\Gamma^\beta_{\alpha 3} &= -\sum\nolimits_{n\geq 0} x_3^n (\mathbf{B}^{n+1})^\beta_\alpha,\\
\Gamma^3_{3\beta} &= \Gamma^p_{33} = 0,
\end{aligned}$$

where the functions $c_{\alpha\beta}$, $(\mathbf{B}^n)^\sigma_\tau$, *and* $b^\tau_\alpha|_\beta$ *are defined as in* parts (i) *and* (ii).

All computations are straightforward. We simply point out that the assumed *Codazzi-Mainardi equations* are needed to conclude that the factor of x_3 in the function $\Gamma_{\alpha\beta\sigma}$ is indeed that announced above. We also note that the computation of the factor of x_3^2 in $\Gamma_{\alpha\beta\sigma}$ relies in particular on the easily established relations

$$\partial_\alpha c_{\beta\sigma} = b^\tau_\beta|_\alpha b_{\sigma\tau} + b^\mu_\sigma|_\alpha b_{\mu\beta} + C^\mu_{\alpha\beta} c_{\sigma\mu} + C^\mu_{\alpha\sigma} c_{\beta\mu}.$$

(iv) *The functions* $\Gamma_{ijq} \in \mathcal{C}^1(\overline{\Omega}_0)$ *and* $\Gamma_{ij}^p \in \mathcal{C}^1(\overline{\Omega}_0)$ *being defined as in* part (iii), *define the functions* $R_{qijk} \in \mathcal{C}^0(\overline{\Omega}_0)$ *by*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}.$$

Then, in order that the relations

$$R_{qijk} = 0 \text{ in } \overline{\Omega}_0$$

hold, it is sufficient that

$$R_{1212} = 0, \quad R_{\alpha 2\beta 3} = 0, \quad R_{\alpha 3\beta 3} = 0 \text{ in } \overline{\Omega}_0.$$

The above definition of the functions R_{qijk} and that of the functions Γ_{ijq} and Γ_{ij}^p (part (iii)) together imply that, for all i, j, k, q,

$$R_{qijk} = R_{jkqi} = -R_{qikj},$$
$$R_{qijk} = 0 \text{ if } j = k \text{ or } q = i.$$

Consequently, the relation $R_{1212} = 0$ implies that $R_{\alpha\beta\sigma\tau} = 0$, the relations $R_{\alpha 2\beta 3} = 0$ imply that $R_{qijk} = 0$ if exactly one index is equal to 3, and finally, the relations $R_{\alpha 3\beta 3} = 0$ imply that $R_{qijk} = 0$ if exactly two indices are equal to 3.

(v) *The functions*

$$R_{\alpha 3\beta 3} := \partial_\beta \Gamma_{33\alpha} - \partial_3 \Gamma_{3\beta\alpha} + \Gamma_{3\beta}^p \Gamma_{3\alpha p} - \Gamma_{33}^p \Gamma_{\beta\alpha p}$$

satisfy

$$R_{\alpha 3\beta 3} = 0 \text{ in } \overline{\Omega}_0.$$

These relations immediately follow from the expressions found in part (iii) for the functions Γ_{ijq} and Γ_{ij}^p. Note that neither the Gauß equations nor the Codazzi-Mainardi equations are needed here.

(vi) *The functions*

$$R_{\alpha 2\beta 3} := \partial_\beta \Gamma_{23\alpha} - \partial_3 \Gamma_{2\beta\alpha} + \Gamma_{2\beta}^p \Gamma_{3\alpha p} - \Gamma_{23}^p \Gamma_{\beta\alpha p}$$

satisfy

$$R_{\alpha 2\beta 3} = 0 \text{ in } \overline{\Omega}_0.$$

The definitions of the functions $g_{\alpha\beta}$ (part (i)) and Γ_{ijq} (part (iii)) show that

$$\partial_\beta \Gamma_{23\alpha} - \partial_3 \Gamma_{2\beta\alpha} = (\partial_2 b_{\alpha\beta} - \partial_\alpha b_{2\beta}) + x_3(\partial_\alpha c_{2\beta} - \partial_2 c_{\alpha\beta}).$$

Then the expressions found in part (iii) show that

$$\begin{aligned}\Gamma_{2\beta}^p \Gamma_{3\alpha p} - \Gamma_{23}^p \Gamma_{\beta\alpha p} &= \Gamma_{3\alpha}^\sigma \Gamma_{2\beta\sigma} - \Gamma_{23}^\sigma \Gamma_{\alpha\beta\sigma} \\ &= C_{\alpha\beta}^\sigma b_{2\sigma} - C_{2\beta}^\sigma b_{\alpha\sigma} \\ &\quad + x_3(b_2^\sigma|_\beta b_{\alpha\sigma} - b_\alpha^\sigma|_\beta b_{2\sigma} + C_{2\beta}^\sigma c_{\alpha\sigma} - C_{\alpha\beta}^\sigma c_{2\sigma}),\end{aligned}$$

and the relations $R_{\alpha 3\beta 3} = 0$ follow by making use of the relations

$$\partial_\alpha c_{\beta\sigma} = b^\tau_\beta|_\alpha b_{\sigma\tau} + b^\mu_\sigma|_\alpha b_{\mu\beta} + C^\mu_{\alpha\beta} c_{\sigma\mu} + C^\mu_{\alpha\sigma} c_{\beta\mu}$$

together with the relations

$$\partial_2 b_{\alpha\beta} - \partial_\alpha b_{2\beta} + C^\sigma_{\alpha\beta} b_{2\sigma} - C^\sigma_{2\beta} b_{\alpha\sigma} = 0,$$

which are special cases of the assumed *Codazzi-Mainardi equations*.

(vii) *The function*

$$R_{1212} := \partial_1 \Gamma_{221} - \partial_2 \Gamma_{211} + \Gamma^p_{21}\Gamma_{21p} - \Gamma^p_{22}\Gamma_{11p}$$

satisfies

$$R_{1212} = 0 \text{ in } \overline{\Omega}_0.$$

The computations leading to this relation are fairly lengthy and they require some care. We simply record the main intermediary steps, which consist in evaluating separately the various terms occurring in the function R_{1212} rewritten as

$$R_{1212} = (\partial_1\Gamma_{221} - \partial_2\Gamma_{211}) + (\Gamma^\sigma_{12}\Gamma_{12\sigma} - \Gamma^\sigma_{11}\Gamma_{22\sigma}) + (\Gamma_{123}\Gamma_{123} - \Gamma_{113}\Gamma_{223}).$$

First, the expressions found in part (iii) for the functions $\Gamma_{\alpha\beta 3}$ easily yield

$$\begin{aligned}\Gamma_{123}\Gamma_{123} - \Gamma_{113}\Gamma_{223} \;=\; & (b^2_{12} - b_{11}b_{22}) \\ & + \; x_3(b_{11}c_{22} - 2b_{12}c_{12} + b_{22}c_{11}) + x^2_3(c^2_{12} - c_{11}c_{22}).\end{aligned}$$

Second, the expressions found in part (iii) for the functions $\Gamma_{\alpha\beta\sigma}$ and $\Gamma^\sigma_{\alpha\beta}$ yield, after some manipulations:

$$\begin{aligned}\Gamma^\sigma_{12}\Gamma_{12\sigma} - \Gamma^\sigma_{11}\Gamma_{22\sigma} = {} & (C^\sigma_{12}C^\tau_{12} - C^\sigma_{11}C^\tau_{22})a_{\sigma\tau} \\ & + x_3\{(C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)a_{\sigma\tau} \\ & + 2(C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})b_{\sigma\tau}\} \\ & + x^2_3\{b^\sigma_1|_1 b^\tau_2|_2 - b^\sigma_1|_2 b^\tau_1|_2)a_{\sigma\tau} \\ & + (C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)b_{\sigma\tau} \\ & + (C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})c_{\sigma\tau}\}.\end{aligned}$$

Third, after somewhat delicate computations, which in particular make use of the relations established in part (ii) about the functions $b^\tau_\alpha|_\beta$ and $b_{\alpha\beta|\sigma}$, it is found that

$$\begin{aligned}\partial_1\Gamma_{221} - \partial_2\Gamma_{211} = {} & \partial_1 C_{221} - \partial_2 C_{211} \\ & - x_3\{S_{1212}b^\alpha_\alpha + (C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)a_{\sigma\tau} \\ & + 2(C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})b_{\sigma\tau}\} \\ & + x^2_3\{S_{\sigma\tau 12}b^\sigma_1 b^\tau_2 + (b^\sigma_1|_1 b^\tau_2|_2 - b^\sigma_1|_2 b^\tau_1|_2)a_{\sigma\tau} \\ & + (C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)b_\sigma \\ & + (C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})c_{\sigma\tau}\},\end{aligned}$$

where the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu}$$

are precisely those appearing in the left-hand sides of the Gauß equations.

It is then easily seen that the above equations together yield

$$\begin{aligned} R_{1212} = \{&S_{1212} - (b_{11}b_{22} - b_{12}b_{12})\} \\ &- x_3\{S_{1212} - (b_{11}b_{22} - b_{12}b_{12})b^\alpha_\alpha\} \\ &+ x_3^2\{S_{\sigma\tau 12} b_1^\sigma b_2^\tau + (c_{12}c_{12} - c_{11}c_{22})\}. \end{aligned}$$

Since

$$\begin{aligned} S_{\sigma\tau 12} b_1^\sigma b_2^\tau &= S_{1212}(b_1^1 b_2^2 - b_1^2 b_2^1), \\ c_{12}c_{12} - c_{11}c_{22} &= (b_{11}b_{12} - b_{11}b_{22})(b_1^1 b_2^2 - b_1^2 b_2^1), \end{aligned}$$

it is finally found that the function R_{1212} has the following remarkable expression:

$$R_{1212} = \{S_{1212} - (b_{11}b_{22} - b_{12}b_{12})\}\{1 - x_3(b_1^1 + b_2^2) + x_3^2(b_1^1 b_2^2 - b_1^2 b_2^1)\}.$$

By the assumed *Gauß equations*,

$$S_{1212} = b_{11}b_{22} - b_{12}b_{12}.$$

Hence $R_{1212} = 0$ as announced.

(viii) *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Then there exist open subsets $\omega_\ell, \ell \geq 0$, of $\mathbb{R}^2$ such that $\overline{\omega}_\ell$ is a compact subset of ω for each $\ell \geq 0$ and*

$$\omega = \bigcup_{\ell \geq 0} \omega_\ell.$$

Furthermore, for each $\ell \geq 0$, there exists $\varepsilon_\ell = \varepsilon_\ell(\omega_\ell) > 0$ such that the symmetric matrices (g_{ij}) are positive definite at all points in $\overline{\Omega}_\ell$, where

$$\Omega_\ell := \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[.$$

Finally, the open set

$$\Omega := \bigcup_{\ell \geq 0} \Omega_\ell$$

is connected and simply-connected.

Let ω_ℓ, $\ell \geq 0$, be open subsets of ω with compact closures $\overline{\omega}_\ell \subset \omega$ such that $\omega = \bigcup_{\ell \geq 0} \omega_\ell$. For each ℓ, a set $\Omega_\ell := \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[$ can then be constructed in the same way that the set Ω_0 was constructed in part (i).

It is clear that the set $\Omega := \bigcup_{\ell \geq 0} \Omega_\ell$ is connected. To show that Ω is simply-connected, let $\boldsymbol{\gamma}$ be a *loop in* Ω, i.e., a mapping $\boldsymbol{\gamma} \in \mathcal{C}^0([0,1]; \mathbb{R}^3)$ that satisfies

$$\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1) \text{ and } \boldsymbol{\gamma}(t) \in \Omega \text{ for all } 0 \leq t \leq 1.$$

Let the projection operator $\boldsymbol{\pi} : \Omega \to \omega$ be defined by $\boldsymbol{\pi}(y, x_3) = y$ for all $(y, x_3) \in \Omega$, and let the mapping $\boldsymbol{\varphi}_0 : [0,1] \times [0,1] \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\varphi}_0(t, \lambda) := (1-\lambda)\boldsymbol{\gamma}(t) + \lambda\boldsymbol{\pi}(\boldsymbol{\gamma}(t)) \text{ for all } 0 \le t \le 1,\ 0 \le \lambda \le 1.$$

Then $\boldsymbol{\varphi}_0$ is a continuous mapping such that $\boldsymbol{\varphi}_0([0,1]\times[0,1]) \subset \Omega$, by definition of the set Ω. Furthermore, $\boldsymbol{\varphi}_0(t,0) = \boldsymbol{\gamma}(t)$ and $\boldsymbol{\varphi}_0(t,1) = \boldsymbol{\pi}(\boldsymbol{\gamma}(t))$ for all $t \in [0,1]$.

The mapping

$$\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\pi} \circ \boldsymbol{\gamma} \in \mathcal{C}^0([0,1]; \mathbb{R}^2)$$

is a *loop in* ω since $\widetilde{\boldsymbol{\gamma}}(0) = \boldsymbol{\pi}(\boldsymbol{\gamma}(0)) = \boldsymbol{\pi}(\boldsymbol{\gamma}(1)) = \widetilde{\boldsymbol{\gamma}}(1)$ and $\widetilde{\boldsymbol{\gamma}}(t) \in \omega$ for all $0 \le t \le 1$. Since ω is simply connected, there exist a mapping $\boldsymbol{\varphi}_1 \in \mathcal{C}^0([0,1] \times [0,1]; \mathbb{R}^2)$ and a point $y^0 \in \omega$ such that

$$\boldsymbol{\varphi}_1(t,1) = \widetilde{\boldsymbol{\gamma}} \text{ and } \boldsymbol{\varphi}_1(t,2) = y^0 \text{ for all } 0 \le t \le 1$$

and

$$\boldsymbol{\varphi}_1(t,\lambda) \in \omega \text{ for all } 0 \le t \le 1,\ 1 \le \lambda \le 2.$$

Then the mapping $\boldsymbol{\varphi} \in \mathcal{C}^0([0,1] \times [0,2]; \mathbb{R}^3)$ defined by

$$\begin{aligned}
\boldsymbol{\varphi}(t,\lambda) &= \boldsymbol{\varphi}_0(t,\lambda) \quad \text{for all} \quad 0 \le t \le 1,\ 0 \le \lambda \le 1,\\
\boldsymbol{\varphi}(t,\lambda) &= \boldsymbol{\varphi}_1(t,\lambda) \quad \text{for all} \quad 0 \le t \le 1,\ 1 \le \lambda \le 2,
\end{aligned}$$

is a *homotopy in* Ω that reduces the loop $\boldsymbol{\gamma}$ to the point $(y^0, 0) \in \Omega$. Hence the set Ω is simply-connected.

(ix) *By* parts (iv) *to* (viii), *the functions* $\Gamma_{ijq} \in \mathcal{C}^1(\Omega)$ *and* $\Gamma_{ij}^p \in \mathcal{C}^1(\Omega)$ *constructed as in* part (iii) *satisfy*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0$$

in the connected and simply-connected open set Ω. *By* Theorem 1.6-1, *there thus exists an immersion* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *such that*

$$g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \text{ in } \Omega,$$

where the matrix field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *is defined by*

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \Omega.$$

Then the mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ *defined by*

$$\boldsymbol{\theta}(y) = \boldsymbol{\Theta}(y, 0) \text{ for all } y \in \omega,$$

satisfies

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \textit{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Let $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$. Then $\partial_{33}\boldsymbol{\Theta} = \partial_3 \boldsymbol{g}_3 = \Gamma^p_{33}\boldsymbol{g}_p = \mathbf{0}$; cf. part (iii). Hence there exists a mapping $\boldsymbol{\theta}^1 \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3\boldsymbol{\theta}^1(y) \text{ for all } (y, x_3) \in \Omega,$$

and consequently, $\boldsymbol{g}_\alpha = \partial_\alpha\boldsymbol{\theta} + x_3\partial_\alpha\boldsymbol{\theta}^1$ and $\boldsymbol{g}_3 = \boldsymbol{\theta}^1$. The relations $g_{i3} = \boldsymbol{g}_i \cdot \boldsymbol{g}_3 = \delta_{i3}$ (cf. part (i)) then show that

$$(\partial_\alpha\boldsymbol{\theta} + x_3\partial_\alpha\boldsymbol{\theta}^1) \cdot \boldsymbol{\theta}^1 = 0 \text{ and } \boldsymbol{\theta}^1 \cdot \boldsymbol{\theta}^1 = 1.$$

These relations imply that $\partial_\alpha\boldsymbol{\theta} \cdot \boldsymbol{\theta}^1 = 0$. Hence either $\boldsymbol{\theta}^1 = \boldsymbol{a}_3$ or $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ in ω, where

$$\boldsymbol{a}_3 := \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}.$$

But $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ is ruled out since

$$\{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}\} \cdot \boldsymbol{\theta}^1 = \det(g_{ij})|_{x_3=0} > 0.$$

Noting that

$$\partial_\alpha\boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \text{ implies } \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{a}_3 = -\partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3,$$

we obtain, on the one hand,

$$\begin{aligned} g_{\alpha\beta} &= (\partial_\alpha\boldsymbol{\theta} + x_3\partial_\alpha\boldsymbol{a}_3) \cdot (\partial_\beta\boldsymbol{\theta} + x_3\partial_\beta\boldsymbol{a}_3) \\ &= \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} - 2x_3\partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3 + x_3^2\partial_\alpha\boldsymbol{a}_3 \cdot \partial_\beta\boldsymbol{a}_3 \text{ in } \Omega. \end{aligned}$$

Since, on the other hand,

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ in } \Omega$$

by part (i), we conclude that

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3 \text{ in } \omega,$$

as desired. This completes the proof. $\square$

Remarks. (1) The functions $c_{\alpha\beta} = b^\tau_\alpha b_{\beta\tau} = \partial_\alpha\boldsymbol{a}_3 \cdot \partial_\beta\boldsymbol{a}_3$ introduced in part (i) are the covariant components of the *third fundamental form* of the surface $\boldsymbol{\theta}(\omega)$.

(2) The series expansion $g^{\alpha\beta} = \sum_{n\geq 0}(n+1)x_3^n a^{\alpha\sigma}(\mathbf{B}^n)^\beta_\sigma$ found in part (i) is known; cf., e.g., Naghdi [1972].

(3) The *Gauß equations* are used only *once* in the above proof, for showing that $R_{1212} = 0$ in part (vii). $\square$

The definitions of the functions $C^\sigma_{\alpha\beta}$ and $C_{\alpha\beta\tau}$ imply that *the sixteen Gauß equations are satisfied if and only if they are satisfied for* $\alpha = 1, \beta = 2, \sigma = 1, \tau = 2$ and that *the Codazzi-Mainardi equations are satisfied if and only* *if they are satisfied* for $\alpha = 1$, $\beta = 2$, $\sigma = 1$ and $\alpha = 1$, $\beta = 2$, $\sigma = 2$ (other choices of indices with the same properties are clearly possible).

In other words, the Gauß equations and the Codazzi-Mainardi equations in fact reduce to *one* and *two* equations, respectively.

The *regularity assumptions* made in Theorem 2.8-1 on the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ can be significantly relaxed in several ways. First, C. Mardare [2003b] has shown by means of an *ad hoc*, but not trivial, modification of the proof given here, that the existence of an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ still holds under the weaker (but certainly more natural, in view of the regularity of the resulting immersion $\boldsymbol{\theta}$) assumption that $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$, all other assumptions of Theorem 2.8-1 holding *verbatim*.

In fact, Hartman & Wintner [1950] have shown the stronger result that the existence theorem still holds if $(a_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^0(\omega; \mathbb{S}^2)$, with a resulting mapping $\boldsymbol{\theta}$ in the space $\mathcal{C}^2(\omega; \mathbf{E}^3)$. Their result has been itself superseded by that of S. Mardare [2004], which asserts that if $(a_{\alpha\beta}) \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^{\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ are two matrix fields that satisfy the Gauß and Codazzi-Mainardi equations in the sense of distributions, then there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\mathrm{loc}}(\omega)$ such that $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are the fundamental forms of the surface $\boldsymbol{\theta}(\omega)$.

2.9 UNIQUENESS UP TO ISOMETRIES OF SURFACES WITH THE SAME FUNDAMENTAL FORMS

In Section 2.8, we have established the *existence* of an immersion $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ giving rise to a surface $\boldsymbol{\theta}(\omega)$ with prescribed first and second fundamental forms, provided these forms satisfy *ad hoc* sufficient conditions. We now turn to the question of *uniqueness* of such immersions.

This is the object of the next theorem, which constitutes another *rigidity theorem*, called the **rigidity theorem for surfaces**. Like its "three-dimensional counterpart" (Theorem 1.7-1), it asserts that, if two immersions $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ share the same fundamental forms, then the surface $\widetilde{\boldsymbol{\theta}}(\omega)$ is obtained by subjecting the surface $\boldsymbol{\theta}(\omega)$ to a *rotation* (represented by an orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = 1$), then by subjecting the rotated surface to a *translation* (represented by a vector $\boldsymbol{c}$). Such a *"rigid"* transformation is thus an *isometry* in $\mathbf{E}^3$.

As shown by Ciarlet & Larsonneur [2001] (whose proof is adapted here), the issue of uniqueness can be resolved as a corollary to its "three-dimensional counterpart", like the issue of existence. We recall that $\mathbb{O}^3$ denotes the set of all orthogonal matrices of order three and that $\mathbb{O}^3_+ = \{\mathbf{Q} \in \mathbb{O}^3; \det \mathbf{Q} = 1\}$ denotes the set of all 3×3 *rotations*.

Theorem 2.9-1. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ be two immersions such that their associated first and second fundamental forms satisfy* (with self-explanatory notations)

$$a_{\alpha\beta} = \widetilde{a}_{\alpha\beta} \text{ and } b_{\alpha\beta} = \widetilde{b}_{\alpha\beta} \text{ in } \omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ *and a rotation* $\mathbf{Q} \in \mathbb{O}^3_+$ *such that*

$$\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y) \text{ for all } y \in \omega.$$

Proof. Arguments similar to those used in parts (i) and (viii) of the proof of Theorem 2.8-1 show that there exist open subsets ω_ℓ of ω and real numbers $\varepsilon_\ell > 0$, $\ell \geq 0$, such that the symmetric matrices (g_{ij}) defined by

$$g_{\alpha\beta} := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3},$$

where $c_{\alpha\beta} := a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$, are positive definite in the set

$$\Omega := \bigcup_{\ell \geq 0} \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[.$$

The two immersions $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ defined by (with self-explanatory notations)

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ and } \widetilde{\boldsymbol{\Theta}}(y, x_3) := \widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y)$$

for all $(y, x_3) \in \Omega$ therefore satisfy

$$g_{ij} = \widetilde{g}_{ij} \text{ in } \Omega.$$

By Theorem 1.7-1, there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and an orthogonal matrix $\mathbf{Q} \in \mathbb{O}^3$ such that

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

Hence, on the one hand,

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) = \det \mathbf{Q} \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

On the other hand, a simple computation shows that

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) = \sqrt{\det(a_{\alpha\beta}(y)}\{1 - x_3(b_1^1 + b_2^2)(y) + x_3^2(b_1^1 b_2^2 - b_1^2 b_2^1)(y)\}$$

for all $(y, x_3) \in \Omega$, where

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y),\ y \in \omega,$$

so that

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) = \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

Therefore $\det \mathbf{Q} = 1$, which shows that the matrix $\mathbf{Q} \in \mathbb{O}^3$ is in fact a rotation. The conclusion then follows by letting $x_3 = 0$ in the relation

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

$\square$

As a preparation to our next result, we note that the second fundamental form of the surface $\boldsymbol{\theta}(\omega)$ can still be defined under the weaker assumptions that $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$, by means of the definition

$$b_{\alpha\beta} := -\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3,$$

which evidently coincides with the usual one when $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$.

Theorem 2.9-1 constitutes the "classical" rigidity theorem for surfaces, in the sense that both immersions $\boldsymbol{\theta}$ and $\widetilde{\boldsymbol{\theta}}$ are assumed to be in the space $\mathcal{C}^2(\omega; \mathbf{E}^3)$.

Following Ciarlet & C. Mardare [2004a], we now show that *a similar result holds under the assumptions that* $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ *and* $\widetilde{\boldsymbol{a}}_3 := \dfrac{\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2}{|\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2|} \in H^1(\omega; \mathbf{E}^3)$ (with self-explanatory notations). Naturally, our first task will be to verify that the vector field $\widetilde{\boldsymbol{a}}_3$, which is not necessarily well defined a.e. in ω for an arbitrary mapping $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$, is nevertheless well defined a.e. in ω for those mappings $\widetilde{\boldsymbol{\theta}}$ that satisfy the assumptions of the next theorem. This fact will in turn imply that the functions $\widetilde{b}_{\alpha\beta} := -\widetilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3$ are likewise well defined a.e. in ω.

Theorem 2.9-2. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \mathcal{C}^1(\omega; \mathbf{E}^3)$. Assume that there exists a vector field $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ that satisfies*

$$\widetilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega, \quad \widetilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbf{E}^3), \quad \text{ and } \quad \widetilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\theta}(y) \text{ for almost all } y \in \omega.$$

Proof. The proof essentially relies on the extension to a Sobolev space setting of the "three-dimensional" rigidity theorem established in Theorem 1.7-3.

(i) To begin with, we record several *technical preliminaries.*

First, we observe that the relations $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω and the assumption that $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ is an immersion together imply that

$$|\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2| = \sqrt{\det(\widetilde{a}_{\alpha\beta})} = \sqrt{\det(a_{\alpha\beta})} > 0 \text{ a.e. in } \omega.$$

Consequently, *the vector field $\widetilde{\boldsymbol{a}}_3$, and thus the functions $\widetilde{b}_{\alpha\beta}$, are well defined a.e. in ω.*

Second, we establish that

$$b_{\alpha\beta} = b_{\beta\alpha} \text{ in } \omega \text{ and } \widetilde{b}_{\alpha\beta} = \widetilde{b}_{\beta\alpha} \text{ a.e. in } \omega,$$

i.e., that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \boldsymbol{a}_\beta \cdot \partial_\alpha \boldsymbol{a}_3$ in ω and $\widetilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3 = \widetilde{\boldsymbol{a}}_\beta \cdot \partial_\alpha \widetilde{\boldsymbol{a}}_3$ a.e. in ω. To this end, we note that either the assumptions $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ together, or the assumptions $\boldsymbol{\theta} \in H^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 \in H^1(\omega; \mathbf{E}^3)$ together, imply that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in L^1_{\mathrm{loc}}(\omega)$, hence that $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in \mathcal{D}'(\omega)$.

Given any $\varphi \in \mathcal{D}(\omega)$, let U denote an open subset of $\mathbb{R}^2$ such that $\operatorname{supp} \varphi \subset U$ and $\overline{U}$ is a compact subset of ω. Denoting by ${}_{X'}\langle \cdot, \cdot \rangle_X$ the duality pairing between a topological vector space X and its dual X', we have

$$\begin{aligned}
{}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3, \varphi \rangle_{\mathcal{D}(\omega)} &= \int_\omega \varphi \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \, \mathrm{d}y \\
&= \int_\omega \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y - \int_\omega (\partial_\beta \varphi) \partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \, \mathrm{d}y.
\end{aligned}$$

Observing that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0$ a.e. in ω and that

$$\begin{aligned}
-\int_\omega \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y &= -\int_U \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y \\
&= {}_{H^{-1}(U;\mathbb{E}^3)}\langle \partial_\beta (\partial_\alpha \boldsymbol{\theta}), \, \varphi \boldsymbol{a}_3 \rangle_{H^1_0(U;\mathbb{E}^3)},
\end{aligned}$$

we reach the conclusion that the expression ${}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3, \, \varphi \rangle_{\mathcal{D}(\omega)}$ is symmetric with respect to α and β since $\partial_{\alpha\beta} \boldsymbol{\theta} = \partial_{\beta\alpha} \boldsymbol{\theta}$ in $\mathcal{D}'(U)$. Hence $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3$ in $L^1_{\mathrm{loc}}(\omega)$, and the announced symmetries are established.

Third, let

$$\widetilde{c}_{\alpha\beta} := \partial_\alpha \widetilde{\boldsymbol{a}}_3 \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3 \text{ and } c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3.$$

Then we claim that $\widetilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω. To see this, we note that the matrix fields $(\widetilde{a}^{\alpha\beta}) := (\widetilde{a}_{\alpha\beta})^{-1}$ and $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$ are well defined and equal a.e. in ω since $\boldsymbol{\theta}$ is an immersion and $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω. The formula of Weingarten (Section 2.6) can thus be applied a.e. in ω, showing that $\widetilde{c}_{\alpha\beta} = \widetilde{a}^{\sigma\tau} \widetilde{b}_{\sigma\alpha} \widetilde{b}_{\tau\beta}$ a.e. in ω.

The assertion then follows from the assumptions $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω.

(ii) *Starting from the set ω and the mapping $\boldsymbol{\theta}$* (as given in the statement of Theorem 2.9-2), *we next construct a set Ω and a mapping $\boldsymbol{\Theta}$ that satisfy the assumptions of* Theorem 1.7-2. More precisely, let

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \omega \times \mathbb{R}.$$

Then the mapping $\boldsymbol{\Theta} := \omega \times \mathbb{R} \to \mathbb{E}^3$ defined in this fashion is clearly continuously differentiable on $\omega \times \mathbb{R}$ and

$$\det \boldsymbol{\nabla\Theta}(y, x_3) = \sqrt{\det(a_{\alpha\beta}(y))} \{1 - x_3(b_1^1 + b_2^2)(y) + x_3^2(b_1^1 b_2^2 - b_1^2 b_2^1)(y)\}$$

for all $(y, x_3) \in \omega \times \mathbb{R}$, where

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y), y \in \omega.$$

Let ω_n, $n \geq 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_n$ is a compact subset of ω and $\omega = \bigcup_{n \geq 0} \omega_n$. Then the continuity of the functions $a_{\alpha\beta}, a^{\alpha\beta}, b_{\alpha\beta}$ and the assumption that $\boldsymbol{\theta}$ is an immersion together imply that, for each $n \geq 0$, there exists $\varepsilon_n > 0$ such that

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) > 0 \text{ for all } (y, x_3) \in \overline{\omega}_n \times [-\varepsilon_n, \varepsilon_n].$$

Besides, there is no loss of generality in assuming that $\varepsilon_n \leq 1$ (this property will be used in part (iii)).

Let then

$$\Omega := \bigcup_{n \geq 0} (\omega_n \times]-\varepsilon_n, \varepsilon_n[).$$

Then it is clear that Ω *is a connected open subset of* $\mathbb{R}^3$ and that *the mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *satisfies* $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ *in* Ω.

Finally, note that the covariant components $g_{ij} \in \mathcal{C}^0(\Omega)$ of the metric tensor field associated with the mapping $\boldsymbol{\Theta}$ are given by (the symmetries $b_{\alpha\beta} = b_{\beta\alpha}$ established in (i) are used here)

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1.$$

(iii) Starting with the mapping $\widetilde{\boldsymbol{\theta}}$ (as given in the statement of Theorem 2.9-2), we *construct a mapping* $\widetilde{\boldsymbol{\Theta}}$ *that satisfies the assumptions of* Theorem 1.7-2. To this end, we define a mapping $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbb{E}^3$ by letting

$$\widetilde{\boldsymbol{\Theta}}(y, x_3) := \widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y) \text{ for all } (y, x_3) \in \Omega,$$

where the set Ω is defined as in (ii). Hence $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$, since $\Omega \subset \omega \times]-1, 1[$. Besides, $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \det \boldsymbol{\nabla}\boldsymbol{\Theta}$ a.e. in Ω since the functions $\widetilde{b}_\alpha^\beta := \widetilde{a}^{\beta\sigma}\widetilde{b}_{\alpha\sigma}$, which are well defined a.e. in ω, are equal, again a.e. in ω, to the functions b_α^β. Likewise, the components $\widetilde{g}_{ij} \in L^1(\Omega)$ of the metric tensor field associated with the mapping $\widetilde{\boldsymbol{\Theta}}$ satisfy $\widetilde{g}_{ij} = g_{ij}$ a.e. in Ω since $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω by assumption and $\widetilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω by part (i).

(iv) By Theorem 1.7-2, there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y) = \boldsymbol{c} + \mathbf{Q}(\boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)) \text{ for almost all } (y, x_3) \in \Omega.$$

Differentiating with respect to x_3 in this equality between functions in $H^1(\Omega; \mathbf{E}^3)$ shows that $\widetilde{\boldsymbol{a}}_3(y) = \mathbf{Q}\boldsymbol{a}_3(y)$ for almost all $y \in \omega$. Hence $\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\theta}(y)$ for almost all $y \in \omega$ as announced. $\square$

Remarks. (1) The existence of $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ satisfying the assumptions of Theorem 2.9-2 implies that $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 \in \mathcal{C}^1(\omega; \mathbf{E}^3)$, and that $\boldsymbol{\theta} \in H^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 \in H^1(\omega; \mathbf{E}^3)$.

(2) It is easily seen that the conclusion of Theorem 2.9-2 is still valid if the assumptions $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbf{E}^3)$ are replaced by the weaker assumptions $\widetilde{\boldsymbol{\theta}} \in H^1_{\mathrm{loc}}(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 \in H^1_{\mathrm{loc}}(\omega; \mathbf{E}^3)$. $\square$

2.10 CONTINUITY OF A SURFACE AS A FUNCTION OF ITS FUNDAMENTAL FORMS

Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Together, Theorems 2.8-1 and 2.9-1 establish the existence of a mapping F that associates to any pair of matrix fields $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ satisfying the Gauß and Codazzi-Mainardi equations in ω a well-defined element $F((a_{\alpha\beta}),(b_{\alpha\beta}))$ in the quotient set $\mathcal{C}^3(\omega;\mathbf{E}^3)/R$, where $(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}}) \in R$ means that there exists a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a rotation $\mathbf{Q} \in \mathbb{O}^3_+$ such that $\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y \in \omega$.

A natural question thus arises as to whether there exist *ad hoc* topologies on the space $\mathcal{C}^2(\omega;\mathbb{S}^2) \times \mathcal{C}^2(\omega;\mathbb{S}^2)$ and on the quotient set $\mathcal{C}^3(\omega;\mathbf{E}^3)/R$ such that the mapping F defined in this fashion is *continuous*.

Equivalently, *is a surface a continuous function of its fundamental forms?*

The purpose of this section, which is based on Ciarlet [2003], is to provide an affirmative answer to the above question, through a proof that relies in an essential way on the solution to the *analogous problem in dimension three* given in Section 1.8.

Such a question is not only relevant to surface theory, but it also finds its source in two-dimensional *nonlinear shell theories*, where the *stored energy functions* are often functions of the *first and second fundamental forms of the unknown deformed middle surface* (for an overview of nonlinear shell theories, see, e.g., Ciarlet [2000]). For instance, the well-known stored energy function w_K proposed by Koiter [1966, Equations (4.2), (8.1), and (8.3)] for modeling nonlinearly elastic shells made with a homogeneous and isotropic elastic material takes the form:

$$w_K = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau}(\widetilde{a}_{\sigma\tau} - a_{\sigma\tau})(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau}(\widetilde{b}_{\sigma\tau} - b_{\sigma\tau})(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta}),$$

where 2ε is the thickness of the shell,

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

$\lambda > 0$ and $\mu > 0$ are the two *Lamé constants* of the constituting material, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the given undeformed middle surface, $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$, and finally $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the *unknown deformed middle surface*.

An inspection of the above stored energy functions thus suggests a tempting approach to shell theory, where the functions $\widetilde{a}_{\alpha\beta}$ *and* $\widetilde{b}_{\alpha\beta}$ *would be regarded as the primary unknowns* in lieu of the customary (Cartesian or curvilinear) components of the displacement. In such an approach, the unknown components $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ must naturally satisfy the classical *Gauß* and *Codazzi-Mainardi equations* in order that they actually define a surface.

To begin with, we introduce the following two-dimensional analogs to the notations used in Section 1.8. Let ω be an open subset of $\mathbb{R}^3$. The notation $\kappa \Subset \omega$

means that κ is a compact subset of ω. If $f \in \mathcal{C}^\ell(\omega;\mathbb{R})$ or $\boldsymbol{\theta} \in \mathcal{C}^\ell(\omega;\mathbf{E}^3)$, $\ell \geq 0$, and $\kappa \Subset \omega$, we let

$$\|f\|_{\ell,\kappa} := \sup_{\left\{\substack{y\in\kappa \\ |\alpha|\leq\ell}\right.} |\partial^\alpha f(y)| \quad , \quad \|\boldsymbol{\theta}\|_{\ell,\kappa} := \sup_{\left\{\substack{y\in\kappa \\ |\alpha|\leq\ell}\right.} |\partial^\alpha \boldsymbol{\theta}(y)|,$$

where ∂^α stands for the standard multi-index notation for partial derivatives and $|\cdot|$ denotes the Euclidean norm in the latter definition. If $\mathbf{A} \in \mathcal{C}^\ell(\omega;\mathbb{M}^3)$, $\ell \geq 0$, and $\kappa \Subset \omega$, we likewise let

$$\|\mathbf{A}\|_{\ell,\kappa} = \sup_{\left\{\substack{y\in\kappa \\ |\alpha|\leq\ell}\right.} |\partial^\alpha \mathbf{A}(y)|,$$

where $|\cdot|$ denotes the matrix spectral norm.

The next *sequential continuity* result constitutes the key step towards establishing the continuity of a surface as a function of its two fundamental forms in *ad hoc* metric spaces (see Theorem 2.10-2).

Theorem 2.10-1. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ be matrix fields satisfying the Gauß and Codazzi-Mainardi equations in ω and let $(a^n_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b^n_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ be matrix fields satisfying for each $n \geq 0$ the Gauß and Codazzi-Mainardi equations in ω. Assume that these matrix fields satisfy*

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbf{E}^3)$ be any mapping that satisfies

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{\frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}\right\} \text{ in } \omega$$

(such mappings exist by Theorem 2.8-1). *Then there exist mappings $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega;\mathbf{E}^3)$ satisfying*

$$a^n_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta}^n \cdot \partial_\beta\boldsymbol{\theta}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta}^n \cdot \left\{\frac{\partial_1\boldsymbol{\theta}^n \wedge \partial_2\boldsymbol{\theta}^n}{|\partial_1\boldsymbol{\theta}^n \wedge \partial_2\boldsymbol{\theta}^n|}\right\} \text{ in } \omega,\ n \geq 0,$$

such that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Proof. For clarity, the proof is broken into five parts.

(i) *Let the matrix fields $(g_{ij}) \in \mathcal{C}^2(\omega\times\mathbb{R};\mathbb{S}^3)$ and $(g^n_{ij}) \in \mathcal{C}^2(\omega\times\mathbb{R};\mathbb{S}^3)$, $n \geq 0$, be defined by*

$$\begin{aligned} g_{\alpha\beta} &:= a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{and} \quad g_{i3} := \delta_{i3}, \\ g^n_{\alpha\beta} &:= a^n_{\alpha\beta} - 2x_3 b^n_{\alpha\beta} + x_3^2 c^n_{\alpha\beta} \quad \text{and} \quad g^n_{i3} := \delta_{i3}, n \geq 0 \end{aligned}$$

(the variable $y \in \omega$ is omitted, x_3 designates the variable in $\mathbb{R}$), *where*

$$c_{\alpha\beta} := b_\alpha^\tau b_{\beta\tau}, \quad b_\alpha^\tau := a^{\sigma\tau} b_{\alpha\sigma}, \quad (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1},$$
$$c_{\alpha\beta}^n := b_\alpha^{\tau,n} b_{\beta\tau}^n, \quad b_\alpha^{\tau,n} := a^{\sigma\tau,n} b_{\alpha\sigma}^n, \quad (a^{\sigma\tau,n}) := (a_{\alpha\beta}^n)^{-1}, n \geq 0.$$

Let ω_0 be an open subset of $\mathbb{R}^2$ such that $\overline{\omega}_0 \Subset \omega$. Then there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the symmetric matrices

$$\mathbf{C}(y, x_3) := (g_{ij}(y, x_3)) \text{ and } \mathbf{C}^n(y, x_3) := (g_{ij}^n(y, x_3)),\ n \geq 0,$$

are positive definite at all points $(y, x_3) \in \overline{\Omega}_0$, where

$$\Omega_0 := \omega_0 \times \left]-\varepsilon_0, \varepsilon_0\right[.$$

The matrices $\mathbf{C}(y, x_3) \in \mathbb{S}^3$ and $\mathbf{C}^n(y, x_3) \in \mathbb{S}^3$ are of the form (the notations are self-explanatory):

$$\mathbf{C}(y, x_3) = \mathbf{C}_0(y) + x_3 \mathbf{C}_1(y) + x_3^2 \mathbf{C}_2(y),$$
$$\mathbf{C}^n(y, x_3) = \mathbf{C}_0^n(y) + x_3 \mathbf{C}_1^n(y) + x_3^2 \mathbf{C}_2^n(y),\ n \geq 0.$$

First, it is easily deduced from the matrix identity $\mathbf{B} = \mathbf{A}(\mathbf{I} + \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A}))$ and the assumptions $\lim_{n\to\infty} \|a_{\alpha\beta}^n - a_{\alpha\beta}\|_{0,\overline{\omega}_0} = 0$ and $\lim_{n\to\infty} \|b_{\alpha\beta}^n - b_{\alpha\beta}\|_{0,\overline{\omega}_0} = 0$ that there exists a constant M such that

$$\|(\mathbf{C}_0^n)^{-1}\|_{0,\overline{\omega}_0} + \|\mathbf{C}_1^n\|_{0,\overline{\omega}_0} + \|\mathbf{C}_2^n\|_{0,\overline{\omega}_0} \leq M \text{ for all } n \geq 0.$$

This uniform bound and the relations

$$\mathbf{C}(y, x_3) = \mathbf{C}_0(y)\{\mathbf{I} + (\mathbf{C}_0(y))^{-1}(-2x_3 \mathbf{C}_1(y) + x_3^2 \mathbf{C}_2(y))\},$$
$$\mathbf{C}^n(y, x_3) = \mathbf{C}_0^n(y)\{\mathbf{I} + (\mathbf{C}_0^n(y))^{-1}(-2x_3 \mathbf{C}_1^n(y) + x_3^2 \mathbf{C}_2^n(y))\},\ n \geq 0,$$

together imply that there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the matrices $\mathbf{C}(y, x_3)$ and $\mathbf{C}^n(y, x_3)$, $n \geq 0$, are invertible for all $(y, x_3) \in \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$.

These matrices are positive definite for $x_3 = 0$ by assumption. Hence they remain so for all $x_3 \in [-\varepsilon_0, \varepsilon_0]$ since they are invertible.

(ii) *Let ω_ℓ, $\ell \geq 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_\ell \Subset \omega$ for each ℓ and $\omega = \bigcup_{\ell \geq 0} \omega_\ell$. By* (i), *there exist numbers $\varepsilon_\ell = \varepsilon_\ell(\omega_\ell) > 0$, $\ell \geq 0$, such that the symmetric matrices $\mathbf{C}(x) = (g_{ij}(x))$ and $\mathbf{C}^n(x) = (g_{ij}^n(x))$, $n \geq 0$, defined for all $x = (y, x_3) \in \omega \times \mathbb{R}$ as in* (i), *are positive definite at all points $x = (y, x_3) \in \overline{\Omega}_\ell$, where $\Omega_\ell := \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[$, hence at all points $x = (y, x_3)$ of the open set*

$$\Omega := \bigcup_{\ell \geq 0} \Omega_\ell,$$

which is connected and simply connected. Let the functions $R_{qijk} \in \mathcal{C}^0(\Omega)$ be defined from the matrix fields $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ by

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma_{ij}^p := g^{pq}\Gamma_{ijq}, \text{ with } (g^{pq}) := (g_{ij})^{-1},$$

and let the functions $R^n_{qijk} \in \mathcal{C}^0(\Omega)$, $n \geq 0$ *be similarly defined from the matrix fields* $(g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$. *Then*

$$R_{qijk} = 0 \text{ in } \Omega \text{ and } R^n_{qijk} = 0 \text{ in } \Omega \text{ for all } n \geq 0.$$

That Ω is connected and simply-connected is established in part (viii) of the proof of Theorem 2.8-1. That $R_{qijk} = 0$ in Ω, and similarly that $R^n_{qijk} = 0$ in Ω for all $n \geq 0$, is established as in parts (iv) to (viii) of the same proof.

(iii) *The matrix fields* $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *and* $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *defined in* (ii) *satisfy* (the notations used here are those of Section 1.8)

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Given any compact subset K of Ω, there exists a finite set Λ_K of integers such that $K \subset \bigcup_{\ell \in \Lambda_K} \Omega_\ell$. Since by assumption,

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0, \; \ell \in \Lambda_K,$$

it follows that

$$\lim_{n\to\infty} \|\mathbf{C}^n_p - \mathbf{C}_p\|_{2,\overline{\omega}_\ell} = 0, \; \ell \in \Lambda_k, \; p = 0, 1, 2,$$

where the matrices $\mathbf{C}_p$ and $\mathbf{C}^n_p$, $n \geq 0$, $p = 0, 1, 2$, are those defined in the proof of part (i). The definition of the norm $\|\cdot\|_{2,\overline{\Omega}_\ell}$ then implies that

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,\overline{\Omega}_\ell} = 0, \; \ell \in \Lambda_K.$$

The conclusion then follows from the finiteness of the set Λ_K.

(iv) *Conclusion.*

Given any mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ that satisfies

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega,$$

let the mapping $\mathbf{\Theta} : \Omega \to \mathbf{E}^3$ be defined by

$$\mathbf{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \Omega,$$

where $\boldsymbol{a}_3 := \dfrac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}$, and let

$$g_{ij} := \partial_i \mathbf{\Theta} \cdot \partial_j \mathbf{\Theta}.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \Omega,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $c_{\alpha\beta} = a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

In other words, *the matrices* (g_{ij}) *constructed in this fashion coincide over the set* Ω *with those defined in* part (i). Since parts (ii) and (iii) of the above proof together show that all the assumptions of Theorem 1.8-3 are satisfied by the fields $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

We now show that the mappings

$$\boldsymbol{\theta}^n(\cdot) := \boldsymbol{\Theta}^n(\cdot, 0) \in \mathcal{C}^3(\omega; \mathbf{E}^3)$$

indeed satisfy

$$a^n_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta}^n \cdot \partial_\beta \boldsymbol{\theta}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n}{|\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n|} \right\} \text{ in } \omega.$$

Dropping the exponent n for notational convenience in this part of the proof, let $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$. Then $\partial_{33} \boldsymbol{\Theta} = \partial_3 \boldsymbol{g}_3 = \Gamma^p_{33} \boldsymbol{g}_p = \mathbf{0}$, since it is easily verified that the functions Γ^p_{33}, constructed from the functions g_{ij} as indicated in part (ii), vanish in Ω. Hence there exists a mapping $\boldsymbol{\theta}^1 \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{\theta}^1(y) \text{ for all } (y, x_3) \in \Omega.$$

Consequently, $\boldsymbol{g}_\alpha = \partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1$ and $\boldsymbol{g}_3 = \boldsymbol{\theta}^1$. The relations $g_{i3} = \boldsymbol{g}_i \cdot \boldsymbol{g}_3 = \delta_{i3}$ then show that

$$(\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1) \cdot \boldsymbol{\theta}^1 = 0 \text{ and } \boldsymbol{\theta}^1 \cdot \boldsymbol{\theta}^1 = 1.$$

These relations imply that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{\theta}^1 = 0$. Hence either $\boldsymbol{\theta}^1 = \boldsymbol{a}_3$ or $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ in ω. But $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ is ruled out since we must have

$$\{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}\} \cdot \boldsymbol{\theta}^1 = \det(g_{ij})|_{x_3=0} > 0.$$

Noting that

$$\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \text{ implies } \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = -\partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3,$$

we obtain, on the one hand,

$$\begin{aligned} g_{\alpha\beta} &= (\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{a}_3) \cdot (\partial_\beta \boldsymbol{\theta} + x_3 \partial_\beta \boldsymbol{a}_3) \\ &= \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} - 2x_3 \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3 + x_3^2 \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \text{ in } \Omega. \end{aligned}$$

Since, on the other hand,

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ in } \Omega,$$

we conclude that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \text{ in } \omega,$$

as desired.

It remains to verify that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

But these relations immediately follow from the relations

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega,$$

combined with the observations that a compact subset of ω is also one of Ω, that $\boldsymbol{\Theta}(\cdot, 0) = \boldsymbol{\theta}$ and $\boldsymbol{\Theta}^n(\cdot, 0) = \boldsymbol{\theta}^n$, and finally, that

$$\|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} \le \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,\kappa}.$$

□

Remark. At first glance, it seems that Theorem 2.10-1 could be established by a proof similar to that of its “three-dimensional counterpart”, viz., Theorem 1.8-3. A quick inspection reveals, however, that the proof of Theorem 1.8-2 does not carry over to the present situation. □

In fact, it is not necessary to assume in Theorem 2.10-1 that the “limit” matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ satisfy the Gauß and Codazzi-Mainardi equations (see the proof of the next theorem). More specifically, another *sequential continuity* result can be derived from Theorem 2.10-1. Its interest is that the assumptions are now made on the immersions $\boldsymbol{\theta}^n$ that define the surfaces $\boldsymbol{\theta}^n(\omega)$ for all $n \ge 0$; besides the existence of a “limit” surface $\boldsymbol{\theta}(\omega)$ is also established.

Theorem 2.10-2. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. For each $n \ge 0$, let there be given immersions $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$, let $a^n_{\alpha\beta}$ and $b^n_{\alpha\beta}$ denote the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}^n(\omega)$, and assume that $b^n_{\alpha\beta} \in \mathcal{C}^2(\omega)$. Let there be also given matrix fields $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ with the property that*

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Then there exist immersions $\widetilde{\boldsymbol{\theta}}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ of the form

$$\widetilde{\boldsymbol{\theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\theta}^n, \ \boldsymbol{c}^n \in \mathbf{E}^3, \ \mathbf{Q}^n \in \mathbb{O}^3_+$$

(hence the first and second fundamental forms of the surfaces $\widetilde{\boldsymbol{\theta}}^n(\omega)$ and $\boldsymbol{\theta}^n(\omega)$ are the same for all $n \geq 0$) *and an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\omega, \mathbf{E}^3)$ *such that* $a_{\alpha\beta}$ *and* $b_{\alpha\beta}$ *are the covariant components of the first and second fundamental forms of the surface* $\boldsymbol{\theta}(\omega)$. *Besides,*

$$\lim_{n\to\infty} \|\widetilde{\boldsymbol{\theta}}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Proof. An argument similar to that used in the proof of Theorem 1.8-4 shows that passing to the limit as $n \to \infty$ is allowed in the Gauß and Codazzi-Mainardi equations, which are satisfied in the spaces $\mathcal{C}^0(\omega)$ and $\mathcal{C}^1(\omega)$ respectively by the functions $a^n_{\alpha\beta}$ and $b^n_{\alpha\beta}$ for each $n \geq 0$ (as necessary conditions; cf. Theorem 2.7-1). Hence the limit functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ also satisfy the Gauß and Codazzi-Mainardi equations.

By the fundamental existence theorem (Theorem 2.8-1), there thus exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\}.$$

Theorem 2.10-1 can now be applied, showing that there exist mappings (now denoted) $\widetilde{\boldsymbol{\theta}}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$a^n_{\alpha\beta} = \partial_\alpha \widetilde{\boldsymbol{\theta}}^n \cdot \partial_\beta \widetilde{\boldsymbol{\theta}}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta} \widetilde{\boldsymbol{\theta}}^n \cdot \left\{ \frac{\partial_1 \widetilde{\boldsymbol{\theta}}^n \wedge \partial_2 \widetilde{\boldsymbol{\theta}}^n}{|\partial_1 \widetilde{\boldsymbol{\theta}}^n \wedge \partial_2 \widetilde{\boldsymbol{\theta}}^n|} \right\} \text{ in } \omega,\ n \geq 0,$$

and

$$\lim_{n\to\infty} \|\widetilde{\boldsymbol{\theta}}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Finally, the rigidity theorem for surfaces (Theorem 2.9-1) shows that, for each $n \geq 0$, there exist $\boldsymbol{c}^n \in \mathbf{E}^3$ and $\mathbf{Q}^n \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\theta}^n \text{ in } \omega,$$

since the surfaces $\widetilde{\boldsymbol{\theta}}^n(\omega)$ and $\boldsymbol{\theta}^n(\omega)$ share the same fundamental forms and the set ω is connected. $\square$

It remains to show how the *sequential continuity* established in Theorem 2.10-1 implies the *continuity of a surface as a function of its fundamental forms* for *ad hoc* topologies.

Let ω be an open subset of $\mathbb{R}^2$. We recall (see Section 1.8) that, for any integers $\ell \geq 0$ and $d \geq 1$, the space $\mathcal{C}^\ell(\omega; \mathbb{R}^d)$ becomes a *locally convex topological space* when its topology is defined by the family of semi-norms $\|\cdot\|_{\ell,\kappa}$, $\kappa \Subset \omega$, and a sequence $(\boldsymbol{\theta}^n)_{n\geq 0}$ converges to $\boldsymbol{\theta}$ with respect to this topology if and only if

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{\ell,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Furthermore, this topology is *metrizable*: Let $(\kappa_i)_{i\geq 0}$ be any sequence of subsets of ω that satisfy

$$\kappa_i \Subset \omega \text{ and } \kappa_i \subset \operatorname{int} \kappa_{i+1} \text{ for all } i \geq 0, \text{ and } \omega = \bigcup_{i=0}^{\infty} \kappa_i.$$

Then

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{\ell,\kappa} = 0 \text{ for all } \kappa \Subset \omega \iff \lim_{n\to\infty} d_\ell(\boldsymbol{\theta}^n, \boldsymbol{\theta}) = 0,$$

where

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\theta}\|_{\ell,\kappa_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\theta}\|_{\ell,\kappa_i}}.$$

Let $\dot{\mathcal{C}}^3(\omega; \mathbf{E}^3) := \mathcal{C}^3(\omega; \mathbf{E}^3)/R$ denote the quotient set of $\mathcal{C}^3(\omega; \mathbf{E}^3)$ by the equivalence relation R, where $(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}) \in R$ means that there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y \in \omega$. Then the set $\dot{\mathcal{C}}^3(\omega; \mathbf{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\psi}}) := \inf_{\begin{cases}\boldsymbol{\kappa} \in \dot{\boldsymbol{\theta}} \\ \boldsymbol{\chi} \in \dot{\boldsymbol{\psi}}\end{cases}} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}) = \inf_{\begin{cases}\boldsymbol{c} \in \mathbf{E}^3 \\ \mathbf{Q} \in \mathbb{O}^3\end{cases}} d_3(\boldsymbol{\theta}, \boldsymbol{c} + \mathbf{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\theta}}$ denotes the equivalence class of $\boldsymbol{\theta}$ *modulo* R.

The announced continuity of a surface as a function of its fundamental forms is then a corollary to Theorem 2.10-1. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 2.10-3. *Let ω be connected and simply connected open subset of $\mathbb{R}^2$. Let*

$$\begin{aligned}
&\mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2) := \{((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^2(\omega; \mathbb{S}^2); \\
&\quad \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} \;=\; b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega, \\
&\qquad\qquad \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} \;=\; 0 \text{ in } \omega\}.
\end{aligned}$$

Given any element $((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2)$, let $F(((a_{\alpha\beta}), (b_{\alpha\beta}))) \in \dot{\mathcal{C}}^3(\omega; \mathbf{E}^3)$ denote the equivalence class modulo R *of any $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ that satisfies*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Then the mapping

$$F : \{\mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2); d_2\} \to \{\dot{\mathcal{C}}^3(\omega; \mathbf{E}^3); \dot{d}^3\}$$

defined in this fashion is continuous.

Proof. Since $\{\mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}); d_2\}$ and $\{\dot{\mathcal{C}}^3(\omega;\mathbf{E}^3); \dot{d}^3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through F into convergent sequences.

Let then $((a_{\alpha\beta}),(b_{\alpha\beta})) \in \mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2)$ and $((a^n_{\alpha\beta}),(b^n_{\alpha\beta})) \in \mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2)$, $n \geq 0$, be such that

$$\lim_{n\to\infty} d_2(((a^n_{\alpha\beta}),(b^n_{\alpha\beta})),((a_{\alpha\beta}),(b_{\alpha\beta}))) = 0,$$

i.e., such that

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Let there be given any $\boldsymbol{\theta} \in F(((a_{\alpha\beta}),(b_{\alpha\beta})))$. Then Theorem 2.10-1 shows that there exist $\boldsymbol{\theta}^n \in F(((a^n_{\alpha\beta}),(b^n_{\alpha\beta})))$, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega,$$

i.e., such that

$$\lim_{n\to\infty} d_3(\boldsymbol{\theta}^n, \boldsymbol{\theta}) = 0.$$

Consequently,

$$\lim_{n\to\infty} \dot{d}_3(F(((a^n_{\alpha\beta}),(b^n_{\alpha\beta}))), F(((a_{\alpha\beta}),(b_{\alpha\beta})))) = 0,$$

and the proof is complete. $\square$

The above continuity results have been extended “up to the boundary of the set ω” by Ciarlet & C. Mardare [2005b].

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