Mathematical Theory of Boltzmann Equation

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Contents

1 Introduction **5**

1.1 Overview . . . 5

1.1.1 Mathematical Aspects of the Boltzmann equation . . . 5

1.1.2 Existence Theory for the Boltzmann Equation . . . 7

1.2 Boltzmann Equation . . . 8

1.2.1 Boltzmann Equation . . . 8

1.2.2 Properties of Q . . . 10

1.2.3 Grad's Angular Cutoff Potential and Estimates of Q . . . 13

2 Solutions in L^∞ Framework **17**

2.1 The Linearized Boltzmann Operator . . . 17

2.2 Spectral Analysis of the Linearized Boltzmann Operator . . . 19

2.2.1 Semi-group e^{tB}. . . . 19

2.2.2 Resolvent and Spectrum of $\hat{B}(k)$ . . . 23

2.2.3 Eigenvalues of $\hat{B}(k)$ near Origin. . . . 27

2.2.4 Asymptotic behaviors of $e^{t\hat{B}(k)}$. . . . 29

2.2.5 Decay Rates of e^{tB} in $\mathbb{R}^n$ . . . 32

2.2.6 Decay Rates of e^{tB} in $\mathbb{T}^n$ . . . 38

2.3 Global Solutions of the Cauchy Problem . . . 39

2.3.1 Global Existence . . . 40

2.3.2 Space Regularity and Decay Rate . . . 46

2.4 Time-Periodic and Stationary Solutions . . . 50

2.4.1 Existence and Stability . . . 50

2.4.2 Proof of Theorem 2.4.1 . . . 54

2.4.3 Proof of Theorem 2.4.3 . . . 58

3 Solutions in L^2 Framework **61**

3.1 Local Existence . . . 62

3.2 Expansions and Decomposition . . . 67

3.2.1 Hilbert Expansion . . . 68

3.2.2 Chapman-Enskog Expansion . . . 70

3.2.3 Macro-Micro Decomposition . . . 73

3.3 Perturbation of Global Maxwellian . . . 76

3.4 Stability of Wave Patterns . . . 87

3.4.1 Basic Wave Patterns 90
3.4.2 Basic Ideas in Stability Analysis 93
3.4.3 Stability of Shock Profile 95
3.4.4 Stability of Rarefaction Wave 99
3.4.5 Stability of Contact Wave 100
3.5 Discussion 103
Bibliography

Preface

Up to the present, three different mathematical frameworks have been developed for the study of the Boltzmann equation. They are the theories in the L^1, L^2, and L^∞ spaces. Their principal ideas and methods are quite different from one another, but have been successfully employed for establishing the existence theorems of global solutions and revealing their various deep structures.

For the further development, however, it will be fairly desirable to merge these different theories. Some attempt has started already. Recently, it has been shown, [47], that an appropriate combination of the L^2 and L^∞ theories gives rise to an almost optimal convergence rates of solutions for the case with the external force.

To pursuit this direction, the authors believe that it is useful to provide clear and brief introduction to each theory. Thus, the aim of these notes is to present the fundamental ideas and methods in the frameworks of the L^2 and L^∞ theories. Due to the limitation of pages, the introduction to the L^1 theory is not included in these notes.

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Chapter 1

Introduction

1.1 Overview

1.1.1 Mathematical Aspects of the Boltzmann equation

The kinetic theory of the gas is a theory devoted to the study of evolutionary behaviors of the gas in the one-particle phase space of position and velocity. To fix the idea, consider a mono-atomic (one species) gas. In the kinetic theory, its state is described by a scalar function $f(t, x, \xi)$ which stands for the mass density function of gas particles having position $x \in \mathbb{R}^3$ and velocity $\xi \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. By definition, f is a non-negative function such that for any region D of the one-particle phase space $\mathbb{R}^3 \times \mathbb{R}^3$, the integral

$$\int\int_D f(t, x, \xi) dx d\xi,$$

gives the expectation value (statistical average) of the total mass of gas contained in D at time t. In some context, f happens to be taken as the number or probability density.

Recently, the kinetic theory is getting more and more recognized to be significant both in mathematics and practical applications as a key theory connecting the microscopic and macroscopic theory of gases and fluids. In this sense, the kinetic theory is in-between or mesoscopic.

In gas and fluid dynamics, there are many famous equations of motion, which have been derived by focusing the attention on different aspects of gases and fluids in different physical scales. Most of them are classical, dating back to the 19th century or earlier.

In the macroscopic scales where the gas and fluid are regarded as a continuum, their motion is described by the macroscopic quantities such as macroscopic mass density, bulk velocity, temperature, pressure, stresses, heat flux and so on. The Euler and Navier-Stokes equations, compressible or incompressible, are the most famous equations among governing equations proposed so far in fluid dynamics.

The extreme contrary is the microscopic scale where the gas, fluid, and hence any matter, are looked at as a many-body system of microscopic particles (atom/molecule). Thus, the motion of the system is governed by the coupled Newton equations, within the framework of the classical mechanics. The number of the involved equations is $6N$ if the total number of the microscopic particles is N.

Although the Newton equation is the first principle of the classical mechanics, it is not of practical use because the number of the equations is so enormous ($N \sim$ the Avogadro number 6×10^{23}) that it is hopeless to specify all the initial data, and we must appeal to statistics. On the other hand, the macroscopic (fluid dynamical) quantities mentioned above are related to statistical average of quantities depending on the microscopic state. Thus, the kinetic theory that gives the mesoscopic descriptions of the gas and fluid is noticed to be a key theory that links the microscopic and macroscopic scales. The Boltzmann equation, which is the subject of these notes, is the most classical but fundamental equation in the mesoscopic kinetic theory.

Except for the Newton equation, all the equations mentioned above are nonlinear partial differential equations which are of different types in the classification of partial differential equations, that is, elliptic, hyperbolic, parabolic type or a mixture, and whose structures of nonlinearity are also different. However, they can be used to describe the motion of one and the same gas/fluid, which means that they are interrelated to one another in certain ways. The mathematical theory on their relations is, indeed, one of the most important issues in gas and fluid dynamics, raising various problems in the asymptotic analysis and theory of singular perturbations, which reveal interesting mathematical relations between the equations of different types that clarify the physical regimes of validity of individual equations.

In physics, the asymptotic diagram in Figure 1.1 has been believed to be true for a long time.

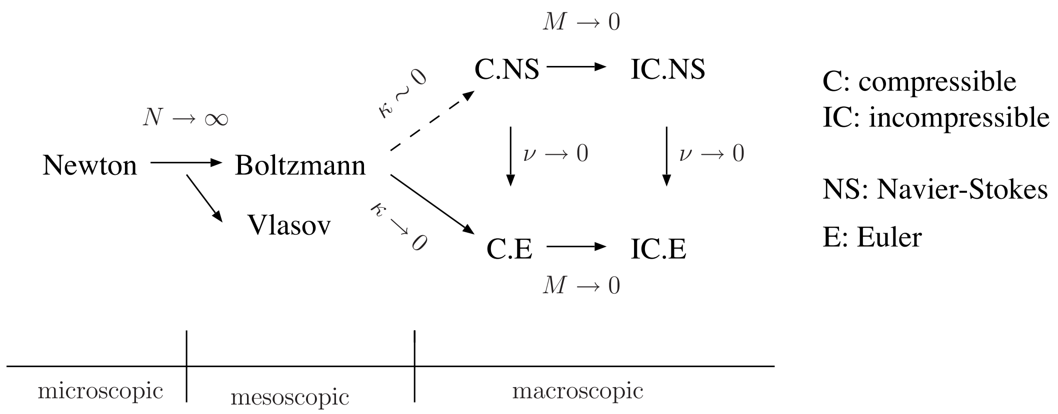

Figure 1.1: N: number of particles, κ: mean free path, ν: viscosity, M: Mach number

However, it is only in these decades that this diagram has been established with a mathematical rigor. The proof shows how the equations in lower scales can be derived from equations in higher scales, and also gives a mathematical explanation of the development of the initial layer. It should be stressed, however, that although the results obtained so far are many, they are yet far from satisfactory because they are mostly for the Cauchy problems. Little is known on the diagram in Figure 1.1 for the boundary and initial boundary value problems and therefore the theory of the boundary layer is not yet fully developed. Certainly, the analysis of these asymptotic relations is a significant issue in the mathematical theory of gas and fluid dynamics. Although this important issue is out of scope of these lecture notes, some fundamental references on this and related subjects are found in the bibliography given at the end of these notes categorized as "Boltzmann-Grad limit" and "Multi-scale analysis".

Another important issue to be worked out is the elucidation of mathematical structures of individual equations. There are still many fundamental open problems on the existence and properties of solutions of those equations. Needless to say that the question on the existence of smooth solutions to the incompressible Navier-Stokes equations is one of seven "Millennium Prize Problems" selected by Clay Mathematics Institute (http://www.claymath.org/millennium/). The equations mentioned above and many other related equations appearing in gas and fluid dynamics are rich resources raising challenging mathematical problems promised with fruitful results. They are strong driving forces for the development of the theory of nonlinear partial differential equations.

1.1.2 Existence Theory for the Boltzmann Equation

The first existence theorem of the solutions to the Boltzmann equation goes back to 1932 when Carleman [22] proved the existence of global (in time) solutions to the Cauchy problem for the spatially homogeneous case. It should be stressed that this is two years before the incompressible Navier-Stokes equation was solved by Leray [34] on the existence of global weak solutions. On the other hand, the research on the spatially inhomogeneous Boltzmann equation started much later. It is only in 1963 when Grad [27] constructed the first local solutions near the Maxwellian, and it is in 1974 when the first author of these notes constructed global solutions that are also near the Maxwellian, extending Grad's mathematical framework, [40].

The progress made afterward was remarkable, however. Up to the present, three different methods have been developed for establishing the global existence theory. The difference is due to difference of function spaces used for solving the Boltzmann equation, and consequently, the methods of proof employed are also different. At the present, it does not seem to be easy to merge these three different methods. It is a big open problem to characterize the "mathematically optimal" space for the Boltzmann equation. Here is a short summary of the solutions established so far.

1. Solutions in L^∞ framework.
Grad's scheme was extended to construct global solutions in the L^∞ space for various initial and initial boundary value problems. See, [40], [41], [37], [38], [44], [21]. The method is a combination of the spectral analysis of the linearized problem and the bootstrap argument based on the smoothing effect of the collision operator. An advantage of this method is that it can be applied to a variety of Cauchy problem, initial-boundary value problems and boundary value problems to provide global classical or strong solutions near Maxwellians. At the present, this is the only effective method applicable not only to the whole space but also to the domain with boundary. The disadvantage is that it does not work for large amplitude solutions which are far from Maxwellians.

2. Solutions in L^1 Framework. DiPerna-Lions [25] constructed global L^1 solutions without smallness assumption on initial data. These solutions are called the renormalized solutions because they are the weak solutions to the Boltzmann equation in the renormalized form.

The method of construction is based on the celebrated H-theorem (see §1.2.2 below) and the velocity averaging lemma (see e.g. [32]). This method was developed by many authors, [18], [24], [39], [49], and is often called the entropy dissipation method. It should be emphasized that the uniqueness of the solutions is an open problem which now seems to be a problem as big as Clay's Millennium Prize Problem on the Navier-Stokes equations.

3. Solutions in L^2 Framework. Recently, the L^2 energy method, which is familiar in the theory of nonlinear PDE's, became available for the Boltzmann equation by introducing a new decomposition of the equation and solutions, called the macro-micro decomposition. This was intiated by Liu-Yu [53] and developed by Liu-Yang-Yu [35]. The flexibility is a well-known feature of the energy method in the theory of PDE's and has been also demonstrated for the Boltzmann equation. The method has been exploited not only for constructing global strong solutions near Maxwellians, but also for analyzing the stability of wave profiles [53], [54], solving the Vlasov-Poisson (Maxwell)-Boltzmann equation [28], [48] and the external force problem [46], etc. Guo [29] developed a different setting of the energy method. Also, Liu-Yu [36] constructed the green function and revealed fine structures of the solutions.

The macro-micro decomposition has been found to have another significance than for the energy method. As will be seen in Chapter 3, it provides a systematic procedure to derive various asymptotic relations depicted in Figure 1.1 between the Boltzmann equation and fluid dynamical equations and also can capture non-classical fluid dynamical equations, just like the classical Hilbert and Chapman-Enskog expansions but without truncation, which facilitates the asymptotic analysis of the fluid dynamical equations.

These notes are concerned with the solutions in the frameworks 1 and 3, which will be presented in Chapter 2 and Chapter 3, respectively.

1.2 Boltzmann Equation

We will deal only with the Boltzmann equation for the monoatomic gas, i.e. the gas of identical particles, but most of mathematical techniques developed here will be applicable to the case of the mixture gas which can be described by coupled Boltzmann equations whose number is equal to the number of species of different gas particles, [3]. The original Boltzmann equation is formulated in the physical space of dimension 3, but we consider it in the space of arbitrary dimension n, which will reveal where and how the space dimension intervenes in the theory.

1.2.1 Boltzmann Equation

The Boltzmann equation for the monoatomic gas in a domain $\Omega \subset \mathbb{R}^n$ is

$$\frac{\partial f}{\partial t} = -\xi \cdot \nabla_x f - \frac{1}{m} F \cdot \nabla_\xi f + \frac{1}{\kappa} Q(f, f), \qquad (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^n. \tag{1.2.1}$$

Here, $f = f(t, x, \xi)$ is the unknown scalar function which stands for the mass density function of gas particles having position $x = (x_1, \cdots, x_n) \in \Omega$ and velocity $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$. (1.2.1) is a balance law for it. The first two terms on the right hand side gives the rate of change of f due to the motion of gas particles in the external force field $F = F(t, x, \xi) = (F_1, \cdots, F_n)$ in which m is the mass of the gas particle, $\cdot$ stands for the inner product of $\mathbb{R}^n$ and

$$\xi \cdot \nabla_x = \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial x_k}, \qquad F \cdot \nabla_\xi = \sum_{k=1}^{n} F_k \frac{\partial}{\partial \xi_k}.$$

On the other hand, the last term on the right hand side gives the rate of change of f due to binary collisions of gas particles, where Q is the nonlinear *collision operator* defined by

$$Q(f, g) = \frac{1}{2} \int_{\mathbb{R}^n \times S^{n-1}} q(v, \theta)(f' g'_* + f'_* g' - f g_* - f_* g) d\xi_* d\omega, \tag{1.2.2}$$
$$f = f(t, x, \xi), \quad f' = f(t, x, \xi'), \quad f'_* = f(t, x, \xi'_*), \quad f_* = f(t, x, \xi_*),$$

and similarly for g, and

$$\xi' = \xi - \big((\xi - \xi_*) \cdot \omega\big)\omega, \quad \xi'_* = \xi_* + \big((\xi - \xi_*) \cdot \omega\big)\omega,$$

where $\omega \in S^{n-1}$ and

$$v = |\xi - \xi_*|, \quad \cos\theta = \frac{1}{v}(\xi - \xi_*) \cdot \omega,$$

while ξ, ξ_* are the velocities of gas particles before collision and ξ', ξ'_* are the velocities after collision, see Figure 1.2.

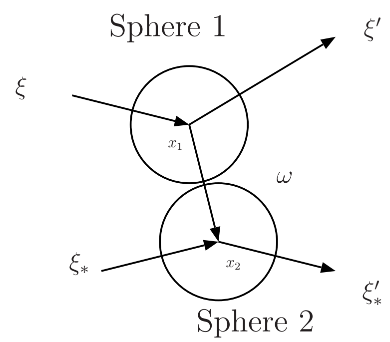

Figure 1.2: Elastic Collision

Notice that the conservation laws of momentum and energy hold through the collision:

$$\begin{aligned} &\xi + \xi_* = \xi' + \xi'_* && \text{(momentum)} \\ &|\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2 && \text{(energy)} \end{aligned} \tag{1.2.3}$$

The function q is the *collision kernel* which is determined by the interaction potential between two colliding particles. Two classical examples are the hard sphere gas for which, ([27]),

$$q(v,\theta) = q_0 v|\cos\theta|, \cos\theta = (\xi - \xi_*)\cdot\omega/v, \tag{1.2.4}$$

where q_0 is the surface area of a hard sphere, and the potential of inverse power s for which ([3], [27])

$$q(v,\theta) = v^{\gamma}|\cos\theta|^{-\gamma'} q_0(\theta), \quad \gamma = 1 - \frac{2(n-1)}{s}, \quad \gamma' = 1 + \frac{n-1}{s}, \tag{1.2.5}$$

where $q_0(\theta)$ is a bounded nonnegative function which does not vanish near $\theta = \pi/2$. The interaction potential is said hard if $s \geq 2(n-1)$ and soft if $0 < s < 2(n-1)$.

Finally, the number $\kappa > 0$ is the *mean free path (mean free time)* of gas particles between collisions or *Knudsen number* that represents the ratio of the mean free path to some characteristic length of the domain containing the gas. As will be seen in §3, it plays an essential role in the analysis of asymptotic relations between the Boltzmann equation and macroscopic fluid equations shown in Figure 1.1. Also, some references are given in the category "Multi-Scale Analysis" of the bibliography on related topics that are not discussed in these notes.

For the existence theory, on the other hand, κ may be fixed, say, to be 1, without loss of generality.

1.2.2 Properties of Q

The existence theory we develop in this note relies largely on the properties of the operator Q. Most of them were deduced by Boltzmann himself. History of the Boltzmann equation (1.2.1) is nicely summarized in the book [3]. Here, we present three fundamental ones.

Define the inner product

$$< f, g >= \int_{\mathbb{R}^n} f(\xi) g(\xi) d\xi. \tag{1.2.6}$$

A function $\varphi(\xi)$ is said a *collision invariant* if

$$< \varphi, Q(f,f) >= 0 \quad (\forall f \in C_0^\infty(\mathbb{R}^n_\xi, \mathbb{R}_+)). \tag{1.2.7}$$

The first property of Q is

[Q1] *Q has $n+2$ collision invariants,*

$$\varphi_0(\xi) = 1, \quad \varphi_i(\xi) = \xi_i \ \ (i = 1, 2, \cdots, n), \quad \varphi_{n+1}(\xi) = \frac{1}{2}|\xi|^2. \tag{1.2.8}$$

This leads to the conservation laws of the Boltzmann equation (1.2.1) as follows. Since $f(t,x,\xi)$ is the mass density in the (x,ξ)-space, that is, the microscopic mass density in the one-particle phase space, its moments with respect to ξ are quantities in the x-space, that is, the macroscopic quantities in the usual physical space. The first few moments are

$$\begin{aligned} \rho &=< \varphi_0, f(t,x,\cdot) >, \\ \rho u_i &=< \varphi_i, f(t,x,\cdot) > \qquad (i = 1, 2, \cdots, n), \\ \rho E &=< \varphi_{n+1}, f(t,x,\cdot) > . \end{aligned} \tag{1.2.9}$$

Here, ρ is the macroscopic mass density, $u = (u_1, u_2, \cdots, u_n)$ is the macroscopic (bulk) velocity, and E is the average energy density per unit mass, of the gas. The temperature θ and the pressure p are related to E by

$$E = \frac{1}{2}|u|^2 + \frac{n}{2}\theta. \qquad p = R\rho\theta. \tag{1.2.10}$$

Here, R is the *gas constant* (the Boltzmann constant divided by the mass of the gas particle). The last equation in (1.2.10) is called the *equation of state* for the ideal gas.

Consider the case $\Omega = \mathbb{R}^n$ and $F = 0$. Let f be a smooth solution to (1.2.1) which vanishes sufficiently rapidly with (x, ξ). Multiply (1.2.1) by φ_j and integrate it over $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. By virtue of (1.2.8) and by integration by parts, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} < \varphi_i, f(t, x, \cdot) > dx = 0, \quad i = 0, 1, \cdots, n+1, \tag{1.2.11}$$

which are, in view of (1.2.9), the conservation laws of total mass ($i = 0$), total momenta ($i = 1, 2, \cdots, n$) and total energy ($i = n + 1$), of the gas.

It is seen that similar conservation laws can hold for the case $\Omega \neq \mathbb{R}^3$ and $F \neq 0$ if some appropriate assumptions are imposed on the boundary conditions on the boundary $\partial\Omega$ and on the external force F.

The second property of Q to be mentioned is

[Q2] $Q(f, f) = 0 \;\Leftrightarrow\; < \log f, Q(f, f) >= 0 \;\Leftrightarrow\; f = \mathbf{M}(\xi)$ *where*

$$\mathbf{M}(\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi) = \frac{\rho}{(2\pi R\theta)^{3/2}} \exp\left(-\frac{|\xi - u|^2}{2R\theta}\right). \tag{1.2.12}$$

$\mathbf{M}$ is called the *Maxwellian* and is known to describe the velocity distribution of a gas in an equilibrium state with the mass density $\rho > 0$, bulk velocity $u = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$, and temperature $\theta > 0$. Here, (ρ, u, θ) are taken to be parameters, and if they are constants, $\mathbf{M}$ is called a global (absolute) Maxwellian while if they are functions of (x, t), it is called a local Maxwellian. Evidently, the global Maxwellian is a stationary solution of (1.2.1) if the external force F is absent.

It should be stressed that the Maxwellian (1.2.12) was discovered independently of the Boltzmann equation (1.2.1). In fact, it is in 1857 when J. C. Maxwell [9] first obtained (1.2.12) while it is in 1872 when L. Boltzmann [1] established (1.2.1). To obtain the Maxwellian, Maxwell relied on physical reasoning whereas Boltzmann solved the equation $Q(f, f) = 0$. Thus, the Maxwellian is built-in in the Boltzmann equation.

The final property of Q to be presented here is

[Q3] $$< \log f, Q(f, f) >\leq 0 \qquad (\forall f \in C_0^\infty(\mathbb{R}^n_\xi, \mathbb{R}_+)).$$

Let f be a density function of a gas. Since it is nonnegative, we may define the *H-function*,

$$H(t) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f \log f dx d\xi, \tag{1.2.13}$$

which gives, according to Boltzmann who is the founder of Statistical Physics, the minus of the entropy of the gas. Consider again the case $\Omega = \mathbb{R}^n$ and $F = 0$ and let f be a non-negative solution to (1.2.1) with rapid decay properties in (x, ξ). Multiply (1.2.1) by $\log f$ and integrate in (x, y). Integration by parts, together with [Q1], yields,

$$\frac{dH}{dt} + \int_{\mathbb{R}^n} D(t, x)dx = 0, \tag{1.2.14}$$

where

$$\begin{aligned} D(t, x) &= - < Q(f, f), \log f > \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}} q(|\xi - \xi_*|, \omega)(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} d\xi d\xi_* d\omega. \end{aligned} \tag{1.2.15}$$

This integral, called the *entropy dissipation integral*, is non-negative as seen from the elementary inequality,

$$(a - b)(\log a - \log b) \geq 0 \qquad (a, b > 0).$$

A consequence of this is the celebrated *H-theorem*

$$\frac{dH}{dt} \leq 0, \tag{1.2.16}$$

where the equality follows if and only if f is a Maxwellian. This theorem provides many physical implications. First, it says that the entropy increases with time. Second, as far as the total entropy dissipation integral $\int Ddx$ is bounded in t, the H-function may play a role of the Lyapounov function, to prove that the solution of the Boltzmann equation converges to a limit. [$Q3$] then asserts that this limit should be a Maxwellian. In other words, the H-theorem asserts that the Maxwellian is the only possible asymptotically stable stationary solution of the Boltzmann equation. Physically, this can be rephrased as the equilibrium state of the gas is uniquely described by the Maxwellian, not by any other distribution functions. Thus, one can say that the Maxwellian is built-in in the Boltzmann equation. As noted above, Maxwell derived the Maxwellian based on the physical argument, much before Boltzmann.

Of course, the above argument is just heuristic. Only deep mathematical analysis of the Boltzmann equation can justify this theorem, and the research is still continuing.

Remark 1.2.1 Having discovered the H-theorem, Boltzmann declared that he constructed the foundation of the second law of the thermodynamics, that is, the law of entropy, based on the classical mechanics. It is a famous episode in the history of science that lots of objections were then raised against him by his contemporaries. The controversies were very keen but it should be stressed that they motivated later developments of various ergodic theories, and it is Boltzmann who eventually won, though more than 100 years later: He was endorsed by Lanford [15] who established the convergence of the Newton equation to the Boltzmann equation depicted in Figure 1.1 as well as by many people who proved the existence of global solutions.

The main point of the controversies raised against Boltzmann is that the H theorem contradicts to the Newton mechanics because the mechanical law is time reversible. This well known time reversibility of the Newton mechanics is phrased for the many-particle system as follows. If we reverse the velocities of all particles at time $t = t_0$ and follow their evolution, we find that all the particles return to their initial positions (the positions at $t = 0$) with the minus of initial velocities, after a lapse of time t_0.

This would be rephrased for the Boltzmann equation as follows. Suppose that a solution $f(t, x, \xi)$ of the Boltzmann equation be obtained. Solve the Boltzmann equation again but with $f(t_0, x, -\xi)$ as the initial data. If this solution is denoted by $g(t, x, \xi)$. then, $g(t_0, x, \xi) = f(0, x, -\xi)$ would hold. Of course, this is false because it contradicts to the H theorem applied to both f and g,

$$H(f(0)) = H(g(t_0)) < H(g(0)) = H(f(t_0)) < H(f(0)),$$

unless f, g are Maxwellian. On the other hand, it has been expected in physics for a long time that, as suggested in Figure 1.1, the Boltzmann equation could be derived from the Newton equation by letting N, the number of particles, tend to ∞. This was eventually justified with a mathematical rigor by Lanford [15], although only for a short time. Therefore, it is not surprising that Lanford's theorem gives a mathematical reasoning of the emergency of the time irreversibility. Lanford's result is one of the most important results established so far in association with the asymptotic diagram in Figure 1.1. See e.g. [3] for a detail.

Remark 1.2.2 The non-negativity of the integral D in (1.2.15) indicates that the Boltzmann equation is a dissipative equation. This dissipativity is essentially used in the L^1 theory and its linearized version plays an important role both in the L^∞ and L^2 theory.

1.2.3 Grad's Angular Cutoff Potential and Estimates of Q

Needless to say that all the properties of Q stated in the previous subsection are valid only when the relevant integrals are convergent. On the other hand, the examples of the collision kernel $q(v, \theta)$ given in §2.1 have two different singular properties. One is the strong singularity at $\theta = \pi/2$ in (1.2.5) due to the grazing collision and the other is the unboundedness for large velocity $|\xi| \to \infty$.

The former does not guarantee the convergence of the integral over S^{n-1} in (1.2.2) under a mild assumption on f, g such that they are bounded. Actually, it is observed in [75] that $Q(f, g)$ is well-defined only for sufficiently smooth f, g as a nonlinear pseudo-differential operator. However, this is a too strong restriction to solve the Boltzmann equation in full generality. In order to avoid this difficulty, Grad [27] introduced an idea to cut off the singularity at $\theta = \pi/2$ assuming that $q_0(\theta)$ vanishes near $\theta = \pi/2$. This assumption was highly successful for the existence theory of the Boltzmann equation in the sense that almost all progresses made after Grad is indebted to his idea. It is now called Grad's angular cutoff assumption.

Throughout this note, we assume that $q(v, \theta)$ is a non-negative measurable function satisfying

$$\int_{S^{n-1}} q(v, \theta) d\omega \geq q_0 v^\gamma, \quad q(v, \theta) \leq q_1 (1 + v)^\gamma |\cos\theta|, \tag{1.2.17}$$

for some constants $q_0, q_1 > 0$ and $\gamma \in [0,1]$. Clearly, this is satisfied by the hard sphere gas (1.2.4) with $\gamma = 1$ and by the inverse power law (1.2.5) under the Grad's cutoff with $\gamma = 1 - 2(n-1)/s$ for $s \geq 2(n-2)$. Thus, (1.2.17) is a slightly generalized version of Grad's cutoff hard potential.

Under this assumption, Q becomes well-defined. To see this, let $\mathbf{M} = \mathbf{M}(\xi)$ be any Maxwellian and introduce the function

$$\nu_{\mathbf{M}}(\xi) = \int_{\mathbb{R}^n} q(|\xi - \xi_*|, \theta)\mathbf{M}(\xi_*)d\xi_* d\omega, \tag{1.2.18}$$

which satisfies under the assumption (1.2.17),

$$\nu_0(1 + |\xi|)^\gamma \leq \nu_{\mathbf{M}}(\xi) \leq \nu_1(1 + |\xi|)^\gamma, \tag{1.2.19}$$

for some positive constants ν_0, ν_1.

Theorem 1.2.3 *For any $p \in [1,\infty]$ and $\alpha \in [0,1]$, there exists a constant $C > 0$ such that*

$$\Big\|\frac{\nu_{\mathbf{M}}^{-\alpha}Q(f,g)}{\mathbf{M}^{1/2}}\Big\|_p \leq C\Big(\Big\|\frac{\nu_{\mathbf{M}}^{1-\alpha}f}{\mathbf{M}^{1/2}}\Big\|_p\Big\|\frac{g}{\mathbf{M}^{1/2}}\Big\|_p + \Big\|\frac{f}{\mathbf{M}^{1/2}}\Big\|_p\Big\|\frac{\nu_{\mathbf{M}}^{1-\alpha}g}{\mathbf{M}^{1/2}}\Big\|_p\Big), \tag{1.2.20}$$

where $\|\cdot\|_p$ is the norm of the space $L^p(\mathbb{R}^n_\xi)$.

Proof. Write

$$Q(f,g) = \frac{1}{2}\{Q_1(f,g) + Q_1(g,f) - Q_2(f,g) - Q_2(g,f)\}, \tag{1.2.21}$$

with

$$\begin{aligned} Q_1(f,g) &= \int_{\mathbb{R}^n \times S^{n-1}} q(|\xi - \xi_*|, \theta) f(\xi')g(\xi'_*)d\xi_* d\omega, \\ Q_2(f,g) &= \int_{\mathbb{R}^n \times S^{n-1}} q(|\xi - \xi_*|, \theta) f(\xi)g(\xi_*)d\xi_* d\omega, \end{aligned} \tag{1.2.22}$$

First, we prove the theorem for Q_1. Put $f = \mathbf{M}^{1/2}u, g = \mathbf{M}^{1/2}v$. Note from the conservation laws of collision (1.2.3) and the definition of Maxwellian (1.2.12) that

$$\mathbf{M}(\xi')\mathbf{M}(\xi'_*) = \mathbf{M}(\xi)\mathbf{M}(\xi_*)$$

holds. The Hölder inequality gives

$$\begin{aligned} |Q_1(f,g)| &\leq \int_{\mathbb{R}^n} q(|\xi - \xi_*|, \theta)\mathbf{M}(\xi)^{1/2}\mathbf{M}(\xi_*)^{1/2}|u(\xi')||v(\xi'_*)|d\xi_* d\omega \\ &\leq \Big(\int_{\mathbb{R}^n} q(|\xi - \xi_*|, \theta)^q\mathbf{M}(\xi)^{q/2}\mathbf{M}(\xi_*)^{q/2}d\xi_* d\omega\Big)^{1/q}\Big(\int_{\mathbb{R}^n} |u(\xi')|^p|v(\xi'_*)|^p d\xi_* d\omega\Big)^{1/p} \\ &\leq C\nu_{\mathbf{M}}(\xi)\mathbf{M}(\xi)^{1/2}\Big(\int_{\mathbb{R}^n} |u(\xi')|^p|v(\xi'_*)|^p d\xi_* d\omega\Big)^{1/p} \end{aligned}$$

with $p \in [1,\infty), 1/p + 1/q = 1$. The last line comes from

$$\int_{\mathbb{R}^n} q(|\xi - \xi_*|, \theta)^q\mathbf{M}(\xi_*)^{q/2}d\xi_* d\omega \leq C(1 + |\xi|)^{\gamma q} \leq C\nu_{\mathbf{M}}(\xi)^q,$$

which holds by virtue of (1.2.17) and (1.2.19), and C is a positive constant depending on $q_1, q_2, \nu_1, \nu_2, \rho, u, \theta$. Consequently,

$$\int_{\mathbb{R}^n} \left|\nu_{\mathbf{M}}(\xi)^{-\alpha}\mathbf{M}(\xi)^{-1/2}Q_1(f,g)(\xi)\right|^p d\xi \leq C \int_{\mathbb{R}^n\times\mathbb{R}^n S^{n-1}} \nu_{\mathbf{M}}(\xi)^{(1-\alpha)p}|u(\xi')|^p|v(\xi'_*)|^p d\xi d\xi_* d\omega.$$

By virtue of (1.2.3) and (1.2.19),

$$\begin{aligned}\nu_{\mathbf{M}}(\xi) &\leq C(1+|\xi|)^\gamma = C(1+|\xi' - \{(\xi'-\xi'_*)\cdot\omega\}\omega|)^\gamma \\ &\leq C(2+|\xi'|+|\xi'_*|)^\gamma \leq C(\nu_{\mathbf{M}}(\xi')+\nu_{\mathbf{M}}(\xi'_*)).\end{aligned}$$

Since the Jacobian of the change of variable $(\xi, \xi_*, \omega) \leftrightarrow (\xi', \xi'_*, -\omega)$ is unity, we finally have

$$\begin{aligned}&\int_{\mathbb{R}^n} \left|\nu_{\mathbf{M}}(\xi)^{-\alpha}\mathbf{M}(\xi)^{-1/2}Q_1(f,g)(\xi)\right|^p d\xi \\ &\qquad\leq C\int_{\mathbb{R}^n\times\mathbb{R}^n S^{n-1}} \left(\nu_{\mathbf{M}}(\xi')^{(1-\alpha)p} + \nu_{\mathbf{M}}(\xi'_*)^{(1-\alpha)p}\right)|u(\xi')|^p|v(\xi'_*)|^p d\xi' d\xi'_* d\omega.\end{aligned}$$

This proves (1.2.20) for Q_1 for the case $p \in [1, \infty)$. The case $p = \infty$ can be proved similarly, and the proof for Q_2 is also similar but much simpler. Now, the proof of the theorem is complete.

The function $\nu_{\mathbf{M}}(\xi)$ is bounded if $\gamma = 0$ and unbounded of order $O(|\xi|^\gamma)$ if $\gamma > 0$. As a consequence, Theorem 1.2.3 asserts that Q is a bounded operator if $\gamma = 0$ whereas it is well-defined but unbounded with the weigh loss of order $O(|\xi|^\gamma)$ if $\gamma > 0$.

Remark 1.2.4 Theorem 1.2.3 was first proved in [27] for the case $p = \infty, \alpha = 1$ and will be used in Chapter 2. The case $p = 2, \alpha = 1/2$ is due to [26] and will be used successfully in Chapter 3.

Chapter 2

Solutions in L^∞ Framework

This chapter is devoted to the L^∞ theory of the Boltzmann equation based on the spectral analysis of the linearized equation around a global Maxwellian. One of advantages of the spectral analysis is that it gives an optimal rate of decay (in time) of the solutions of the linearized equation. Our method for constructing solutions for the nonlinear problems in this chapter is a combination of the good decay estimates for the linearized equation and the contraction mapping principle. Thus, after deriving the sharp decay rate of the semi-group generated by the linearized Boltzmann equation, the global solutions to the Cauchy problem and time-periodic solutions for the case with the time-periodic source term will be constructed. The same decay rates are used coupled with the contraction mapping principle, but of course, in different context. This method has been developed also for the case where the domain Ω has a boundary. Actually, at the present, this is the only effective method for the boundary value problems. In this note, however, we will restrict ourself just to the case without boundary, that is, the whole space case $\Omega = \mathbb{R}^n$ and the torus case $\Omega = \mathbb{T}^n$. Moreover, we deal just with the case without external force F. This makes the spectral analysis much easier.

2.1 The Linearized Boltzmann Operator

Recall that if the external force F is absent, any global Maxwellian $\mathbf{M}$ is a stationary solution of the Boltzmann equation (1.2.1). We will look for the solution f near $\mathbf{M}$, that is, the solution having the form

$$f = \mathbf{M} + \mathbf{M}^{1/2}u. \tag{2.1.1}$$

By a suitable Galilean translation and scaling of the velocity variable ξ, $\mathbf{M}$ can be taken, without loss of generality, to be the standard Maxwellian,

$$\mathbf{M} = \mathbf{M}_{[1,0,1]}(\xi) = \frac{1}{(2\pi)^{n/2}} \exp\Big(-\frac{|\xi|^2}{2}\Big). \tag{2.1.2}$$

Plug this into (1.2.1). If $F = 0$, the equation for the new unknown $u = u(t, x, \xi)$ becomes

$$\frac{\partial u}{\partial t} = Bu + \Gamma[u, u], \quad (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^n, \tag{2.1.3}$$

where

$$Bu = -\xi \cdot \nabla_x u + \mathbf{L}u, \tag{2.1.4}$$
$$\mathbf{L}u = 2\mathbf{M}^{-1/2}Q(\mathbf{M}, \mathbf{M}^{1/2}u), \tag{2.1.5}$$
$$\Gamma[u,v] = \mathbf{M}^{-1/2}Q(\mathbf{M}^{1/2}u, \mathbf{M}^{1/2}v). \tag{2.1.6}$$

Here, $\mathbf{L}$, *the linearized collision operator*, is a linearized operator of Q around $\mathbf{M}$ whereas Γ, being its remainder, is a bilinear symmetric operator. The operator B is called *the linearized Boltzmann operator*.

In the rest of this section, we summarize some properties of $\mathbf{L}$ and Γ which hold under the cutoff assumption (1.2.17). For the proof, see, e.g. [3], [10], [27]. Define the spaces

$$L^p = L^p(\mathbb{R}^n_\xi) \qquad (p \in [1,\infty]),$$
$$L^\infty_\beta = L^\infty(\mathbb{R}^n_\xi; (1+|\xi|)^\beta d\xi) \qquad (\beta \in \mathbb{R}).$$

Note that $L^\infty_0 = L^\infty$.

Proposition 2.1.1 *Under the assumption* (1.2.17), *the following holds.*
(i) $\mathbf{L}$ *has the expression*

$$\mathbf{L}u = -\nu(\xi)u + Ku, \tag{2.1.7}$$

where $\nu(\xi)$ is a nonnegative measurable function of ξ satisfying

$$\nu_0(1+|\xi|)^\gamma \le \nu(\xi) \le \nu_1(1+|\xi|)^\gamma, \qquad \xi \in \mathbb{R}^3, \tag{2.1.8}$$

for some constants $\nu_0, \nu_1 > 0$ and $\gamma \in [0,1]$, while K is a linear integral operator in ξ and is bounded as the operators

$$K : L^2 \to L^2 \cap L^\infty, \qquad L_\beta \to L_{\beta+1} \quad (\forall \beta \ge 0). \tag{2.1.9}$$

(ii) *In the space L^2, if $\gamma \le 0$, the operator $\mathbf{L}$ is a linear bounded operator while if $\gamma > 0$, $\mathbf{L}$ is a linear closed unbounded operator with a dense domain*

$$D(\mathbf{L}) = \{u \in L^2 \mid \nu(\xi)u \in L^2\}.$$

Moreover, it is self-adjoint and non-positive in L^2 whose null space, denoted by $\mathcal{N}$, is (n+2)-dimensional and spanned by collision invariants weighted by $\mathbf{M}^{1/2}$,

$$\mathcal{N} = span\{\mathbf{M}^{1/2}, \quad \xi_i\mathbf{M}^{1/2}\ (i = 1,2,\cdots,n), \quad \frac{1}{2}|\xi|^2\mathbf{M}^{1/2}\}. \tag{2.1.10}$$

Let $\{\psi_i\}_{i=0}^{n+1}$ be an orthonormal basis of $\mathcal{N}$ in L^2 and put

$$\mathbf{P}u = \sum_{i=0}^{n+1}(u,\psi_i)_{L^2}\psi_i, \tag{2.1.11}$$

which defines an orthogonal projection

$$\mathbf{P} : L^2 \to \mathcal{N}. \tag{2.1.12}$$

We have,

$$\mathbf{P}\mathbf{L}u = 0 \quad (\forall u \in D(\mathbf{L})). \tag{2.1.13}$$

(iii) $\mathbf{P} : L^2 \to L^\infty_\beta$ *is a bounded operator for any $\beta \ge 0$.*

The decomposition (2.1.7) is due to [27]. The function $\nu(\xi)$ is nothing but the function in (1.2.18) and is called the *collision frequency.* (2.1.9) provides *smoothing* properties of K which will be useful in the sequel. For the hard sphere gas, the function $\nu(\xi)$ and integral kernel of K have the following explicit expressions for $n = 3$, see, e.g. [3], [27].

$$\nu(\xi) = (2\pi)^{1/2}\Big\{(|\xi| + |\xi|^{-1})\int_0^{|\xi|}\exp(-u^2/2)du + \exp(-|\xi|^2/2)\Big\}. \tag{2.1.14}$$

$$\begin{aligned} K(\xi,\xi_*) =&(2\pi)^{1/2}|\xi-\xi_*|^{-1}\exp\Big(-\frac{1}{8}\frac{(|\xi|^2-|\xi_*|^2)^2}{|\xi-\xi_*|^2}-\frac{1}{8}|\xi-\xi_*|^2\Big) \\ &-\frac{1}{2}|\xi-\xi_*|\exp\Big(-(|\xi|^2+|\xi_*|^2)/4\Big). \end{aligned} \tag{2.1.15}$$

Finally, the property of Γ needed in the sequel is

Proposition 2.1.2 (i) *Let $p \in [1,\infty]$ and $\alpha \in [0,1]$. There is a constant $c_0 > 0$ such that for any $u, v \in L^p$, it holds that*

$$\|\nu^{-\alpha}\Gamma[u,v]\|_{L^p} \le c_0(\|\nu^{1-\alpha}u\|_{L^p}\|v\|_{L^p} + \|u\|_{L^p}\|\nu^{1-\alpha}v\|_{L^p}). \tag{2.1.16}$$

(ii) $\mathbf{P}\Gamma[u,v] = 0$ *for any $u, v \in L^p$.*

Proof. (i) is just Theorem 1.2.3 while (ii) is a simple consequence of [Q1].

2.2 Spectral Analysis of the Linearized Boltzmann Operator

The main goal of this section is to establish the decay rates of the semi-group generated by the linearized Boltzmann operator B defined by (2.1.4), which will be used in an essential way in the next sections for solving the nonlinear Boltzmann equation.

2.2.1 Semi-group e^{tB}.

First, we shall recall the celebrated Hille-Yosida theorem which is the core of the theory of semi-groups. See [107], [108], [109] for the detail. To this end, we introduce some notation. Let X be a Banach space with norm $\|\cdot\|$, and $C^k(J;X)$ the set of continuous functions defined on the interval $J \subset \mathbb{R}$ with value in X having continuous Fréchet derivatives up to order $k = 0, 1, 2, \cdots$. Let T be a linear operator having the domain of definition and the range both in X, and denote its domain of definition by $D(T)$. The set of complex numbers

$$\begin{aligned} \rho(T) = \{\lambda \in \mathbb{C} \mid \lambda I - B \text{ has a bounded inverse } (\lambda I - T)^{-1} \\ \text{defined on the whole space } X\} \end{aligned} \tag{2.2.1}$$

is called the *resolvent set* of T and the inverse $(\lambda I - T)^{-1}$ is called the *resolvent* of T at $\lambda \in \rho(T)$. Here and hereafter, I denotes the identity operator and will be omitted occasionally.

A family $\{S(t)\}_{t\geq 0}$ parametrized by t is called a *strongly continuous semi-group*, or simply C_0 *semi-group*, of operators if the following three conditions are fulfilled.

(a) For each $t \geq 0$, $S(t)$ is a linear bounded operator on X.
(b) $S(0) = I, \qquad S(t+s) = S(t)S(s) \quad (t, s \geq 0)$.
(c) $S(h) \to I$ as $h \to +0$ strongly in X.

It is easily seen that under these conditions, $S(t)u \in C^0([0,\infty); X)$ for any $u \in X$. The second property of (b) is called the *semi-group property.*

The *generator* of a C_0 semi-group $\{S(t)\}$ is the operator T defined by

$$D(T) = \{u \in X \mid \exists s - \lim_{h\to+0} \frac{1}{h}(S(h) - I)u \ \ (\text{strong limit in } X)\}, \tag{2.2.2}$$
$$Tu = s - \lim_{h\to+0} \frac{1}{h}(S(h) - I)u, \qquad u \in D(T).$$

If T is a generator and if $u_0 \in D(T)$, then, $u = S(t)u_0$ is in $D(T)$ for all $t \geq 0$ as well as in $C^1((0,\infty); X)$ and solves the Cauchy problem,

$$\frac{du}{dt} = Tu \quad (t > 0), \qquad u(+0) = u_0. \tag{2.2.3}$$

Since this can be regarded as the initial value problem for the first order linear ordinary differential equation in the Banach space X, it is customary to express the solution operator $S(t)$ by the exponential function e^{tT}. The following is the most fundamental theorem in the theory of semi-groups.

Theorem 2.2.1 *Let $M \geq 1$ and $\lambda_0 \in \mathbb{R}$. The following two properties are equivalent.*

(1) *T is a linear, densely defined, closed operator in X with $\rho(T) \supset (\lambda_0, \infty)$, and the resolvent satisfies*

$$\|(\lambda - T)^{-k}\| \leq M(\lambda - \lambda_0)^{-k},$$

for all $\lambda > \lambda_0$ and $k = 1, 2, \cdots$.

(2) *T is a generator of a C_0 semi-group $S(t)$ satisfying*

$$\|S(t)u\| \leq Me^{\lambda_0 t}\|u\|, \tag{2.2.4}$$

for any $t \geq 0$ and $u \in X$.

The case for $M = 1$ and $\lambda_0 = 0$ is due to Hille and Yoshida and the corresponding semi-group is called the contraction semi-group. The general case is due to Fellar, Miyadera and Phillips. See e.g. [109].

Now, we shall consider the linearized Boltzmann operator B in (2.1.4) in the space $L^2 = L^2(\Omega_x \times \mathbb{R}^n_\xi)$ where $\Omega = \mathbb{R}^n$ or $\mathbb{T}^n$. More precisely, under the cutoff assumption (1.2.17), we define the operator B by

$$\begin{aligned} &D(B) = \left\{u \in L^2 \mid \xi \cdot \nabla_x u, \quad \nu(\xi)u \in L^2\right\}, \\ &Bu = -\xi \cdot \nabla_x u + \mathbf{L}u = -\xi \cdot \nabla_x u - \nu(\xi)u + Ku, \quad u \in D(B), \end{aligned} \tag{2.2.5}$$

where ∇_x is taken in the distribution sense.

Theorem 2.2.2 *Under the assumption* (1.2.17), *B generates a C_0 semi-group e^{tB} satisfying* (2.2.4) *with $M = 1$ and $\lambda = -\nu_0 + \|K\|$, where ν_0 is the constant in* (2.1.8) *and $\|K\|$ is the operator norm of K.*

For the proof, first, note from Proposition 2.1.1 that K can be taken to be a bounded operator on L^2. Let A be the operator obtained by dropping K from (2.2.5),

$$\begin{aligned} &Au = -\xi \cdot \nabla_x u - \nu(\xi)u, \\ &D(A) = \left\{u \in L^2 \mid \xi \cdot \nabla_x u, \quad \nu(\xi)u \in L^2\right\}. \end{aligned} \tag{2.2.6}$$

Proposition 2.2.3 *Let ν_0 be the constant in* (2.1.8). *The operator A fulfills the conditions of Theorem 2.2.1 with $M = 1$ and $\lambda_0 = -\nu_0$. The semi-group e^{tA} has an explicit expression,*

$$e^{tA}u_0 = e^{-\nu(\xi)t}u_0(x - t\xi, \xi), \qquad t \geq 0, \tag{2.2.7}$$

and the estimate (2.2.4) *reads*

$$\|e^{tA}u_0\|_{L^2} \leq e^{-\nu_0 t}\|u_0\|_{L^2}. \tag{2.2.8}$$

Proof. Write the right hand side of (2.2.7) as

$$S(t)u_0 = e^{-\nu(\xi)t}u_0(x - t\xi, \xi), \qquad t \geq 0. \tag{2.2.9}$$

We now show that this $S(t)$ is a C_0 semi-group and its generator is just given by (2.2.6). Evidently, the estimate (2.2.4) holds for $S(t)$ with $M = 1, \lambda_0 = -\nu_0$, implying that the condition (a) is fulfilled. It is easy to check (b). To check (c), introduce

$$\hat{u}(k, \xi) = \mathcal{F}(u)(k, \xi) = (2\pi)^{-n/2}\int_\Omega e^{-ik\cdot x}u(x, \xi)dx, \tag{2.2.10}$$

which is the Fourier transform for the case $\Omega = \mathbb{R}^n$ and the Fourier coefficient for the case $\Omega = \mathbb{T}^n$, of a function $u(x, \xi)$ with respect to x, where $k = (k_1, k_2, \cdots, k_n) \in \hat{\Omega}$ is the dual variable to x, $\cdot$ is the inner product of $\mathbb{R}^n$, and $\hat{\Omega} = \mathbb{R}^n$ or $\mathbb{Z}^n$. Obviously, we get

$$\mathcal{F}(S(t)u_0) = s(t, k, \xi)\mathcal{F}(u_0)(k, \xi), \quad s(t, k, \xi) = e^{-(ik\cdot\xi + \nu(\xi))t}. \tag{2.2.11}$$

This is seen to be a function in $C^0([0, \infty); L^2(\hat{\Omega} \times \mathbb{R}^n)$ if $\mathcal{F}(u_0) \in L^2(\hat{\Omega} \times \mathbb{R}^n)$. In fact, note that for any $z \in \mathbb{C}$ with $\mathrm{Re} z < 0$,

$$|e^{tz} - e^{t'z}| \to 0 \quad (t' \to t), \qquad |e^{tz} - e^{t'z}| \leq 2 \quad (t, t' \geq 0),$$

holds. Then, it suffices to put $z = -(ik\cdot\xi + \nu(\xi))$ and use Lebesgue's dominated convergence theorem. Parseval's relation then shows that $S(t)u_0$ is in $C^0([0,\infty); L^2)$ if $u_0 \in L^2$. Thus, $S(t)$ is a C_0 semi-group on L^2 satisfying the estimate (2.2.4).

It remains to prove that its generator is just (2.2.6). Observe for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$,

$$\frac{1}{h}(e^{zh} - 1) - z \to 0 \quad (h \to 0), \qquad |\frac{1}{h}(e^{zh} - 1) - z| \le 2|z| \quad (\forall h > 0).$$

Again, put $z = -(ik\cdot\xi + \nu(\xi))$ and use Lebesgue's dominated convergence theorem, to see that

$$\mathcal{F}(\frac{1}{h}(S(h) - I)u_0) = \frac{1}{h}(s(h,k,\xi) - 1)\mathcal{F}(u_0)(k,\xi) \tag{2.2.12}$$

converges to $\hat{v} = -(ik\cdot\xi + \nu(\xi))\mathcal{F}(u_0)$ as $h \to +0$ in the norm of $L^2(\hat{\Omega}\times\mathbb{R}^n)$ if and only if $\hat{v} \in L^2(\hat{\Omega}\times\mathbb{R}^n)$, or since $\nu(\xi)$ is a real function, if and only if

$$k\cdot\xi\mathcal{F}(u_0), \quad \nu(\xi)\mathcal{F}(u_0) \in L^2(\hat{\Omega}\times\mathbb{R}^n),$$

which is equivalent, again by Parseval's relation, to the condition $u_0 \in D(A)$. Thus, we are done.

Proof of Theorem 2.2.2. Since K is a bounded operator on L^2, (2.1.4) can be rewritten as

$$B = A + K, \qquad D(B) = D(A). \tag{2.2.13}$$

Thus, B is a bounded perturbation of a C_0 semi-group generator A, and hence [108] implies Theorem 2.2.2 with $M = 1$ and $\lambda_0 = -\nu_0 + \|K\|$.

We shall study the asymptotic behavior of e^{tB}. According to the theory of semi-groups, if B is a C_0 semi-group generator, the resolvent and semi-group are related to each other by the Laplace transform,

$$(\lambda - B)^{-1} = \int_0^\infty e^{-\lambda t}e^{tB}dt \qquad (\mathrm{Re}\lambda > \lambda_0), \tag{2.2.14}$$

and by the inverse Laplace transform

$$e^{tB} = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\lambda - B)^{-1}d\lambda \qquad (\sigma > \lambda_0). \tag{2.2.15}$$

It is also known that the resolvent set is an open subset of $\mathbb{C}$ and that the resolvent is an analytic function of λ on the resolvent set in the operator norm, so that the formula (2.2.15) can provide information on the asymptotic behaviors of e^{tB} if information on the singularities of $(\lambda - B)^{-1}$ is available.

The set of singular points of the analytic function $(\lambda - B)^{-1}$ is called the *spectrum* of B and denoted by $\sigma(B)$, which is the complementary set of the resolvent set $\rho(B)$ in the complex plane $\mathbb{C}$. A customary classification of the spectrum (e.g. [106]), is,

(a) Point spectrum: The set of eigenvalues of B. The point λ is an eigenvalue if and only if $\lambda - B$ is not one-to-one. Moreover, an isolated eigenvalue is a pole of the resolvent.

(b) Continuous spectrum: The set of points λ for which $\lambda - B$ is one-to-one and the image $(\lambda - B)D(B)$ is dense in X but $(\lambda - B)D(B) \neq X$

(c) Residual spectrum: The set of points λ for which $\lambda - B$ is one-to-one but the image $(\lambda - B)D(B)$ is not dense in X.

In order to study the asymptotic behavior of e^{tB}, we use the Fourier transformation (2.2.10). For $u \in D(B)$, we obtain

$$\mathcal{F}(Bu) = (-i\xi \cdot k + \mathbf{L})\hat{u}. \tag{2.2.16}$$

For $w = w(\xi)$, put

$$\hat{B}(k)w = (-i\xi \cdot k + \mathbf{L})w. \tag{2.2.17}$$

Here, we regard $k \in \hat{\Omega}$ as a parameter and consider $\hat{B}(k)$ as an operator in the space $L^2(\mathbb{R}^n_\xi)$ with the domain of definition

$$D(\hat{B}(k)) = \{w \in L^2(\mathbb{R}^n_\xi) \mid k \cdot \xi w, \nu(\xi)w \in L^2(\mathbb{R}^n_\xi)\}. \tag{2.2.18}$$

This generates a C_0 semi-group for each $k \in \hat{\Omega}$, with $M = 1$ and $\lambda_0 = -\nu_0 + \|K\|$, as seen by a similar but much simpler argument to that for Theorem 2.2.2. Set

$$\Phi(t,k) = e^{t\hat{B}(k)}. \tag{2.2.19}$$

Clearly, it follows from (2.2.16) that

$$e^{tB} = \mathcal{F}^{-1}\Big\{\Phi(t,k)\Big\}\mathcal{F}. \tag{2.2.20}$$

Now, we shall establish the asymptotic behavior of Φ by use of (2.2.15) applied to $\hat{B}(k)$,

$$\Phi(t,k) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\lambda - \hat{B}(k))^{-1}d\lambda \qquad (\sigma > \lambda_0). \tag{2.2.21}$$

If we can shift the integration path into the left half plane, we will be able to deduce a decay property of $\Phi(t,k)$. This will be possible by the Cauchy's integration theorem because the resolvent is an analytic function of λ on the resolvent set and has poles at isolated eigenvalues. Thus, we need to study the resolvent and spectrum of $\hat{B}(k)$ near the imaginary axis of the complex plane $\mathbb{C}$.

2.2.2 Resolvent and Spectrum of $\hat{B}(k)$

We start by the study of the operator

$$\hat{A}(k)w = (-ik \cdot \xi - \nu(\xi))w, \qquad D(\hat{A}(k)) = D(\hat{B}(k)), \tag{2.2.22}$$

which comes from (2.2.17) by dropping K.

Lemma 2.2.4 *Theorem 2.2.1 (2) holds for $\hat{A}(k)$ with $M = 1$ and $\lambda_0 = -\nu_0$. Furthermore, we have*

$$\rho(\hat{A}(k)) \supset \{\lambda \in \mathbb{C} \mid Re\,\lambda > -\nu_0\}, \tag{2.2.23}$$

and the explicit expressions

$$(\lambda - \hat{A}(k))^{-1}w = (\lambda + ik\cdot\xi + \nu(\xi)))^{-1}w, \tag{2.2.24}$$

$$e^{t\hat{A}(k)}w = e^{-(ik\cdot\xi+\nu(\xi))t}w. \tag{2.2.25}$$

Proof. Note that (2.2.24) and (2.2.25) hold , at least, formally. The right hand side of (2.2.24) becomes a bounded operator on $L^2(\mathbb{R}^n_\xi)$ when $\mathrm{Re}\lambda > -\nu_0$, whence (2.2.23) follows, which ensures the rest of the lemma.

To go further, we need some properties of the integral kernel of K.

Proposition 2.2.5 *Under the cutoff assumption* (1.2.17), *K is an integral operator*

$$Ku(\xi) = \int_{\mathbb{R}^n} K(\xi,\xi')u(\xi')d\xi',$$

with an integral kernel satisfying
(1) *K is real, measurable, and symmetric with respect to ξ, ξ'.*
(2) $\displaystyle\int_{\mathbb{R}^n} |K(\xi,\xi')|d\xi' \le k_0(1+|\xi|)^{-1}$,
(3) $\displaystyle\int_{\mathbb{R}^n} |K(\xi,\xi')|^2 d\xi' \le k_1$,
with some constants $k_0, k_1 > 0$.

For the proof, see [27].

In the below, we write $\lambda = \sigma + i\tau$ with σ, $\tau \in \mathbb{R}$. The following is a key proposition in the sequel.

Proposition 2.2.6 *There is a constant $C > 0$ such that the following holds.*

(1) *For any $k \in \mathbb{R}^n$ and $\delta > 0$, we have*

$$\sup_{\sigma\ge -\nu_0+\delta, \tau\in\mathbb{R}} \|K(\sigma + i\tau - \hat{A}(k))^{-1}\| \le C\delta^{-1-2/(3n+4)}(1+|k|)^{-2/(3n+4)}.$$

(2) *For any $\delta, \kappa_0 > 0$, there is a constant $\tau_0 > 0$ such that if $|\tau| > \tau_0$, we have,*

$$\sup_{\sigma\ge -\nu_0+\delta, |k|\le\kappa_0} \|K(\sigma + i\tau - \hat{A}(k))^{-1}\| \le C\delta^{-1-2/(n+2)}(1+|\tau|)^{-2/(n+2)}.$$

Here, $\|\cdot\|$ is the operator norm of $L^2(\mathbb{R}^n_\xi)$.

Remark 2.2.7 (1) of the above states that if $u = u(x,\xi) \in L^2_{x,\xi}$ is a solution of the *transport equation*

$$(\lambda - A)u = f(x,\xi) \in L^2_{x,\xi},$$

then, $Ku = K(\lambda - A)^{-1}f$ gains a regularity in x in the sense that $Ku \in L^2(\mathbb{R}^n_\xi, H^\ell_x)$ with $\ell = 2/(3n+4)$, where H^ℓ_x is the usual Sobolev space. This property was used first in [40], [41] and its various versions are now called the "Velocity Averaging Lemma", see, e.g. [32].

Proof. Put $G(\lambda, k) = K(\lambda - \hat{A}(k))^{-1}$ and let $\chi(D)$ be the characteristic function of the domain $D \subset \mathbb{R}^n$. We get by virtue of Proposition 2.2.5 (3),

$$\|G(\lambda, k)\chi(|\xi| < R)\| \le k_1^{1/2}\Big(\int_{|\xi|<R} |\lambda + ik\cdot\xi + \nu(\xi)|^{-2} d\xi\Big)^{1/2}.$$

Denote the last integral by J. Set

$$\Sigma_1 = \{\xi \in \mathbb{R}^n \mid |\xi| < R, \ \ |\tau + k\cdot\xi| \le \epsilon|k|\}, \quad \Sigma_2 = \{\xi \in \mathbb{R}^n \mid |\xi| < R\} - \Sigma_1,$$

for any $\epsilon > 0$. It is easy to see that there is a constant $C > 0$ such that for any k, R, ϵ, τ,

$$\text{mes } \Sigma_1 \le C\epsilon R^{n-1}, \qquad \text{mes } \Sigma_2 \le CR^n,$$

mes being the Lebesgue measure in $\mathbb{R}^n$. Let $\sigma \ge -\nu_0 + \delta$. Then, we get,

$$J = \int_{\Sigma_1} + \int_{\Sigma_2} \le C(\delta^{-2}\epsilon R^{n-1} + (\epsilon|k|)^{-2}R^n).$$

Choose $\epsilon = R^{1/3}(\delta/|k|)^{2/3}$ to deduce

$$\|G(\lambda, k)\chi(|\xi| < R)\| \le C\delta^{-2/3}R^{(3n-2)/6}|k|^{-1/3}.$$

On the other hand, by virtue of Proposition 2.2.5 (2),

$$\|G(\lambda, k)\chi(|\xi| > R)\| \le C\delta^{-1}R^{-1}.$$

Choosing $R = (|k|/\delta)^{2/(3n+4)}$ leads to (1) of the proposition.

To prove (2), let $|\tau| > 2\kappa_0 R$ for which

$$|\tau + k\cdot\xi| \ge |\tau| - |k||\xi| \ge |\tau| - \kappa_0 R \ge |\tau|/2,$$

whenever $|k| \le \kappa_0$ and $|\xi| \le R$, whence

$$J \le C(\delta^2 + |\tau|^2)^{-1}R^n.$$

Choosing $R = (|\tau|/\delta)^{2/(n+2)}$ leads to (2) with the choice $\tau_0 = (2\kappa_0)^{1+2/n}\delta^{-2/n}$. Thus, we are done.

We can write

$$\hat{B}(k) = \hat{A}(k) + K, \qquad D(\hat{B}(k)) = D(\hat{A}(k)). \tag{2.2.26}$$

It is now clear that $\hat{B}(k)$ is a semi-group generator. Let us discuss its spectrum. For any $a \in \mathbb{R}$, set

$$\mathbb{C}_+(a) = \{\lambda \in \mathbb{C} \mid \sigma > a\}, \quad \mathbb{C}_-(a) = \mathbb{C} - \mathbb{C}_+(a). \tag{2.2.27}$$

Proposition 2.2.8 *For any $\delta > 0$, there exists a positive number $\tau_1 > 0$ such that the following holds for all $k \in \mathbb{R}^n$.*

(1) $\rho(\hat{B}(k)) \supset \mathbb{C}_+(0) \cup \{\lambda \in \mathbb{C} \mid \sigma > -\nu_0 + \delta,\ |\tau| \geq \tau_1\}$.

(2) *$\sigma(\hat{B}(k)) \cap \mathbb{C}_+(-\nu_0 + \delta)$ consists of a finite number of eigenvalues of $\hat{B}(k)$.*

(3) *When $k \neq 0$, $\hat{B}(k)$ has no eigenvalues on the imaginary axis. When $k = 0$, the point $\lambda = 0$ is the only eigenvalue on the imaginary axis and its eigenspace is just $\mathcal{N}$ (null space of* **L** *in Proposition 2.1.1).*

Proof. We write the second resolvent equation for $\hat{A}(k)$ and $\hat{B}(k)$,

$$(\lambda - \hat{B}(k))^{-1} = (\lambda - \hat{A}(k))^{-1} + (\lambda - \hat{B})^{-1} K (\lambda - \hat{A}(k))^{-1}, \tag{2.2.28}$$

which holds for $\lambda \in \rho(\hat{A}(k)) \cap \rho(\hat{B}(k))$, and implies that, with $G(\lambda, k) = K(\lambda - \hat{A}(k))^{-1}$,

$$(\lambda - \hat{B}(k))^{-1} = (\lambda - \hat{A}(k))^{-1} (I - G(\lambda, k))^{-1} \tag{2.2.29}$$

is valid if $\lambda \in \rho(\hat{A}(k))$ and if $I - G(\lambda, k)$ has the bounded inverse. Now, Proposition 2.2.6 proves that for each $\delta > 0$, there exist positive numbers $\sigma_1,\ \tau_1 > 0$ ($\tau_1 > \tau_0$) such that $\|G(\lambda, k))\| \leq 1/2$ holds for all k, $\sigma \geq \sigma_1$, and $|\tau| \geq \tau_1$, so that the Neumann series gives the inverse $(I - G(\lambda, k))^{-1}$. This shows

$$\rho(\hat{B}(k)) \supset \mathbb{C}_+(\sigma_1) \cup \{\lambda \in \mathbb{C} \mid \sigma > -\nu_0 + \delta,\ |\tau| \geq \tau_1\}.$$

On the other hand, since K is a compact operator on $L^2(\mathbb{R}^n_\xi)$, $\hat{B}(k)$ is a compact perturbation of $\hat{A}(k)$, and so, thanks to [108, Theorem IV. 1.9] and Lemma 2.2.4, $\sigma(\hat{B}(k)) \cap \mathbb{C}_+(-\nu_0)$ consists of discrete eigenvalues with possible accumulation points only on the boundary of $\mathbb{C}_+(-\nu_0)$. And, we proved already that

$$\sigma(\hat{B}(k)) \cap \overline{\mathbb{C}_+(-\nu_0 + \delta)} \subset \{\lambda \in \mathbb{C} \mid -\nu_0 + \delta \leq \sigma \leq \sigma_1, |\tau| \leq \tau_1\}.$$

Since this is a compact set of $\mathbb{C}_+(-\nu_0)$ and since it does not touch the boundary of $\mathbb{C}_+(-\nu_0)$, the number of eigenvalues in it is necessarily finite. This proves (2) of the proposition.

Finally, we shall show that there are no eigenvalues with non-positive real part for all $k \in \mathbb{R}^n$, except for $\lambda = 0$ eigenvalue for $k = 0$. Let $\lambda = \sigma + i\tau$ and w be an eigenvalue and its eigenfunction, respectively:

$$(\lambda - \hat{B}(k))\varphi = 0, \quad \varphi \in D(\hat{B}(k)), \quad \varphi \neq 0. \tag{2.2.30}$$

Recall that **L** is non-positive self-adjoint and compute

$$0 = \mathrm{Re}\ ((\lambda - \hat{B}(k))\varphi, \varphi)_{L^2} = \sigma \|\varphi\|^2 - (L\varphi, \varphi) \geq \sigma \|\varphi\|^2,$$

which is a contradiction if $\sigma > 0$. Suppose, in this turn, $\sigma = 0$. The above computation shows that $(L\varphi, \varphi)_{L^2} = 0$. By virtue of Proposition 2.1.1, this means $\varphi \in \mathcal{N}$. Then, the eigenvalue equation (2.2.30) is reduced to

$$(\tau + k \cdot \xi)\varphi = 0,$$

which is impossible for $\varphi \neq 0$ unless $\tau = 0$ and $k = 0$. This completes the proof of both (1) and (3).

Now, we shall study the eigenvalues stated in Proposition 2.2.8 (2).

2.2.3 Eigenvalues of $\hat{B}(k)$ near Origin.

We shall consider the eigenvalue problem (2.2.30) for small k and λ. We start with

Proposition 2.2.9 *For any $\sigma_1 \in (0, \nu_0)$, there is a positive number κ_1 such that $\hat{B}(k)$ has no eigenvalues in $\mathbb{C}_+(-\sigma_1)$ if $|k| \geq \kappa_1$. Moreover, $\kappa_1 \to 0$ as $\sigma_1 \to 0$.*

Proof. By virtue of Proposition 2.2.6 (1), it holds that $\|G(\lambda, k)\| \leq 1/2$ if $\lambda \in \mathbb{C}_+(-\sigma_1)$ and if $|k|$ is large, proving $\lambda \in \rho(\hat{B}(k))$. The second assertion of the proposition comes from Proposition 2.2.8(3).

Thus, we shall solve (2.2.30) for small $k \in \mathbb{R}^n$ near $\lambda = 0$. We put

$$k = \kappa\tilde{k}, \qquad \kappa = |k|, \qquad \tilde{k} = k/|k|,$$

and for $r > 0$, $S_1(r) = \{k \in \mathbb{R}^n \mid |k| \leq r\}$, $S_2(r) = \mathbb{R}^n - S_1(r)$.

The following theorem is due to [62].

Theorem 2.2.10 *There exist positive numbers κ_0, σ_0, and functions*

$$\lambda_j(\kappa) \in C^\infty([-\kappa_0, \kappa_0]), \qquad j = 0, 1, \cdots, n+1,$$

such that for any $k \in S_1(\kappa_0)$, the following holds.

(i) $\sigma(\hat{B}(k)) \cap \mathbb{C}_+(-\sigma_0) = \{\lambda_j(\kappa)\}_{j=0}^{n+1}$.

(ii) $\lambda_j(\kappa)$ *has the asymptotic expansion,*

$$\lambda_j(\kappa) = i\lambda_j^{(1)}\kappa - \lambda_j^{(2)}\kappa^2 + 0(\kappa^2) \qquad (\kappa \to 0),$$

with the coefficients

$$\lambda_j^{(1)} \in \mathbb{R}, \qquad \lambda_j^{(2)} > 0.$$

(iii) *Denote the eigenprojection and eigennilpotent (see, e.g. [108] for the definition) corresponding to the eigenvalue $\lambda_j(k)$ by $\mathbf{P}_j(k)$ and $Q_j(k)$ respectively. It holds that*

$$\mathbf{P}_j(k) = \mathbf{P}_j^{(0)}(\tilde{k}) + \kappa\mathbf{P}_j^{(1)}(k), \quad Q_j(k) = 0,$$

for $j = 0, \cdots, n+1$, where $\mathbf{P}_j^{(0)}(\tilde{k})$ are orthogonal projections on L^2_ξ with

$$\mathbf{P} = \sum_{j=0}^{n+1} \mathbf{P}_j^{(0)}(\tilde{k}),$$

$\mathbf{P}$ *being the orthogonal projection in Proposition 2.1.1. Further, the operator norm $\|\mathbf{P}_j^{(1)}(k)\|$ is uniformly bounded for $k \in S_1(\kappa_0)$.*

Proof. It is know [27] that $\mathbf{L}$ is invariant with respect to the rotation R of $\xi \in \mathbb{R}^n$. Therefore, $R\hat{B}(k) = \hat{B}(R^{-1}k)R$ holds, which, applied to (2.2.30), shows that the eigenvalue λ depends only on $\kappa = |k|$. Now we consider our eigenvalue problem in the form

$$\hat{B}(k)\varphi = \kappa\eta(\kappa)\varphi. \tag{2.2.31}$$

Decomposing $\varphi = \mathbf{P}\varphi + (I - \mathbf{P})\varphi = \varphi^0 + \varphi^1$ and applying $\mathbf{P}$ and $\mathbf{P}^1 = I - \mathbf{P}$ to (2.2.31), we get

$$-i\mathbf{P}\xi \cdot \tilde{k}\varphi^0 - i\mathbf{P}\xi \cdot \tilde{k}\varphi^1 = \eta\varphi^0,$$
$$-i\mathbf{P}^1 k \cdot \xi\varphi^0 - i\mathbf{P}^1 k \cdot \xi\varphi^1 + \mathbf{L}\varphi^1 = \kappa\eta\varphi^1,$$

where we have used properties of $\mathbf{L}$ in Proposition 2.1.1. Since $\mathbf{L}^{-1}$ exists on $\mathcal{N}^\perp$, and since κ can be assumed small in our situation, the second equation can be solved as

$$\varphi^1 = i(\mathbf{L} - \kappa\eta\mathbf{P}^1 - i\mathbf{P}^1\kappa\tilde{k} \cdot \xi\mathbf{P}^1)^{-1}\mathbf{P}^1\kappa\tilde{k} \cdot \xi\varphi^0.$$

Insert this to the first equation to get

$$\eta\varphi^0 = i\mathbf{A}(\tilde{k})\varphi^0 - i\kappa\mathbf{P}\tilde{k} \cdot \xi\mathbf{P}^1(\mathbf{L} - i\kappa\eta\mathbf{P}^1 - i\mathbf{P}^1\kappa\tilde{k} \cdot \xi\mathbf{P}^1)^{-1}\mathbf{P}^1\tilde{k} \cdot \xi\mathbf{P}\varphi^0, \tag{2.2.32}$$

where

$$\mathbf{A}(\tilde{k}) = \mathbf{P}\tilde{k} \cdot \xi\mathbf{P}. \tag{2.2.33}$$

If we plug the forms

$$\eta = i\eta_1 + \kappa\eta_2, \qquad \varphi^0 = \varphi^0_0 + \kappa\varphi^0_1 \tag{2.2.34}$$

and put $\kappa = 0$, we get the eigenvalue problem for $A(\tilde{k})$:

$$A(\tilde{k})\varphi^0_0 = \eta_1\varphi^0_0. \tag{2.2.35}$$

This eigenvalue problem can be solved explicitly. To state this, introduce an orthonormal basis of the null space $\mathcal{N}$ defined by

$$\left\{ \begin{array}{l} \psi_0(\xi) = M^{1/2}(\xi), \\ \psi_i(\xi) = \xi_i M^{1/2}(\xi) \quad (i = 1, 2, \cdots, n), \\ \psi_{n+1}(\xi) = \frac{1}{\sqrt{2n}}(|\xi|^2 - n)M^{1/2}(\xi). \end{array} \right. \tag{2.2.36}$$

Note that it suffices to solve the above eigenvalue problem on $\mathcal{N}$.

Lemma 2.2.11 *Put*

$$c = \sqrt{\frac{n+2}{n}}. \tag{2.2.37}$$

The eigenvalues and normalized eigenfunctions of $\mathbf{A}(\tilde{k})$ on $\mathcal{N}$ are given by

$$\left\{ \begin{array}{ll} \eta_{1,0} = c, & \varphi^0_{0,0}(\tilde{k}) = \dfrac{1}{\sqrt{2}c}(\psi_0 + \tilde{k} \cdot \psi' + \gamma\psi_{n+1}), \\ \eta_{1,1} = 0, & \varphi^0_{0,1}(\tilde{k}) = \dfrac{1}{c}(\gamma\psi_0 - \psi_{n+1}), \\ \eta_{1,j} = 0, & \varphi^0_{0,j}(\tilde{k}) = C^j(\tilde{k}) \cdot \psi' \quad (j = 2, \cdots, n), \\ \eta_{1,n+1} = -c, & \varphi^0_{0,n+1}(\tilde{k}) = \dfrac{1}{\sqrt{2}c}(\psi_0 - \tilde{k} \cdot \psi' + \gamma\psi_{n+1}), \end{array} \right. \tag{2.2.38}$$

where ψ' is the vector valued function

$$\psi' = (\psi_1, \cdots, \psi_n) = \xi\psi_0(\xi), \tag{2.2.39}$$

while C^i are n-dimensional normalized vectors such that

$$C^i(\tilde{k}) \cdot C^j(\tilde{k}) = \tilde{k} \cdot C^j(\tilde{k}) = 0 \quad (i, j = 2, \cdots, n,\ i \neq j). \tag{2.2.40}$$

Proof. If we write $\mathbf{P}$ explicitly in the form

$$\mathbf{P}w = \sum_{i=0}^{n+1} < w, \psi_i > \psi_i(\xi), \tag{2.2.41}$$

we have a matrix representation of $\mathbf{A}$,

$$\begin{pmatrix} 0 & \tilde{k} & 0 \\ \tilde{k}^t & 0 & \gamma\tilde{k}^t \\ 0 & \gamma\tilde{k} & 0 \end{pmatrix} \tag{2.2.42}$$

where $\gamma = \sqrt{2/n}$, and $\tilde{k}$ is taken to be a row vector and ^{t} denotes its transpose. It is easy to compute its eigenvalues and normalized eigenvectors. The detail is omitted.

Proof of Proposition 2.2.10 continued. We have obtained $\eta_1 = \eta_{1,j}$ and $\varphi_0^0 = \varphi_{0,j}^0$, $j = 0, 1, \cdots, n+1$. Expand (2.2.32) in the power series of κ. Then, the term of order κ gives

$$\eta_2\varphi_0^0 + i\eta_1\varphi_1^0 = i\mathbf{A}(\tilde{k})\varphi_1^0 - i\mathbf{P}\tilde{k}\cdot\xi\mathbf{P}^1\mathbf{L}^{-1}\mathbf{P}^1\tilde{k}\cdot\xi\mathbf{P}\varphi_0^0. \tag{2.2.43}$$

Take the inner product in $L^2(\mathbb{R}_v^3)$ of this and $\varphi_{0,j}^0$ in (2.2.38) to arrive at

$$\eta_2 = \eta_{2,j} = - < \mathbf{L}^{-1}\mathbf{P}^1 v\cdot\tilde{\xi}\varphi_{0,j}^0, \mathbf{P}^1 v\cdot\tilde{\xi}\varphi_{0,j}^0 > \quad (j = 0, \cdots, n+1). \tag{2.2.44}$$

Since $\mathbf{L}$ is negative definite on $\mathcal{N}^\perp$, we have

$$\eta_{2,j} > 0, \qquad j = 0, 1, \cdots, n+1. \tag{2.2.45}$$

Now, the theorem follows with

$$\lambda_j^{(1)} = \eta_{1,j}, \qquad \lambda_j^{(2)} = \eta_{2,j}, \qquad \mathbf{P}_j^{(0)} =< \cdot, \varphi_{0,j} > \varphi_{0,j}.$$

Remark 2.2.12 It is possible to show that

$$\eta_{2,0} = \eta_{2,n+1} \equiv \mu, \qquad \eta_{2,j} \equiv \nu \qquad (j = 1, \cdots, n).$$

Physically, c in (2.2.37) is the sound speed in the equilibrium gas governed by the Maxwellian $\mathbf{M}$, while μ and ν are the thermal diffusivity and the viscosity coefficient corresponding to $\mathbf{M}$.

2.2.4 Asymptotic behaviors of $e^{t\hat{B}(k)}$.

We shall now use the inverse Laplace transformation (2.2.21). To this end, we need,

Lemma 2.2.13 *For any $w \in L_\xi^2$ and $\sigma > -\nu_0$, it holds that*

$$\int_{-\infty}^{\infty} \|(\sigma + i\tau - \hat{A}(k))^{-1}w\|^2 d\tau \le \pi(\sigma + \nu_0)^{-1}\|w\|^2.$$

Proof. Recall that the resolvent $(\lambda - \hat{A}(k))^{-1}$ is the Laplace transform

$$(\lambda - \hat{A}(k))^{-1} = \int_0^\infty e^{-\lambda t} e^{t\hat{A}(k)} dt, \qquad \lambda \in \mathbb{C}_+(-\nu_0),$$

which can be rewritten as

$$(\sigma + i\tau - \hat{A}(k))^{-1} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\tau t} \Big\{ (2\pi)^{1/2} \chi(t) e^{-\sigma t} e^{t\hat{A}(k)} \Big\} dt,$$

where $\chi(t) = 1$ $(t \geq 0)$, $= 0$ $(t < 0)$. This is a Fourier transform, so that by Parseval's equality, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \|(\sigma + i\tau - \hat{A}(k))^{-1} w\|^2 d\tau &= \int_{-\infty}^{\infty} \|(2\pi)^{1/2} \chi(t) e^{-\sigma t} e^{t\hat{A}(k)} w\|^2 dt \\ = 2\pi \int_0^\infty e^{-2\sigma t} \|e^{t\hat{A}(k)} w\|^2 dt &\leq 2\pi \int_0^\infty e^{-2(\sigma+\nu_0)t} dt \|w\|^2, \end{aligned}$$

which proves the lemma.

We now recall (2.2.21):

$$\Phi(t,k) = e^{t\hat{B}(k)} = \frac{1}{2i\pi} \int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} e^{\lambda t} (\lambda - \hat{B}(k))^{-1} d\lambda. \tag{2.2.46}$$

Combining (2.2.28) and (2.2.29), we have,

$$\begin{aligned} (\lambda - \hat{B}(k))^{-1} &= (\lambda - \hat{A}(k))^{-1} + Z(\lambda), \\ Z(\lambda) &= (\lambda - \hat{A}(k))^{-1} (I - G(\lambda))^{-1} G(\lambda), \end{aligned} \tag{2.2.47}$$

with

$$G(\lambda) = K(\lambda - \hat{A}(k))^{-1}.$$

Proposition 2.2.8(1) says that this is valid if $\sigma > 0$. Substitute this into (2.2.46) to deduce

$$\begin{aligned} \Phi(t,k) &= e^{t\hat{A}(k)} + \lim_{a\to\infty} \frac{1}{2\pi} U_{\sigma,a}(t,k), \\ U_{\sigma,a} &= \int_{-a}^{a} e^{(\sigma+i\tau)t} Z(\sigma + i\tau) d\tau, \end{aligned} \tag{2.2.48}$$

The main ingredient is to shift the integration path from the line Re $\lambda = \sigma > 0$ to Re $\lambda = -\sigma_0 < 0$ where $\sigma_0 > 0$ is the constant in Theorem 2.2.10. Let $\tau_1 > 0$ be the constant of Proposition 2.2.8(2) and choose $a > \tau_1$. Proposition 2.2.6 says that $Z(\lambda)$ is meromorphic in $\mathbb{C}_+(-\sigma_0)$ with only a finite number of singularities at the eigenvalues $\lambda_j(k)$ in Theorem 2.2.10, so that the contour integral of $e^\lambda Z(\lambda)$ on the rectangular path connecting the vertex

$$\sigma - ia, \quad \sigma + ia, \quad \sigma_0 + ia, \quad \sigma_0 - ia$$

can be computed, thanks to Residue Theorem, to deduce

$$U_{\sigma,a} = 2\pi i \sum_{j=0}^{n+1} \mathrm{Res}\Big\{e^{\lambda t} Z(\lambda); \lambda = \lambda_j(k)\Big\} + H + U_{-\sigma_0,a}, \tag{2.2.49}$$

where Res means the residue and

$$H = \Big(\int_{-\sigma_0+ia}^{\sigma+ia} - \int_{-\sigma_0-ia}^{\sigma-ia}\Big) e^{\lambda t} Z(\lambda) d\lambda.$$

First, since $\lambda_j(k) \in \rho(\hat{A}(k))$ and by [108, p.181], we find,

$$\mathrm{Res}\Big\{e^{\lambda t} Z(\lambda); \lambda = \lambda_j(k)\Big\} = \mathrm{Res}\Big\{e^{\lambda t}(\lambda - \hat{B}(k))^{-1}; \lambda = \lambda_j(k)\Big\} = e^{\lambda_j(k)t} P_j(k),$$

for $k \in S_1(\kappa_0)$ and otherwise this residue is 0.

Second, from Proposition 2.2.6, it can be seen that

$$\|H\| \to 0 \qquad (a \to \infty).$$

Finally, we can assert that

$$\|(I - G(-\sigma_0 + i\tau))^{-1}\| \leq C_1, \qquad \tau \in \mathbb{R},$$

holds for a positive constant C_1 independent of τ. This is seen from Proposition 2.2.6 for large τ, while for small τ, it comes since we can assume $\mathrm{Re}\lambda_j(k) \neq -\sigma_0$ by taking a smaller κ_0 if necessary and since G is a compact operator. Hence, for any $u, v \in L^2_\xi$,

$$| < U_{-\sigma_0,a} u, v > | \leq e^{-\sigma_0 t} \int_{-a}^{a} | < Z(-\sigma_0 + i\tau) u, v > | d\tau$$
$$\leq C_1 \|K\| e^{-\sigma_0 t} \int_{-a}^{a} \|(\lambda - \hat{A}(k))^{-1} u\| \|(\lambda - \hat{A}^*(k))^{-1} v\| d\tau, \quad \lambda = -\sigma_0 + i\tau,$$

where * means the adjoint. Since Lemma 2.2.13 applies to $\hat{A}^*(k)$ as well, we have,

$$| < U_{-\sigma_0,a} u, v > | \leq C_0(-\sigma_0 + \nu_0)^{-1} e^{-\sigma_0 t} \|u\| \|v\|.$$

This implies not only that $U_{-\sigma_0,a}$ converges as $a \to \infty$ in a weak operator topology, but also that the limit operator satisfies the estimate

$$\|U_{-\sigma_0,\infty}(t)\| \leq C_2 e^{-\sigma_0 t}, \qquad t \geq 0.$$

Summarizing, we have proved the

Theorem 2.2.14 *The semi-group $\Phi(t,k) = e^{t\hat{B}(k)}$ has the following decomposition.*

$$\Phi(t,k) = \sum_{j=0}^{n+2} \Phi_j(t,k), \tag{2.2.50}$$

where

$$\Phi_j(t,k) = e^{\lambda_j(k)t} \mathbf{P}_j(k) \chi(|k| < \kappa_0), \qquad j = 0, 1, \cdots, n+1, \tag{2.2.51}$$
$$\|\Phi_{n+2}(t,k)\| \leq C e^{-\sigma_0 t}, \qquad t \geq 0. \tag{2.2.52}$$

Note that we put $\Phi_{n+2} = e^{t\hat{A}(k)} + U_{-\sigma_0,\infty}(t,k)$.

2.2.5 Decay Rates of e^{tB} in $\mathbb{R}^n$

In Theorem 2.2.2, B in (2.1.4) was shown to be a C_0 semi-group generator in the space $L^2 = L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. The formula (2.2.50) enables us to establish explicit decay rates of this semi-group in various function spaces, which are used in an essential way for solving nonlinear problems in later sections. All the decay estimates are derived by bootstrap argument starting from the decay estimates in the L^2 Sobolev space that are established in this subsection. The cases $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{T}^n$ will be discussed separately. Most of computation is the same for both cases, though the resulting decay properties are quite different.

Let $H^\ell = H^\ell(\mathbb{R}^n)$, $\ell \in \mathbb{R}$, be the usual L^2 Sobolev space. Define a space of functions $u = u(x,\xi)$ by

$$H_\ell = L^2(\mathbb{R}^n_\xi; H^\ell_x), \tag{2.2.53}$$

with the norm

$$\|u\|_{H_\ell} = \Big(\int_{\mathbb{R}^n} \|u(\cdot,\xi)\|^2_{H^\ell} d\xi\Big)^{1/2} = \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} (1+|k|^2)^\ell |\hat{u}(k,\xi)|^2 dk d\xi\Big)^{1/2}.$$

It is seen that B is a C_0 semi-group generator also in H_ℓ for any $\ell \in \mathbb{R}$ if it is defined by (2.1.4) with L^2 replaced by H_ℓ. The proof is similar and omitted. Now, we derive various decay estimates of e^{tB}. First, substituting (2.2.50) into (2.2.20), we have the decomposition,

$$e^{tB} = E_1(t) + E_2(t), \tag{2.2.54}$$

$$E_1(t) = \sum_{j=0}^{n+1} \mathcal{F}^{-1}\Big\{\Phi_j(t,k)\Big\}\mathcal{F}, \quad E_2(t) = \mathcal{F}^{-1}\Big\{\Phi_{n+2}(t,k)\Big\}\mathcal{F}.$$

Both components are continuous functions of t with values in the space of bounded operators on H_ℓ. We shall show that $E_1(t)$ has the algebraic decay while $E_2(t)$ has the exponential decay, as $t \to \infty$.

To this end, introduce,

$$\begin{aligned} &\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \\ &\partial^\alpha_x = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\cdots\partial^{\alpha_n}_{x_n}. \end{aligned}$$

As usual, $\alpha' \le \alpha$ for $\alpha, \alpha' \in \mathbb{N}^n$ means that $\alpha'_i \le \alpha_i$ for all $i = 1, 2, \cdots, n$. For $q \in [1,2]$ and $m \ge 0$, set

$$\sigma_{q,m} = \frac{n}{2}\Big(\frac{1}{q} - \frac{1}{2}\Big) + \frac{m}{2}, \tag{2.2.55}$$

and

$$Z_q = L^2(\mathbb{R}^n_\xi; L^q(\mathbb{R}^n_x)), \quad \|u\|_{Z_q} = \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |u(x,\xi)|^q dx\Big)^{2/q} d\xi\Big)^{1/2}. \tag{2.2.56}$$

All the decay estimates to be derived in this subsection relies on the

Theorem 2.2.15 (i) *For any $q \in [1,2]$ and $\ell \in \mathbb{R}$, $E_1(t)$ can be regarded as a bounded operator from Z_q to H_ℓ for each t, and for any $\alpha, \alpha' \in \mathbb{N}^n$ with $\alpha' \leq \alpha$ and for any u satisfying $\partial_x^{\alpha'} u \in Z_q$,*

$$\|\partial_x^\alpha E_1(t)u\|_{L^2} \leq b_1(1+t)^{-\sigma_{q,m}}\|\partial_x^{\alpha'}u\|_{Z_q}, \tag{2.2.57}$$

$$\|\partial_x^\alpha E_1(t)(I-\mathbf{P})u\|_{L^2} \leq b_2(1+t)^{-\sigma_{q,m+1}}\|\partial_x^{\alpha'}u\|_{Z_q} \tag{2.2.58}$$

hold for $t \geq 0$ with $m = |\alpha - \alpha'|$ where $\mathbf{P}$ is the orthogonal projection (2.1.11) *while b_1, b_2 are positive constants depending on q and m only.*
(ii) *For any $\alpha \in \mathbb{N}^n$, $E_2(t)$ satisfies*

$$\|\partial_x^\alpha E_2(t)u\|_{L^2} \leq b_3 e^{-\sigma_0 t}\|\partial_x^\alpha u\|_{L^2}, \tag{2.2.59}$$

where σ_0 and b_3 are positive constants independent of α, u, and t.

Remark 2.2.16 The part (i) shows that higher derivatives in x of $E_1(t)$ decay faster than lower derivatives as $t \to \infty$. The derivatives in ξ, on the other hand, have no such property. It will be shown in §3.4.2, that this feature is inherited by the nonlinear problem. The fact that the constants σ_0 and b_3 are independent of α is crucial for the proof. Notice that the heat kernel enjoys the same theorem.

Proof of Theorem 2.2.15. Write $k^\alpha = k_1^{\alpha_1}k_1^{\alpha_2}\cdots k_n^{\alpha_n}$ and note that

$$k^\alpha \Phi_j(t,k)\hat{u}(k,\cdot) = \Phi_j(t,k)(k^\alpha \hat{u}(k,\cdot)), \quad j = 0,1,\cdots,n+2$$

hold point wise for $k \in \mathbb{R}^n$ in the space L^2_ξ. Put

$$I_j(t,k) = \|\Phi_j(t,k)\hat{u}(k,\cdot)\|_{L^2(\mathbb{R}^n_\xi)}.$$

It follows from (2.2.52) and by the aid of Parseval's relation that

$$\|k^\alpha I_{n+2}(t,k)\|_{L^2(\mathbb{R}^n_k)} \leq C_1 e^{-\sigma_0 t}\|k^\alpha \hat{u}\|_{L^2(\mathbb{R}^n_k \times \mathbb{R}^n_\xi)} = C_1 e^{-\sigma_0 t}\|\partial_x^\alpha u\|_{L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)},$$

which proves the part (ii).

On the other hand, for $j = 0,\cdots,n+1$, we have from Theorem 2.2.10 (iii),

$$\|\hat{\mathbf{P}}_j(k)\hat{u}(k,\cdot)\|_{L^2(\mathbb{R}^n_\xi)} \leq c_0\|\hat{u}(k,\cdot)\|_{L^2(\mathbb{R}^n_\xi)} \quad (|k| \leq \kappa_0),$$

and choosing κ_0 sufficiently small if necessary,

$$\mathrm{Re}\lambda_j(k) = -\lambda_j^{(2)}|k|^2(1+O(|k|)) \geq -a_0|k|^2 \quad (|k| \leq \kappa_0), \tag{2.2.60}$$

with some constants $a_0, c_0 > 0$ independent of k, so that

$$\begin{aligned}\|k^\alpha I_j(t,k)\|^2_{L^2(\mathbb{R}^n_k)} &\leq c_0\int_{|k|\leq\kappa_0}|k^{\alpha-\alpha'}|^2 e^{2\mathrm{Re}\lambda_j(k)}\|k^{\alpha'}\hat{u}(k,\cdot)\|^2_{L^2(\mathbb{R}^n_\xi)}dk \\ &\leq c_0\Big(\int_{|k|\leq\kappa_0}|k|^{2p'm}e^{-2p'a_0|k|^2 t}dk\Big)^{1/p'}\Big(\int_{\mathbb{R}^n}\|k^{\alpha'}\hat{u}(k,\cdot)\|^{2q'}_{L^2(\mathbb{R}^n_\xi)}dk\Big)^{1/q'},\end{aligned} \tag{2.2.61}$$

where $m = |\alpha - \alpha'|$ and $p' \in [1, \infty)$ with $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$. Note that

$$\int_{|k|\le\kappa_0} |k|^{2p'm} e^{-2p'a_0|k|^2 t} dk \le c_2(1+t)^{-n/2-p'm},$$

and that, by the well known property of the Fourier transformation,

$$\Big(\int_{\mathbb{R}^n} \|k^{\alpha'}\hat{u}(k,\cdot)\|^{2q'}_{L^2(\mathbb{R}^n_\xi)} dk\Big)^{1/q'} \le \|\partial_x^{\alpha'} u\|^2_{Z_q}, \qquad \frac{1}{q} + \frac{1}{2q'} = 1.$$

Thus, (2.2.57) follows.

Similarly, if $\mathbf{P}u = 0$, we have $\mathbf{P}\hat{u}(k,\xi) = 0$ for all k, and hence from Theorem 2.2.10 (iii),

$$\|\hat{\mathbf{P}}_j(k)\hat{u}(k,\cdot)\|_{L^2(\mathbb{R}^n_\xi)} \le c_0|k|\|\hat{u}(k,\cdot)\|_{L^2(\mathbb{R}^n_\xi)} \quad (|k| \le \kappa_0).$$

Consequently, the same computation as above with m replaced by $m+1$ gives (2.2.58). This completes the proof of the theorem.

In (2.2.58), the extra decay rate 1/2 is obtained under the assumption $u \in \mathcal{N}^\perp$. This will be used in an essential way in later sections for nonlinear problems, in conjunction with the property (2.1.2) of the nonlinear operator Γ. The same extra decay rate can be obtained under a different assumption, which will be also essential in §2.3 for constructing time-periodic solutions for the case $n = 3, 4$.

Proposition 2.2.17 *Suppose that u satisfy*

$$(1+|x|)u \in Z_1, \quad \int_{\mathbb{R}^3} \mathbf{P}u\, dx = 0, \qquad a.e.\ \xi \in \mathbb{R}^n. \tag{2.2.62}$$

Then, for any $\alpha \in \mathbb{N}^n$,

$$\|\partial_x^\alpha E_1(t)u\|_{L^2} \le b_4(1+t)^{-\sigma_{1,|\alpha|+1}}\|(1+|x|)u\|_{Z_1} \tag{2.2.63}$$

holds for $t \ge 0$ with a positive constant b_4 independent of u and t.

Proof. The first assumption in (2.2.62) implies that $\hat{u}$ is Lipschitz continuous in k,

$$\|\hat{u}(k,\cdot) - \hat{u}(k',\cdot)\|_{L^2_\xi} \le C_0|k-k'|\|(1+|x|)u\|_{Z_1},$$

and the second assumption implies that $\mathbf{P}\hat{u}(0,\cdot) = 0$, so that

$$\|\hat{\mathbf{P}}_j(\tilde{k})\hat{u}(k,\cdot)\|_{L^2_\xi} = \|\hat{\mathbf{P}}_j(\tilde{k})(\hat{u}(k,\cdot) - \hat{u}(0,\cdot))\|_{L^2(\mathbb{R}^n_\xi)} \le c_0|k|\|(1+|x|)u\|_{Z_1}.$$

Hence, (2.2.63) is obtained by a similar computation as in the proof of Theorem 2.2.15 (i). Thus, the proposition was proved.

The decay estimates in the Hilbert space H_ℓ established above, however, are not applicable directly to the nonlinear problems which are usually manipulated in a Banach algebra that is a property that H_ℓ does not possesses. One of the Banach algebras which are suitable for the Boltzmann equation is

$$\dot{H}_{\ell,\beta} = \left\{ u \in L^\infty_{loc}(\mathbb{R}^n_\xi; H^\ell_x) \mid \|u\|_{\ell,\beta} < \infty,\ \limsup_{|\xi|\to\infty} (1+|\xi|)^\beta \|u(\cdot,\xi)\|_{H^\ell} = 0 \right\}, \tag{2.2.64}$$
$$\|u\|_{\ell,\beta} = \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^\beta \|u(\cdot,\xi)\|_{H^\ell}.$$

Actually, this is a Banach algebra if $\ell > n/2$ and $\beta \geq 0$. The condition

$$\limsup_{|\xi|\to\infty} (1+|\xi|)^\beta \|u(\cdot,\xi)\|_{H^\ell} = 0 \tag{2.2.65}$$

in the above is introduced to ensure the continuity property of the semi-group e^{tB}.

We shall now reestablish the previous decay estimates in this space. This is possible by bootstrap argument based on the smoothing property (2.1.9) of K. First, let us show that the operator B is a C_0 semi-group generator also in the space $\dot{H}_{\ell,\beta}$, if $D(B)$ is defined by (2.2.5) with L^2 replaced by $\dot{H}_{\ell,\beta}$.

Proposition 2.2.18 *For any $\ell, \beta \in \mathbb{R}$, B generates a C_0 semi-group e^{tB} in $\dot{H}_{\ell,\beta}$.*

Proof. Proposition 2.1.1 says that K can be taken to be a bounded operator on $\dot{H}_{\ell,\beta}$. Therefore, as before, it suffices to show that Proposition 2.2.3 holds for A in this new domain of definition. We proceed just in the same way. Recall $S(t)$ in (2.2.9). It satisfies (2.2.8) also in $\dot{H}_{\ell,\beta}$. To prove its continuity in t, note, first, from (2.2.65) that $\dot{H}_{\ell',\beta'}$ is dense in $\dot{H}_{\ell,\beta}$ as long as $\ell' > \ell$ and $\beta' > \beta$. Therefore, the continuity of $S(t)$ on $\dot{H}_{\ell,\beta}$ follows if

$$\|(S(t') - S(t))u\|_{\ell,\beta} \to 0 \qquad (t,\ t' \geq 0,\ t' \to t) \tag{2.2.66}$$

holds for each $u \in \dot{H}_{\ell',\beta'}$. For this, let $w(t,k,\xi)$ be the Fourier transform (2.2.11), that is,

$$w(t,k,\xi) = s(t,x,k)\hat{u}(k,\xi), \qquad s(t,x,k) = e^{-(ik\cdot\xi + \nu(\xi))t}. \tag{2.2.67}$$

We get,

$$\begin{aligned}
&\|(S(t') - S(t))u\|^2_{\ell,\beta} \\
&\quad \leq \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^{2\beta} \int_{\mathbb{R}^n} (1+|k|)^{2\ell} |s(t',k,\xi) - s(t',k,\xi)|^2 |\hat{u}(k,\xi)|^2 dk \\
&\quad \leq b(t,t')^2 \|u\|^2_{\ell',\beta'},
\end{aligned}$$

where

$$b(t,t') = \sup_{k\in\mathbb{R}^n, \xi\in\mathbb{R}^n} \frac{|s(t',k,\xi) - s(t',k,\xi)|}{(1+|k|)^{\ell'-\ell}(1+|\xi|)^{\beta'-\beta}}.$$

By a simple computation, we get for any $\delta \in (0,1)$,

$$|e^{ik\cdot\xi t'} - e^{ik\cdot\xi t}| \leq C(|k||\xi||t'-t|)^\delta,$$

$$|e^{-\nu(\xi)t'} - e^{-\nu(\xi)t}| \leq C((1+|\xi|)^\gamma |t'-t|)^\delta,$$

for some constant $C>0$, where (2.1.8) was used. Choosing a sufficiently small δ concludes (2.2.66). The rest of proof is the same as in Proposition 2.2.3.

To derive decay estimates of e^{tB} on $\dot{H}_{\ell,\beta}$, apply Duhamel's formula to (2.2.13) to deduce

$$e^{tB} = e^{tA} + (e^{tA}K) * e^{tB}, \tag{2.2.68}$$

where $*$ means the convolution in t,

$$g * h = \int_0^t g(t-s)h(s)ds.$$

Iterate this to deduce

$$e^{tB} = \sum_{k=0}^{N} G_j(t) + (G_N(t)K) * e^{tB}, \tag{2.2.69}$$

for $N \in \mathbb{N}$ where

$$G_0(t) = e^{tA}, \qquad G_j(t) = (e^{tA}K) * G_{j-1}(t) = G_{j-1}(t) * (Ke^{tA}) \quad (j = 1, 2, \cdots).$$

Lemma 2.2.19 *Let $\ell \in \mathbb{R}$ and $\beta \ge 0$.*
(i) *For any $j \in \mathbb{N}$, $G_j(t)$ is a bounded operator on $\dot{H}_{\ell,\beta}$, and for any $\epsilon > 0$, there is a constant $C > 0$ such that*

$$\|G_j(t)u\|_{\ell,\beta} \le Ce^{-(\nu_0-\epsilon)t}\|u\|_{\ell,\beta}.$$

(ii) *There is an integer $N \in \mathbb{N}$ such that $G_N(t)K$ is a bounded operator from H_ℓ to $\dot{H}_{\ell,\beta}$, and for any $\epsilon > 0$, there is a constant $C > 0$ such that*

$$\|G_N(t)Ku\|_{\ell,\beta} \le Ce^{-(\nu_0-\epsilon)t}\|u\|_{H_\ell}.$$

(iii) *$G_j(t)$ commutes with ∂_x^α.*

Proof. Note that for any $\epsilon, \kappa > 0$,

$$e^{-\kappa t} * e^{-\kappa t} = te^{-\kappa t} \le Ce^{-(\kappa-\epsilon)t}$$

holds for a constant $C = C_\epsilon$ such that $C_\epsilon \to \infty \ (\epsilon \to 0)$. Then, (i) is an easy consequence of the estimate (2.2.8) in $\dot{H}_{\ell,\beta}$ since K is a bounded operator on $\dot{H}_{\ell,\beta}$ in virtue of Proposition 2.1.1. Further, the same proposition says that K can be also taken to be a bounded operator from $\dot{H}_{\ell,\beta-1}$ to $\dot{H}_{\ell,\beta}$ as well as from H_ℓ to $\dot{H}_{\ell,0}$. This proves (ii) with $N \ge [\beta]+1$. (iii) is obvious from (2.2.7).

Substituting the decomposition (2.2.54) into the last term of (2.2.69) yields a decomposition of e^{tB} in the space $\dot{H}_{\ell,\beta}$:

$$e^{tB} = D_1(t) + D_2(t), \tag{2.2.70}$$

$$D_1(t) = (G_N(t)K)*E_1(t), \qquad D_2(t) = \sum_{k=0}^{N} G_j(t) + (G_N(t)K) * E_2(t),$$

with $N \geq [\beta] + 1$.

By virtue of Lemma 2.2.19, the L^2 decay estimates in Theorem 2.2.15 can be transferred into the space $\dot{H}_{\ell,\beta}$. Recall the definitions (2.2.55) and (2.2.56) for $\sigma_{q,m}$ and Z_q, respectively.

Theorem 2.2.20 *Let $\ell \in \mathbb{R}$ and $\beta \geq 0$.*
(i) *For any $q \in [1,2]$, $D_1(t)$ is a bounded operator from Z_q to $\dot{H}_{\ell,\beta}$ for each t, and for any $\alpha, \alpha' \in \mathbb{N}^n$ with $\alpha' \leq \alpha$ and for any u satisfying $\partial_x^{\alpha'} u \in Z_q$,*

$$\|\partial_x^\alpha D_1(t)u\|_{\dot{H}_{\ell,\beta}} \leq b_1(1+t)^{-\sigma_{q,m}} \|\partial_x^{\alpha'} u\|_{Z_q}, \tag{2.2.71}$$

$$\|\partial_x^\alpha D_1(t)(I-\mathbf{P})u\|_{\dot{H}_{\ell,\beta}} \leq b_2(1+t)^{-\sigma_{q,m+1}} \|\partial_x^{\alpha'} u\|_{Z_q} \tag{2.2.72}$$

hold for $t \geq 0$ with $m = |\alpha - \alpha'|$ where b_1, b_2 are positive constants depending on m, ℓ, β, q, but not on α, α' themselves nor on u, t.
(ii) *$D_2(t)$ is a bounded operator from $\dot{H}_{\ell,\beta} \cap H_\ell$ into $\dot{H}_{\ell,\beta}$, and for any $\alpha \in \mathbb{N}^n$, it satisfies*

$$\|\partial_x^\alpha D_2(t)u\|_{\dot{H}_{\ell,\beta}} \leq b_3 e^{-\sigma_0 t}(\|\partial_x^\alpha u\|_{H_\ell} + \|\partial_x^\alpha u\|_{\dot{H}_{\ell,\beta}}) \tag{2.2.73}$$

where σ_0 and b_3 are positive constants independent of α, u, and t.

Proof. Notice that for any numbers $\kappa_1 > 0$ and $\kappa_2 \geq 0$,

$$e^{-\kappa_1 t} * (1+t)^{-\kappa_2} \leq C(1+t)^{-\kappa_2}, \qquad e^{-\kappa_1 t} * e^{-\kappa_2 t} \leq Ce^{-\kappa_2 t} \quad (\kappa_1 > \kappa_2)$$

hold for $t \geq 0$ with some positive constant $C > 0$. Whereas the second inequality comes by a direct computation, The first inequality can be concluded by the computation,

$$\begin{aligned}\int_0^t e^{-\kappa_1(t-s)}(1+s)^{-\kappa_2} ds &= \int_0^{t/2} + \int_{t/2}^t \\ &\leq e^{-\kappa_1 t/2} \int_0^{t/2} (1+s)^{-\kappa_2} ds + (1+t/2)^{-\kappa_2} \int_{t/2}^t e^{-\kappa_1(t-s)} ds \\ &\leq C(1+t)^{-\kappa_2}.\end{aligned}$$

Take $\kappa_1 = \nu_0 - \epsilon$ and $\kappa_2 = \sigma_{q,m}$ or $= \sigma_0$, and combine Theorem 2.2.15 with Lemma 2.2.19. whence follows parts (i), (ii). For (ii), σ_0 is to be taken smaller than in Theorem 2.2.15 (ii), if necessary.

The following theorem summarizes the decay estimates of e^{tB} that are used for solving the nonlinear problems in the later sections. All the part except for (2.2.77) is a direct consequence of the above theorem with $\alpha' = 0$, and (2.2.77) is obtained by noticing that $D_1(t)$ enjoys a similar estimate to that of Proposition 2.2.17, with a due modification. The estimates for $\alpha' \neq 0$ are essentially used in §3.1.2 where the decay of space derivatives of solutions to the Cauchy problem is discussed.

Theorem 2.2.21 *Let $q \in [1,2]$, $\ell \in \mathbb{R}$ and $\beta \geq 0$. Then, there is a positive constant b_0 such that for any $u \in H_{\ell,\beta} \cap Z_q$, it holds that*

$$\|e^{tB}u\|_{\ell,\beta} \leq b_0(1+t)^{-\sigma_{q,0}} \Big\{ \|u\|_{\ell,\beta} + \|u\|_{Z_q} \Big\}, \tag{2.2.74}$$

$$\|e^{tB}(I-\mathbf{P})u\|_{\ell,\beta} \leq b_0(1+t)^{-\sigma_{q,1}} \Big\{ \|u\|_{\ell,\beta} + \|u\|_{Z_q} \Big\}, \tag{2.2.75}$$

while if in addition,

$$(1+|x|)u \in Z_1, \qquad \int_{\mathbb{R}^3} \mathbf{P}u\ dx = 0, \qquad a.e.\ \xi \in \mathbb{R}^n, \tag{2.2.76}$$

it holds that

$$\|e^{tB}u\|_{\ell,\beta} \le b_0(1+t)^{-\sigma_{1,1}}\Big\{\|u\|_{\ell,\beta} + \|(1+|x|)u\|_{Z_1}\Big\}. \tag{2.2.77}$$

Note that for $q = 2$, (2.2.71) does not assure the decay because $\sigma_{2,0} = 0$. However, the following decay property is still available.

Theorem 2.2.22 *Let $\ell \ge 0$ and $\beta > n/2$.*

$$\|e^{tB}u\|_{\ell,\beta} \to 0 \qquad (t \to \infty) \tag{2.2.78}$$

holds for any $u \in H_\ell$.

Proof. Note that $Z_2 = L^2$. Hence, for $q = 2$, (2.2.71) asserts that e^{tB} is a uniformly bounded semi-group in the space $L^2 \cap \dot{H}_{\ell,\beta}$ for any $\beta \in \mathbb{R}$, and hence so is in $\dot{H}_{\ell,\beta}$ if $\ell \ge 0$ and $\beta > n/2$ since then $L^2 \supset \dot{H}_{\ell,\beta}$. On the other hand, it is easy to show that for any $q \in [1,2)$ $(q \ne 2)$, the space $\dot{H}_{\ell,\beta} \cap Z_q$ has a subset which is dense in $\dot{H}_{\ell,\beta}$, whence, together with (2.2.71), follows the theorem.

2.2.6 Decay Rates of e^{tB} in $\mathbb{T}^n$

The computation in the previous subsection leads to the exponential decays for the case $\Omega = \mathbb{T}^n$, under some extra restriction on u. To see this, note that the spaces H_ℓ and $\dot{H}_{\ell,\beta}$ are still defined respectively by (2.2.53) and (2.2.64), except the Sobolev space H^ℓ which is to be defined on $\mathbb{T}^n$, $H^\ell = H^\ell(\mathbb{T}^n)$. Thus, the norm in (2.2.53) is to be replaced by

$$\|u\|_{H_\ell} = \Big(\int_{\mathbb{R}^n} \|u(\cdot,\xi)\|^2_{H^\ell} d\xi\Big)^{1/2} = \Big(\sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} (1+|k|^2)^\ell |\hat{u}(k,\xi)|^2 d\xi\Big)^{1/2}, \tag{2.2.79}$$

where $\hat{u}(k,\xi)$ is the Fourier coefficient defined by (2.2.10). This changes Theorem 2.2.15. Indeed, now, the computation (2.2.61) for $j = 0, \cdots, n+1$ should be modified as

$$\begin{aligned} \|k^\alpha I_j(k)\|^2_{\ell^2(\mathbb{Z}^n)} &= \sum_{k\in\mathbb{Z}^n, |k|\le\kappa_0} |k^\alpha I_j(k)|^2 \\ &\le c_1\delta_{|\alpha|,0}\|\mathbf{P}\hat{u}(0,\cdot)\|^2_{L^2(\mathbb{R}^n_\xi)} + c_2 \sum_{k\in\mathbb{Z}^n, 0<|k|\le\kappa_0} e^{2\mathrm{Re}\lambda_j(k)t}\|k^\alpha \hat{u}(k,\cdot)\|^2_{L^2(\mathbb{R}^n_\xi)}. \end{aligned} \tag{2.2.80}$$

Here, c_0, c_1, c_2 are positive constants independent of u, t, and $\delta_{|\alpha|,0}$ is Kronecker's symbol. And we used Theorem 2.2.10 saying $\lambda_j(0) = 0$ and $\sum_{j=0}^{n+1} P_j(0) = \mathbf{P}$. Recall (2.2.60). Then, the rest of the computation in the proof of Theorem 2.2.15 leads to

Theorem 2.2.23 (i) *$E_1(t)$ is a uniformly bounded operator on H_ℓ for any $\ell \in \mathbb{R}$, and for all $\alpha \in \mathbb{N}^n$, it holds that*

$$\|\partial^\alpha E_1(t)u\|_{L^2} \le b_1 \|\partial^\alpha u\|_{L^2} \qquad (\forall t \ge 0). \tag{2.2.81}$$

Furthermore, either if u satisfies an extra condition

$$\int_{\mathbb{T}^3} \mathbf{P}u \; dx = \mathbf{P}\hat{u}(0,\cdot) = 0, \qquad a.e. \;\xi \in \mathbb{R}^n, \tag{2.2.82}$$

or if $\alpha \ne 0$, then, $E_1(t)$ enjoys the exponential decay

$$\|\partial_x^\alpha E_1(t)u\|_{L^2} \le b_2 e^{-a_1 t} \|\partial_x^{\alpha'} u\|_{L^2} \qquad (\forall t \ge 0,\; \alpha' \le \alpha). \tag{2.2.83}$$

Here, a_1, b_1, b_2 are positive constants independent of u, t, α.
(ii) *For any $\alpha \in \mathbb{N}^n$, $E_2(t)$ satisfies*

$$\|\partial_x^\alpha E_2(t)u\|_{L^2} \le b_3 e^{-\sigma_0 t} \|\partial_x^\alpha u\|_{L^2} \qquad (\forall t \ge 0), \tag{2.2.84}$$

where σ_0 and b_3 are positive constants independent of α, u, and t.

The remaining argument in the previous subsection then yields the counterpart of Theorem 2.2.20.

Theorem 2.2.24 *For any $\ell \in \mathbb{R}$ and $\beta > n/2$, there are positive constants b_0 and σ_1 such that for all $u \in \dot{H}_{\ell,\beta}$, it holds that*

$$\|e^{tB}u\|_{\ell,\beta} \le b_0 \|u\|_{\ell,\beta} \qquad (\forall t \ge 0), \tag{2.2.85}$$

and if, in addition, either u satisfies (2.2.82) *or has a form $u = \partial_{x_j} v$ for some j, it holds that*

$$\|e^{tB}u\|_{\ell,\beta} \le b_0 e^{-\sigma_1 t}\Big(\|u\|_{\ell,\beta} \quad or \quad \|v\|_{\ell,\beta}\Big) \qquad (\forall t \ge 0). \tag{2.2.86}$$

Here, $\sigma_1 = \min(a_1, \sigma_0)$ where a_1, σ_0 are those in Theorem 2.2.23.

Remark 2.2.25 In contrast to the case $\Omega = \mathbb{R}^n$, the non-trivial decay follows only under the condition (2.2.82). No analogue to Theorem 2.2.22 is possible.

2.3 Global Solutions of the Cauchy Problem

The decay estimates established in the previous section have many applications to the nonlinear Boltzmann equation. This sections shows that they can be used to construct global solutions to the Cauchy problem and also time-periodic solutions for the case with time-periodic external source. This is done by combining with the contraction mapping principle, but in quite different contexts. Also, they will be used to establish a new decay property of space derivatives of the global solutions to the Cauchy problem.

In this subsection, we construct global solutions to the Cauchy problem for the the Boltzmann equation (2.1.3) and then discuss some decay property of their space derivatives.

2.3.1 Global Existence

We discuss the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Bu + \Gamma[u,u], & t>0,\ x\in\Omega, \xi\in\mathbb{R}^n, \\ u|_{t=0} = u_0(x,\xi), & x\in\Omega,\ \xi\in\mathbb{R}^n. \end{cases} \tag{2.3.1}$$

for the case $\Omega=\mathbb{R}^n$ and $\Omega=\mathbb{T}^n$. Throughout this section, we assume

$$n\ge 3, \qquad \ell > \frac{n}{2}, \qquad \beta > \frac{n}{2}+1. \tag{2.3.2}$$

Recall $\sigma_{q,k}$ of (2.2.55):

$$\sigma_{q,k} = \frac{n}{2}(\frac{1}{q}-\frac{1}{2}) + \frac{k}{2}.$$

The existence theorem to be proved here is,

Theorem 2.3.1 *Assume the cutoff hard potential* (1.2.17)*. Then, there are two positive constants* $a_0,\ a_1$ *such that the following holds.*
(1) The case $\mathbb{R}^n$. *For any initial data* $u_0\in \dot{H}_{\ell,\beta}$ *satisfying*

$$\|u_0\|_{\ell,\beta} \le a_0,$$

then, (2.3.1) *admits a global unique solution* u *in the function class*

$$BC^0([0,\infty);\dot{H}_{\ell,\beta}) \cap BC^1([0,\infty);\dot{H}_{\ell-1,\beta-1}). \tag{2.3.3}$$

It has the estimate

$$\|u(t)\|_{\ell,\beta} \le a_1\|u_0\|_{\ell,\beta}, \qquad t\in[0,\infty). \tag{2.3.4}$$

and the decay property

$$\|u(t)\|_{\ell,\beta} \to 0 \qquad (t\to\infty). \tag{2.3.5}$$

Further, non-trivial decay rates are possible provided the initial data u_0 *satisfy additional conditions.*

(a) *Suppose that for* $q\in[1,2]$*,* u_0 *satisfies*

$$u_0\in \dot{H}_{\ell,\beta}\cap Z_p, \qquad \|u_0\|_{\ell,\beta} + \|u_0\|_{Z_p} \le a_0. \tag{2.3.6}$$

Then, u *has an algebraic decay*

$$\|u(t)\|_{\ell,\beta} \le a_1(1+t)^{-\sigma_{q,0}}\Big(\|u_0\|_{\ell,\beta} + \|u_0\|_{Z_q}\Big), \qquad t\in[0,\infty). \tag{2.3.7}$$

(b) *If further* $\mathbf{P}u_0=0$*, the decay rate* $\sigma_{q,0}$ *in* (2.3.7) *can be replaced with* $\sigma_{q,1}$*.*

(c) *Suppose that*

$$u_0 \in \dot{H}_{\ell,\beta}, \qquad (1+|x|)u_0 \in Z_1, \qquad \int_{\mathbb{R}^n} \mathbf{P}u_0 dx = 0, \tag{2.3.8}$$

$$\|u_0\|_{\ell,\beta} + \|(1+|x|)u_0\|_{Z_1} \le a_0.$$

Then, the decay estimate (2.3.7) *is replaced by*

$$\|u(t)\|_{\ell,\beta} \le a_1(1+t)^{-\sigma_{1,1}} \Big(\|u_0\|_{\ell,\beta} + \|(1+|x|)u_0\|_{Z_1} \Big), \qquad t \in [0,\infty). \tag{2.3.9}$$

(2) The case $\mathbb{T}^n$. *For any initial data $u_0 \in \dot{H}_{\ell,\beta}$ satisfying*

$$\|u_0\|_{\ell,\beta} \le a_0,$$

(2.3.1) *admits a global unique solution u in the function class*

$$BC^0([0,\infty); \dot{H}_{\ell,\beta}) \cap BC^1([0,\infty); \dot{H}_{\ell-1,\beta-1}), \tag{2.3.10}$$

and satisfies the estimate

$$\|u(t)\|_{\ell,\beta} \le a_1 \|u_0\|_{\ell,\beta}, \qquad t \in [0,\infty). \tag{2.3.11}$$

If, in addition, u_0 satisfies

$$\int_{\mathbb{T}^n} \mathbf{P}u_0 dx = 0, \tag{2.3.12}$$

the solution u has an exponential decay

$$\|u(t)\|_{\ell,\beta} \le a_1 e^{-\sigma_0 t} \|u_0\|_{\ell,\beta}, \qquad t \in [0,\infty). \tag{2.3.13}$$

where the constant $\sigma_0 > 0$ is the same as in (2.2.86).

Remark 2.3.2 In part (2), the counter part of (2.3.5) is not available.

Remark 2.3.3 For the space dimension $n = 1$ or 2, i.e. $x \in \mathbb{R}$ or $\mathbb{R}^2$, (1.2.1) has a physical sense only if the velocity variable ξ is kept three-dimensional, i.e. $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. In this case, $\xi \cdot \nabla_x$ is to be taken $\xi_1 \partial_x$ $(n = 1)$ or $\xi_1 \partial_{x_1} + \xi_2 \partial_{x_2}$ $(n = 2)$. Then, the above theorem also holds. However, the proof of Theorem 2.3.1 will be given only for the case $n \ge 3$, since the case $n = 1, 2$ can be proved with a slight modification.

The rest of this subsection is devoted to the proof of Theorem 2.3.1. First, we rewrite (2.3.1) in the form of the integral equation

$$u(t) = e^{tB} u_0 + \int_0^t e^{(t-s)B} \Gamma[u(s), u(s)] ds, \tag{2.3.14}$$

which is deduced by means of Duhamel's formula. We shall show that this has a global solution in the function class $BC^0([0,\infty);\dot{H}_{\ell,\beta})$. Once it is proved, it follows from the definition of domain $D(B)$ in (2.2.5) that the right hand side of (2.3.14) is differentiable in t in the space $BC^1([0,\infty);\dot{H}_{\ell-1,\beta-1})$, which implies that, in turn, the solution u of the integral equation (2.3.14) is a classical solution of the Cauchy problem (2.3.1).

Now, we set

$$\Phi[u](t) = e^{tB}u_0 + \int_0^t e^{(t-s)B}\Gamma[u(s),u(s)]ds. \tag{2.3.15}$$

This defines a nonlinear map and (2.3.14) is written as $u = \Phi[u]$, that is, the solution u to (2.3.14) is a fixed point of the map Φ. We now show that the decay estimates obtained in the previous section ensures that Φ is a contraction map if u_0 is small.

In the sequel, we assume (2.3.2). For $\sigma \geq 0$, set

$$\rho_\sigma(t) = \begin{cases} (1+t)^{-\sigma}, & \Omega = \mathbb{R}^n, \\ e^{-\sigma t}, & \Omega = \mathbb{T}^n, \end{cases} \tag{2.3.16}$$

and introduce the function space

$$X_{\ell,\beta,\sigma} = \Big\{ u \in BC^0([0,\infty);\dot{H}_{\ell,\beta}) \mid |||u|||_{\ell,\beta,\sigma} < +\infty \Big\}, \tag{2.3.17}$$
$$|||u|||_{\ell,\beta,\sigma} = \sup_{t\geq 0}\Big(\rho_\sigma(t)^{-1}\|u(t)\|_{\ell,\beta}\Big),$$

Recall the function $\nu(\xi)$ in (2.1.8), and as before, denote the multiplication operator by this function by ν. Given a function $h = h(t,x,\xi)$, define the integral

$$\Psi[h] = \int_0^t e^{(t-s)B}(\nu h(s))ds. \tag{2.3.18}$$

Lemma 2.3.4 *Let $\sigma \geq 0$ and assume (2.3.2). Put*

$$\sigma^* = \begin{cases} \min(\sigma_{1,1},\sigma), & \Omega = \mathbb{R}^n, \\ \min(\sigma_0,\sigma), & \Omega = \mathbb{T}^n, \end{cases} \tag{2.3.19}$$

where $\sigma_{1,1} = n/4 + 1/2$ from Theorem 2.2.20 and σ_0 from Theorem 2.2.24.
(1) *If*

$$h \in X_{\ell,\beta,2\sigma}, \quad \rho_{2\sigma}(t)(\nu h)(t) \in BC^0([0,\infty);Z_1), \quad \mathbf{P}(\nu h) = 0, \tag{2.3.20}$$

then, $\Psi[h] \in X_{\ell,\beta,\sigma^*}$ and

$$|||\Psi[h]|||_{\ell,\beta,\sigma^*} \leq C_0[[h]]_{\ell,\beta,2\sigma}, \tag{2.3.21}$$

holds for a positive constant C_0 where

$$[[h]]_{\ell,\beta,2\sigma} = |||h|||_{\ell,\beta,2\sigma} + \sup_{t\geq 0}\rho_{2\sigma}(t)\|\nu h(t)\|_{Z_1}.$$

(2) *For the case $\Omega = \mathbb{R}^n$, if $\sigma = \sigma^* = 0$ but $[[\chi(t>R)h]]_{\ell,\beta,0} \to 0$ $(R\to\infty)$, it holds that*

$$|||\chi(t>R)\Psi[h]|||_{\ell,\beta,0} \to 0 \qquad (R\to\infty), \tag{2.3.22}$$

where $\chi(t>R)$ is the characteristic function of the interval $[R,\infty)$.

Remark 2.3.5 (1) A point here is, among others, that although the function $\nu(\xi)$ is an unbounded function as is stated in (2.1.8), there is no loss of weight $(1+|\xi|)^\beta$ in the space $\dot{H}_{\ell,\beta}$, thanks to the smoothing effect of K stated in (2.1.9).

Proof of Lemma 2.3.4. First, by virtue of Theorems 2.2.20 and 2.2.24, we get

$$\|\Psi[h](t)\|_{\ell,\beta-1} \le C\int_0^t \rho_{\sigma_*}(t-s)\rho_{2\sigma}(s)ds[[h]]_{\ell,\beta,2\sigma}, \tag{2.3.23}$$

for some positive constant C and σ_* is $\sigma_{1,1} = n/4+1/2$ for $\Omega = \mathbb{R}^n$ or σ_0 from Theorem 2.2.24 for $\Omega = \mathbb{T}^n$. Note that we have used

$$|||\nu h|||_{\ell,\beta-1,2\sigma} \le C|||h|||_{\ell,\beta,2\sigma},$$

which comes from (2.1.8), showing the loss of the weight β mentioned in the above remark. Compute

$$\begin{aligned}\int_0^t \rho_{\sigma_*}(t-s)\rho_{2\sigma}(s)ds &= \int_0^{t/2} + \int_{t/2}^t \\ &\le \rho_{\sigma_*}(t/2)\int_0^{t/2}\rho_{2\sigma}(s)ds + \rho_{2\sigma}(t/2)\int_{t/2}^t \rho_{\sigma_*}(t-s)ds \\ &\le C\rho_\sigma(t),\end{aligned} \tag{2.3.24}$$

whence follows

$$|||\Psi[h]|||_{\ell,\beta-1,\sigma^*} \le C_0[[h]]_{\ell,\beta,2\sigma}.$$

The loss of weight is to be recovered by the smoothing property (2.1.9). To this end, substitute Duhamel's formula (2.2.68) into (2.3.18) to deduce

$$\Psi[h] = \int_0^t e^{(t-s)A}(\nu h(s))ds + \int_0^t e^{(t-s)A}K\Psi[h](s)ds \equiv I_1 + I_2.$$

Using the explicit formula (2.2.7) of e^{tA}, we get

$$(1+|\xi|)^\beta \|I_1\|_{H^\ell}(\xi,t) \le \int_0^t e^{-(t-s)\nu(\xi)}\nu(\xi)\rho_\sigma(s)ds|||h|||_{\ell,\beta,\sigma}.$$

Compute

$$\begin{aligned}\int_0^t e^{-(t-s)\nu(\xi)}\nu(\xi)\rho_\sigma(s)ds &= \int_0^{t/2} + \int_{t/2}^t \\ &\le e^{-\nu_0 t/4}\int_0^t e^{-(t-s)\nu(\xi)/2}\nu(\xi)ds + \rho_\sigma(t/2)\int_0^t e^{-(t-s)\nu(\xi)}\nu(\xi)ds \\ &\le C_0\rho_{\sigma^*}(t),\end{aligned}$$

where we assume, without loss of generality, $\sigma_0 \le \nu_0/4$ for $\Omega = \mathbb{T}^n$. Then,

$$|||I_1|||_{\ell,\beta,\sigma^*} \le C_0|||h|||_{\ell,\beta,\sigma}.$$

Finally, by a similar computation and (2.1.9), we have

$$|||I_2|||_{\ell,\beta,\sigma^*} \le C|||K\Psi[h]|||_{\ell,\beta,\sigma*)} \le C|||\Psi[h]|||_{\ell,\beta-1,\sigma*}.$$

Combining all of these estimates completes the proof of part (1).

For the proof of (2), take $t > R/2$ and write (2.3.23) as

$$\begin{aligned}\|\Psi[h](t)\|_{\ell,\beta-1} \le C_1 \int_0^{R/2} \rho_{\sigma_*}(t-s)ds[[h]]_{\ell,\beta,0} \\ + C_2 \int_{R/2}^{t} \rho_{\sigma_*}(t-s)ds[[\chi(t > R/2)h(t)]]_{\ell,\beta,0} \quad .\end{aligned} \tag{2.3.25}$$

Take $t > R$ and note

$$\int_0^{R/2} \rho_{\sigma_*}(t-s)ds = \int_{t-R/2}^{t} \rho_{\sigma_*}(s)ds \le \int_{R/2}^{\infty} \rho_{\sigma_*}(s)ds \to 0 \quad (R \to \infty)$$

and

$$\int_{R/2}^{t} \rho_{\sigma_*}(t-s)ds \le \int_0^{\infty} \rho_{\sigma_*}(s)ds < +\infty,$$

which yields

$$|||\chi(t > R)\Psi[h]|||_{\ell,\beta-1,0} \to 0 \qquad (R \to \infty).$$

The loss of weight can be recovered similarly. The detail is omitted.

Finally, we need

Lemma 2.3.6 *Under the assumption* (2.3.2),
(1) *for any* $u, v \in X_{\ell,\beta,\sigma}$,

$$[[\nu^{-1}\Gamma[u,v]]]_{\ell,\beta,2\sigma} \le C_1|||u|||_{\ell,\beta,\sigma}|||v|||_{\ell,\beta,\sigma}$$

holds with some constant $C_1 > 0$ *independent of* u, v, *and*
(2) $\mathbf{P}\Gamma[u,v] = 0$.

Proof. (2) was already stated in Proposition 2.1.2. (1) is proved by 3 steps.
Step 1: Recall $Q_j, j = 1,2$ in (1.2.21) and define

$$\Gamma_j[u,v] = \mathbf{M}^{-1/2}Q_j(\mathbf{M}^{1/2}u, \mathbf{M}^{1/2}v).$$

Obviously, it suffices to prove (i) for these operators separately. H^ℓ is a Banach algebra, so that

$$\|\Gamma_j[u,v](t,\cdot,\xi)\|_{H^\ell} \le |\Gamma_j[u^0,v^0](t,\xi)|$$

where $u^0(t,\xi) = \|u(t,\cdot,\xi)\|_{H^\ell}$.
Step 2: Proposition 2.1.2 (1) applies to Γ_j so that

$$\|\nu^{-1}\Gamma_j[u^0.v^0](t,\cdot)\|_{L^\infty_\beta} \le C\|u^0(t,\cdot)\|_{L^\infty_\beta}\|v^0(t,\cdot)\|_{L^\infty_\beta}.$$

By definition,

$$\|u^0(t,\cdot)\|_{L^\infty_\beta} = \|u(t)\|_{\ell,\beta}.$$

Step 3: By Schwartz's inequality, we have

$$\|\Gamma_j[u,v](t,\cdot,\xi)\|_{L^1(\mathbb{R}^n)} \le |\Gamma_j[u^1,v^1](t,\xi)|,$$

where $u^1(t,\xi) = \|u(t,\cdot,\xi)\|_{L^2(\mathbb{R}^n)}$, and since $L^2 \supset L^\infty_{\beta_0}$ for $\beta_0 > n/2$ and owing again to Proposition 2.1.2,

$$\|\nu\nu^{-1}\Gamma_j[u^1,v^1](t,\cdot)\|_{L^2} \le C\|\nu^{-1}\Gamma_j[u^1,v^1](t,\cdot)\|_{L^\infty_{\beta_0+\gamma}} \le C\|u^1\|_{L^\infty_{\beta_0+\gamma}}\|v^1\|_{L^\infty_{\beta_0+\gamma}}.$$

And, $\|u^1\|_{L^\infty_{\beta_0+\gamma}} \le C\|u\|_{\ell,\beta}$ if $\ell \ge 0$ and $\beta \ge \beta_0+\gamma$. Therefore, we can conclude

$$\|\Gamma_j[u,v]\|_{Z_1} \le C\|u\|_{\ell,\beta}\|v\|_{\ell,\beta}. \tag{2.3.26}$$

Combining these estimates and recalling the definition of the norm $[[\cdot]]_{\ell,\beta,\sigma}$ in (2.3.21) complete the proof of the lemma.

Now we are in the position to prove Theorem 2.3.1. We start with
Proof of Part (1)(a). Note that we can write

$$\Phi[u] = e^{tB}u_0 + \Psi(\nu^{-1}\Gamma[u,u]).$$

We now use Lemma 2.3.4 with $\sigma = \sigma_{q,0}$ in (2.3.19) so that $\sigma^* = \sigma_{q,0}$ holds. Combine this lemma with Theorem 2.2.20 and Lemma 2.3.6, to deduce

$$|||\Phi[u]|||_{\ell,\beta,\sigma_{q,0}} \le C_0U_0 + C_1|||u|||^2_{\ell,\beta,\sigma_{q,0}}, \qquad U_0 \equiv \|u_0\|_{\ell,\beta} + \|u_0\|_{Z_q}$$

and

$$|||\Phi[u]-\Phi[v]|||_{\ell,\beta,\sigma_{q,0}} \le C_1(|||u|||_{\ell,\beta,\sigma_{q,0}} + |||v|||_{\ell,\beta,\sigma_{q,0}})|||u-v|||^2_{\ell,\beta,\sigma_{q,0}},$$

for some constants $C_0, C_1 > 0$ independent of u, v. In the last inequality, we used the fact that Γ is bilinear symmetric, or,

$$\Gamma[u,u] - \Gamma[v,v] = \Gamma[u+v, u-v].$$

Now, choose U_0 so small that

$$D \equiv 1 - 4C_0C_1U_0 > 0$$

can hold, and put

$$a_1 = \frac{1}{2C_1}(1-\sqrt{D}),$$

which is a smaller positive root of the quadratic equation

$$C_1 a^2 - a + C_0 U_0 = 0.$$

With this a_1, set

$$W = \{u \in X_{\ell,\beta,\sigma_{q,0}} \mid |||u|||_{\ell,\beta,\sigma_{q,0}} \le a_1\}.$$

Clearly, W is a complete metric space with the metric induced by the norm $|||\cdot|||_{\ell,\beta,\sigma_{q,0}}$. From above estimates, it follows that for any $u, v \in W$,

$$|||\Phi[u]|||_{\ell,\beta,\sigma_{q,0}} \le C_0 U_0 + C_1 |||u|||^2_{\ell,\beta,\sigma_{q,0}} \le C_0 U_0 + C_1 a_1^2 = a_1,$$

and

$$\begin{aligned} |||\Phi[u] - \Phi[v]|||_{\ell,\beta,\sigma_{q,0}} &\le C_1(|||u|||_{\ell,\beta,\sigma_{q,0}} + |||v|||_{\ell,\beta,\sigma_{q,0}})|||u - v|||^2_{\ell,\beta,\sigma_{q,0}} \\ &\le 2C_1 a_1 |||u - v|||_{\ell,\beta,\sigma_{q,0}}. \end{aligned}$$

The first inequality shows that Φ maps W into itself whereas, since $2C_1 a_1 = 1 - \sqrt{D} < 1$, the second inequality proves that Φ is a contraction mapping. Thus, we are done.

Proof of parts (1)(b),(c) and (2). The same proof gives the remaining parts of Theorem 2.3.1 except for (2.3.5) by using different decay rates in Theorems 2.2.20 and Theorem 2.2.24 conforming to the relevant assumptions.

Proof of (2.3.5). We shall show that the same proof is still valid if the space $X_{\ell,\beta,\sigma}$ is replaced by the space

$$\dot{X}_{\ell,\beta,0} = \Big\{u \in X_{\ell,\beta,0} \mid |||\chi(t > R)u|||_{\ell,\beta,0} \to 0 (R \to \infty)\Big\}. \tag{2.3.27}$$

For this, in view of (2.2.78), it suffices to show that $\Psi(\nu^{-1}\Gamma[\cdot,\cdot])$ maps $\dot{X}_{\ell,\sigma,0}$ into itself, which comes, in turn, from Lemma 2.3.4 (2).

2.3.2 Space Regularity and Decay Rate

An analogue of Theorem 2.2.20 is possible for the solutions obtained in Theorem 2.3.1 for the case $\Omega = \mathbb{R}^n$. Recall

$$\sigma_{q,k} = \frac{n}{4}(\frac{1}{q} - \frac{1}{2}) + \frac{k}{2}$$

from (2.2.55).

Theorem 2.3.7 *Let $\Omega = \mathbb{R}^n$ and assume* (2.3.2)*. Suppose that the initial data u_0 satisfy the condition* (2.3.6) *for some $q \in [1, 2)$, and further that for some $N \in \mathbb{N}$,*

$$u_0 \in \dot{H}_{\ell+N,\beta}. \tag{2.3.28}$$

Then, the solution u in Theorem 2.3.1 satisfies

$$u \in BC([0,\infty); \dot{H}_{\ell+N,\beta}), \tag{2.3.29}$$

and for each $k = 1, 2, \cdots, N$,

$$\|\partial_x^\alpha u(t)\|_{\ell,\beta} \le C_0(1+t)^{-\sigma_{q,k}}, \qquad |\alpha| = k \quad (\forall t \ge 0), \tag{2.3.30}$$

holds where C_0 *is a positive constant independent of* t.

Remark 2.3.8 (1) In contrast to the space derivatives $\partial_x^\alpha u$, the velocity derivatives $\partial_\xi^\alpha u$ have not faster decay than $O(t^{-\sigma_{q,0}})$. Thus, the regularity of solutions diffuses fast in the $x-$space but not in the $\xi-$space. As noted already in Remark 2.2.16, the linearized Boltzmann operator has a smoothing property similar to the space Laplacian $\triangle_x$, but has not a counterpart for ξ.

This is a feature of the Boltzmann equation to be compared with other kinetic equations having a smoothing property in ξ such as the Fokker-Planck-Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \mu \nabla_\xi \cdot (\xi f) - \nu \triangle_\xi f = Q(f),$$

the classical Landau equation which is the same as the Boltzmann equation except

$$Q(f) = \nabla_\xi \cdot \Big\{ \int_{\mathbb{R}^3} \phi(\xi - \xi')[f(v')\nabla_\xi f(v) - f(v)\nabla_\xi f(\xi')]d\xi' \Big\}$$

with $\phi^{ij}(\xi) = \frac{1}{|\xi|}(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2})$, and so on. It is also open whether the same holds for the Vlasov-Poisson (Maxwell)-Boltzmann equation.

(2) A point of Theorem 2.3.7 is that in (2.3.28), any smallness condition is not imposed on higher derivatives of u_0. This is in contrast to the recent result by Guo [29] on the almost exponential decay of u for the case $\mathbb{T}^n$ with the cutoff soft potential: Let $N \ge 4$. For any k, it holds that if $a_k = \|u_0\|_{H^{N+k}_{x,\xi}}$ is sufficiently small, then

$$\text{(AED)} \qquad \|u(t)\|_{H^N_{x,\xi}} \le C a_k (1 + \tfrac{t}{k})^{-k}.$$

Here, it is required that $a_k \to 0$ as $k \to \infty$.

Desvillettes-Villani [24] also established (AED) but in a quite different context: (AED) holds if u is a smooth global solution satisfying

$$u(t) \in BC^0([0,\infty); H^\ell_{x,\xi})$$

for sufficiently large $\ell > k$. The smallness condition on u_0 is not assumed, but the existence of such smooth global solutions is a big open problem at the present moment.

Remark 2.3.9 Theorem 2.3.7 does not cover the case $q = 2$. However, we can recover the decay rate $\sigma_{2,N}$ if we choose a_0 smaller with N. The point here is again that the derivatives of the initial data need not be small. For the proof, see Remark 2.3.10 below.

The proof of Theorem 2.3.7 will be done by induction with respect to N. The case $k = 0$ is just Theorem 2.3.1. Suppose that Theorem 2.3.7 is true up to $k = N - 1$ and that (2.3.28) is fulfilled for $k = N$. For the map Φ in (2.3.15), put

$$\partial_x^\alpha \Phi[u] = \partial_x^\alpha e^{tB} u_0 + \int_0^t \partial_x^\alpha e^{(t-s)B} \Gamma[u(s), u(s)] ds = \Phi_1^\alpha + \Phi_2^\alpha.$$

The first term Φ_1^α for $|\alpha| = N$ is evaluated by combining parts (i) and (ii) of Theorem 2.2.20 as

$$\|\Phi_1^\alpha\|_{\ell,\beta} \leq (b_1 + b_3)(1+t)^{-\sigma_{q,N}} (\|\partial^\alpha u_0\|_{\ell,\beta} + \|u_0\|_{Z_q}) \leq c(1+t)^{-\sigma_{q,N}}. \tag{2.3.31}$$

Here and hereafter, c denotes various positive constants that may depend on N and the norms $\|\partial^\alpha u_0\|_{\ell,\beta} + \|u_0\|_{Z_q}$ for $|\alpha| \leq N$.

Decompose Φ_2^α using (2.2.70) as follows.

$$\Phi_2^\alpha = \Phi_{21}^\alpha + \Phi_{22}^\alpha + \Phi_{23}^\alpha,$$

$$\begin{aligned}
\Phi_{21}^\alpha &= \int_0^{t/2} \partial_x^\alpha D_1(t-s) \Gamma[u(s), u(s)] ds, \\
\Phi_{22}^\alpha &= \int_{t/2}^t \partial_x^\alpha D_1(t-s) \Gamma[u(s), u(s)] ds, \\
\Phi_{23}^\alpha &= \int_0^t \partial_x^\alpha D_2(t-s) \Gamma[u(s), u(s)] ds.
\end{aligned}$$

Use Theorem 2.2.20 (i) with $|\alpha| = N$ and $\alpha' = 0$ to deduce

$$\begin{aligned}
\|\Phi_{21}^\alpha\|_{\ell,\beta} &\leq b_2 \int_0^{t/2} (1+t-s)^{-\sigma_{1,N+1}} \|\Gamma[u(s), u(s)]\|_{Z_1} ds \qquad (2.3.32) \\
&\leq c \int_0^{t/2} (1+t-s)^{-\sigma_{1,N+1}} \|u(s)\|_{\ell,\beta}^2 ds \qquad \text{(by (2.3.26))} \\
&\leq c \int_0^{t/2} (1+t-s)^{-\sigma_{1,N+1}} (1+s)^{-2\sigma_{q,0}} ds |||u|||_{\ell,\beta,\sigma_{q,0}}^2 \qquad \text{(Theorem 2.3.1)} \\
&\leq c(1+t/2)^{-\sigma_{1,N+1} + \max(0, 1-2\sigma_{q,0})}.
\end{aligned}$$

Note that $\sigma_{1,N+1} - \max(0, 1-2\sigma_{q,0}) \geq \sigma_{q,N}$.

Use Theorem 2.2.20 (i) again, but this time with $\alpha = \alpha'$, $|\alpha| = N$, to deduce

$$\begin{aligned}
\|\Phi_{22}^\alpha\|_{\ell,\beta} &\leq b_2 \sum_{\alpha' \leq \alpha} \int_{t/2}^t (1+t-s)^{-\sigma_{1,1}} \|\Gamma[\partial_x^{\alpha'} u(s), \partial_x^{\alpha-\alpha'} u(s)]\|_{Z_1} ds \qquad \text{(Leipnitz)} \\
&\leq b_2 \sum_{\alpha' \leq \alpha} \int_{t/2}^t (1+t-s)^{-\sigma_{1,1}} \|\partial^{\alpha'} u(s)\|_{\ell,\beta} \|\partial^{\alpha-\alpha'} u(s)\|_{\ell,\beta} ds \qquad \text{(by (2.3.26))} \\
&= b_2(J_1 + J_2),
\end{aligned}$$

where the constant b_2 is independent of N because $m = |\alpha - \alpha'| = 0$ (see Theorem 2.2.20), and

$$J_1 = \sum_{\alpha'=0,\alpha'=\alpha}, \qquad J_2 = \sum_{0\neq\alpha'<\alpha}. \tag{2.3.33}$$

By virtue of Theorem 2.3.1, we get

$$\begin{aligned}
J_1 &= 2\int_{t/2}^{t}(1+t-s)^{-\sigma_{1,1}}\|\partial^\alpha u(s)\|_{\ell,\beta}\|u(s)\|_{\ell,\beta}ds \\
&\leq 2\int_{t/2}^{t}(1+t-s)^{-\sigma_{1,1}}(1+s)^{-\sigma-\sigma_{q,0}}ds|||\partial_x^\alpha u||_{\ell,\beta,\sigma}|||u|||_{\ell,\beta,\sigma_{q,0}} \\
&\leq c(\sigma)(1+t)^{-\sigma-\sigma_{q,0}}|||\partial_x^\alpha u||_{\ell,\beta,\sigma}|||u|||_{\ell,\beta,\sigma_{q,0}},
\end{aligned}$$

for any $\sigma \geq 0$. Here and herafter, $c(\sigma)$ stands for various constants which depend only on σ. On the other hand, by the induction hypothesis for $k \leq N-1$,

$$\begin{aligned}
J_2 &\leq c\sum_{m=1}^{N-1}\int_{t/2}^{t}(1+t-s)^{-\sigma_{1,1}}(1+s)^{-\sigma_{q,m}-\sigma_{q,N-m}}ds \\
&\leq c(1+t/2)^{-\sigma_{q,N}-\sigma_{q,0}}\int_{t/2}^{t}(1+t-s)^{-\sigma_{1,1}}ds \\
&\leq c(1+t)^{-\sigma_{q,N}}.
\end{aligned}$$

In the last line, we used $\sigma_{1,1} > 1$.

Furthermore, by Theorem 2.2.20 (ii) for $|\alpha| = N$,

$$\begin{aligned}
\|\Phi_{23}^\alpha\|_{\ell,\beta-1} &\leq b_3\sum_{\alpha'\leq\alpha}\int_0^t e^{-\sigma_0(t-s)}\|\Gamma[\partial_x^{\alpha'}u(s), \partial_x^{\alpha-\alpha'}u(s)]\|_{\ell,\beta-1}ds \qquad (2.3.34)\\
&\leq b_3\sum_{\alpha'\leq\alpha}\int_0^t e^{-\sigma_0(t-s)}\|\partial^{\alpha'}u(s)\|_{\ell,\beta}\|\partial^{\alpha-\alpha'}u(s)\|_{\ell,\beta}ds \qquad \text{(Leipnitz)}\\
&= b_3(J_3+J_4),
\end{aligned}$$

where b_3 is also independent of N (see Theorem 2.2.20 (ii)), and

$$J_3 = \sum_{\alpha'=0,\alpha'=\alpha}, \qquad J_4 = \sum_{0\neq\alpha'<\alpha}. \tag{2.3.35}$$

By virtue of Theorem 2.3.1,

$$\begin{aligned}
J_3 &\leq 2\int_0^t e^{-\sigma_0(t-s)}(1+s)^{-\sigma-\sigma_{q,0}}ds|||\partial_x^\alpha u||_{\ell,\beta,\sigma}|||u|||_{\ell,\beta,\sigma_{q,0}} \qquad (2.3.36)\\
&\leq c(\sigma)(1+t)^{-\sigma-\sigma_{q,0}}|||\partial_x^\alpha u||_{\ell,\beta,\sigma}|||u|||_{\ell,\beta,\sigma_{q,0}} \equiv J_3^*,
\end{aligned}$$

while by induction hypothesis,

$$J_4 \leq c\sum_{m=1}^{N-1}\int_0^t e^{-\sigma_0(t-s)}(1+s)^{-\sigma_{q,N-m}-\sigma_{q,m}}ds \leq c(1+t)^{-\sigma_{q,N}} \equiv J_4^*. \tag{2.3.37}$$

Finally, the weight loss in (2.3.34) can be recovered in the same way as in the previous subsection so that

$$\|\Phi_{23}^\alpha\|_{\ell,\beta} \le b_3(J_3^* + J_4^*) \tag{2.3.38}$$

holds with another constant b_3 independent of N.

Now, combining these estimates yields

$$\|\Phi^\alpha(t)\|_{\ell,\beta} \le c(1+t)^{-\sigma_{q,N}} + c(\sigma)(1+t)^{-\sigma-\sigma_{q,0}}|||\partial_x^\alpha u||_{\ell,\beta,\sigma}|||u|||_{\ell,\beta,\sigma_{q,0}}, \tag{2.3.39}$$

for $|\alpha| = N$. For simplicity of notation, fix ℓ, β and and write

$$[[u]]_\sigma = |||u|||_{\ell,\beta,\sigma}.$$

Since u is a fixed point $u = \Phi[u]$ and since $|||u|||_{\ell,\beta,\sigma_{q,0}} \le a_1 U_0 \le a_0 a_1$, (2.3.39) implies

$$[[\partial_x^\alpha u]]_{\sigma+\sigma_{q,0}} \le c + c(\sigma) a_0 a_1 [[\partial_x^\alpha u]]_\sigma \tag{2.3.40}$$

for $\sigma \ge 0$ such that $\sigma + \sigma_{q,0} \le \sigma_{q,N}$. First, we put $\sigma = 0$ and choose the constant a_0 smaller if necessary so that $c(0)a_0a_1 < 1$ holds. Then, we get

$$[[\partial_x^\alpha u]]_0 \le \frac{c}{1 - c(0)a_0a_1}.$$

On the other hand, since $q \in [1, 2)$ is assumed, we know $\sigma_{q,0} > 0$ and therefore, we can solve the recurrence inequality (2.3.40) and after a finite number of steps (actually, $[\sigma_{q,N}/\sigma_{q,0}] + 1$ steps) we get

$$[[\partial_x^\alpha u]]_{\sigma_{q,N}} \le c_1 + c_2[[\partial_x^\alpha u]]_0.$$

Combining these two estimates confirms the induction hypothesis for $k = N$, and the proof of Theorem 2.3.7 is now complete.

Remark 2.3.10 The proof of the statement in Remark 2.3.9 follows directly from (2.3.40). If $q = 2$, then $\sigma_{q,0} = 0$. Take $\sigma = \sigma_{2,N}$ and assume that a_0 is so small that $c(\sigma_{2,N})a_0a_1 < 1$ holds. Then, (2.3.40) gives

$$[[\partial_x^\alpha u]]_{\sigma_{2,N}} \le \frac{c}{1 - c(\sigma_{2,N})a_0a_1}.$$

=

2.4 Time-Periodic and Stationary Solutions

2.4.1 Existence and Stability

The aim of this subsection is to study the Boltzmann equation (2.1.3) with an inhomogeneous term,

$$\frac{\partial u}{\partial t} = Bu + \Gamma[u, u] + S, \quad (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^n, \tag{2.4.1}$$

for $\Omega = \mathbb{R}^3$ or $\mathbb{T}^n$. Here, $S = S(t, x, \xi)$ is a given function. Our main goal is to show the dual applicability to (2.4.1) of the decay estimates derived in the previous section: It will be shown that if S is a time-periodic (resp. time-independent) function, (2.4.1) possesses a unique time-periodic solution (resp. stationary solution) and that the time-periodic (resp. stationary solution) is asymptotically stable. Both will be done with a combination of the decay estimates and the contraction mapping principle, of course in different contexts.

The inhomogeneous term S stands for the distributional density of an external source of gas particles. Thus, in the case where S is time-periodic, (2.4.1) is the most basic model problem in the study of the generation and propagation of sound waves in a gas with an oscillating source. The problems of sound waves have been studied deeply in the fluid mechanics [111], but little is known in the kinetic theory. The result given here is from [43].

Assume (2.3.2) again, that is,

$$n \geq 3, \qquad \ell > \frac{n}{2} + 1, \qquad \beta > \frac{n}{2} + 1. \tag{2.4.2}$$

Recall the spaces $\dot{H}_{\ell,\beta}$ and Z_q as in §2.2.5, and define, for functions $u = u(t, x, \xi)$,

$$Y^k_{\ell,\beta} = BC^k(\mathbb{R}; \dot{H}_{\ell,\beta}), \quad k = 0, 1, 2, \cdots, \tag{2.4.3}$$

and the norm for $k = 0$,

$$|||u|||_{\ell,\beta} = \sup_{t \in \mathbb{R}} \|u(t, \cdot, \cdot)\|_{\ell,\beta}. \tag{2.4.4}$$

Further, set

$$Z^q = BC^0(\mathbb{R}; Z_q), \qquad \|u\|_{Z^q} = \sup_{t \in \mathbb{R}} \|u(t, \cdot, \cdot)\|_{Z_q}. \tag{2.4.5}$$

The norm (2.4.4) must not be confused with the norm (2.3.17). Our first result is the existence theorem of periodic solutions.

Theorem 2.4.1 *Assume* (2.4.2). *Then, there are two positive constants* a_0 *and* a_1 *with which the following holds. In the below,* $S = S(t, x, \xi)$ *is a periodic function in* t *and its period is denoted by* T_0.

(1) The case $\Omega = \mathbb{R}^n$.

(a) *The case* $n = 3, 4$. *Suppose*

$$S \in Y^0_{\ell,\beta}, \quad (1 + |x|)S \in Z^1, \quad |||S|||_{\ell,\beta} + ||(1 + |x|)S||_{Z^1} \leq a_0, \tag{2.4.6}$$

$$\int_{\mathbb{R}^n} (\mathbf{P}S)(t, x, \xi)dx = 0, \qquad a.e. \ (t, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{2.4.7}$$

Then, (2.4.1) *admits a solution* $u^{per} = u^{per}(t, x, \xi)$ *which is periodic in* t *with the same period* T_0 *and satisfies*

$$u^{per} \in Y^0_{\ell,\beta} \cap Y^1_{\ell-1,\beta-1}, \tag{2.4.8}$$

$$|||u^{per}|||_{\ell,\beta} \leq a_1 \Big(|||S|||_{\ell,\beta} + ||(1 + |x|)S||_{Z^1} \Big). \tag{2.4.9}$$

Moreover, this is a unique T_0*-periodic solution in the function class defined by* (2.4.8).

(b) *The case $n \geq 5$. Suppose*

$$S \in Y^0_{\ell,\beta} \cap Z^1, \qquad |||S||_{\ell,\beta} + \|S\|_{Z^1} \leq a_0. \tag{2.4.10}$$

Then, (2.4.1) *admits a solution $u^{per} = u^{per}(t,x,\xi)$ which is periodic in t with the same period T_0 and satisfies*

$$u^{per} \in Y^0_{\ell,\beta} \cap Y^1_{\ell-1,\beta-1}, \tag{2.4.11}$$
$$|||u^{per}|||_{\ell,\beta} \leq a_1(|||S|||_{\ell,\beta} + \|S\|_{Z^1}). \tag{2.4.12}$$

Moreover, this is a unique T_0-periodic solution in the function class defined by (2.4.11).

(2) The case $\Omega = \mathbb{T}^n$. *Suppose*

$$S \in Y^0_{\ell,\beta}, \qquad \|S\|_{Y^0_{\ell,\beta}} \leq a_0, \tag{2.4.13}$$
$$\int_{\mathbb{T}^n} (\mathbf{P}S)(t,x,\xi)dx = 0, \qquad a.e. \ \ (t,\xi) \in \mathbb{R} \times \mathbb{T}^n. \tag{2.4.14}$$

Then, (2.4.1) *admits a solution $u^{per} = u^{per}(t,x,\xi)$ which is periodic in t with the same period T_0 and satisfies*

$$u^{per} \in Y^0_{\ell,\beta} \cap Y^1_{\ell-1,\beta-1}, \tag{2.4.15}$$
$$|||u^{per}|||_{\ell,\beta} \leq a_1|||S|||_{\ell,\beta}. \tag{2.4.16}$$

Moreover, this is a unique T_0-periodic solution in the function class defined by (2.4.15).

Remark 2.4.2 As will be seen in the proof below, the extra condition (2.4.7) is necessary for the case $n = 3, 4$ because of a shortage of the decay rate $\sigma_{1,0}$ in Theorem 2.2.20. This condition is not necessary for $n \geq 5$ because $\sigma_{1,0} > 1$. In contrast, the condition (2.4.14) cannot be dropped for any space dimension n.

In order to study the stability of the periodic solutions u^{per}, we shall consider the Cauchy problem to (2.4.1) without fixing the initial time t_0, that is, we shall consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = Bu + \Gamma[u,u] + S(t), & t > t_0, \\ u(t_0) = u_0, \end{cases} \tag{2.4.17}$$

for each $t_0 \in \mathbb{R}$. We shall solve this Cauchy problem in the function space

$$V_{\ell,\beta,t_0} = BC^0([t_0,\infty); \dot{H}_{\ell,\beta}) \cap BC^1([t_0,\infty); \dot{H}_{\ell-1,\beta-1}). \tag{2.4.18}$$

Note that the periodic solutions u^{per} in Theorem 2.4.1 is in V_{ℓ,β,t_0} for any $t_0 \in \mathbb{R}$, if restricted to the time interval $[t_0,\infty)$.

Theorem 2.4.3 *Under the same condition of Theorem 2.4.1, let $u^{per} = u^{per}(t)$ be the time-periodic solution obtained there. Then, for each $t_0 \in \mathbb{R}$, there are positive constant δ_0, δ_1 such that the following holds.*

(1) *The case $\Omega = \mathbb{R}^n$. For any initial data u_0 satisfying*

$$u_0 \in \dot{H}_{\ell,\beta}, \quad \|u_0 - u^{per}(t_0)\|_{\ell,\beta} \le \delta_0, \tag{2.4.19}$$

a global solution u exists to the Cauchy problem (2.4.17) *which is unique in the function class*

$$u = u(t) \in V_{\ell,\beta,t_0}, \tag{2.4.20}$$

and satisfies

$$\|u(t) - u^{per}(t)\|_{\ell,\beta} \le \delta_1 \|u_0 - u^{per}(t_0)\|_{\ell,\beta} \quad (t \ge t_0), \tag{2.4.21}$$

$$\|u(t) - u^{per}(t)\|_{\ell} \to 0 \qquad (t \to \infty). \tag{2.4.22}$$

If the assumption (2.4.19) *is strengthened to*

$$u_0 \in \dot{H}_{\ell,\beta} \cap Z_q, \quad U_0 \equiv \|u_0 - u^{per}(t_0)\|_{\ell,\beta} + \|u_0 - u^{per}(t_0)\|_{Z_q} \le \delta_0, \tag{2.4.23}$$

for some $q \in [1, 2)$, u enjoys an algebraic decay,

$$\|u(t) - u^{per}(t)\|_{\ell,\beta} \le \delta_1 (1 + t - t_0)^{-\sigma_{q,0}} U_0, \quad (t \ge t_0), \tag{2.4.24}$$

where $\sigma_{q,0}$ is the same as before;

$$\sigma_{q,0} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right).$$

(2) *The case $\Omega = \mathbb{T}^n$. For any initial data u_0 satisfying*

$$u_0 \in \dot{H}_{\ell,\beta}, \quad \|u_0 - u^{per}(t_0)\|_{\ell,\beta} \le \delta_0, \tag{2.4.25}$$

a global solution u exists to the Cauchy problem (2.3.1) *which is unique in the function class*

$$u \in V_{\ell,\beta,t_0}, \tag{2.4.26}$$

and satisfies the estimate

$$|||u - u^{per}|||_{\ell,\beta} \le \delta_1 \|u_0 - u^{per}(t_0)\|_{\ell,\beta}. \tag{2.4.27}$$

If, further, u_0 satisfies an additional condition

$$\int_{\mathbb{T}^n} (\mathbf{P}u_0)(x,\xi)dx = 0 \quad a.e. \quad \xi \in \mathbb{R}^n, \tag{2.4.28}$$

u enjoys the exponential decay

$$\|u(t) - u^{per}(t)\|_{\ell,\beta} \le \delta_1 e^{-\sigma_0 t} \|u_0 - u^{per}(t_0)\|_{\ell,\beta}. \tag{2.4.29}$$

Remark 2.4.4 (2.4.22), (2.4.24), and (2.4.29) imply the asymptotic stability of the periodic solution u^{per} but not the orbital stability because (2.4.17) is not an autonomous system. In the case $\mathbb{T}^n$, if the condition (2.4.28) is dropped, (2.4.29) does not hold, and u^{per} is stable but not asymptotically stable, a contrast to the case $\mathbb{R}^n$.

Remark 2.4.5 Any time-independent function can be regarded as a time-periodic function with arbitray period. Thus, even if S is t-independent, the above two theorems apply, implying the existence and stability of stationary solutions.

Remark 2.4.6 The above two theorems are valid also for the case $\Omega = \mathbb{T}^n$ with the space dimension $n = 1, 2$, but it is not true for $\Omega = \mathbb{R}^n$. See Remark 2.3.3.

As already stated, the decay estimates obtained in §2 can be successfully used in the proof of these two theorems, but in different usage.

2.4.2 Proof of Theorem 2.4.1

A method for establishing the existence of time-periodic solutions to the evolution equation is to solve the boundary value problem under the the periodic boundary condition in t. This strategy has been adopted by many authors for many nonlinear problem. However, it does not seem to work for our problem. Instead, we use again a combination of the contraction mapping principle and time decay estimates obtained in §2.3. Our method is applicable to a wide class of semi-linear evolution equations.

Our setting start from the Cauchy problem (2.4.17);

$$\begin{cases} \dfrac{du}{dt} = Bu + \Gamma(u,u) + S(t) \qquad (t > t_0), \\ \qquad\qquad u(t_0) = u_0, \end{cases} \tag{2.4.30}$$

for $t_0 \in \mathbb{R}$. Recall the integral formula (2.3.14). The corresponding integral equation for (2.4.30) is,

$$u(t) = e^{(t-t_0)B}u_0 + \int_{t_0}^{t} e^{(t-\tau)B}\Big\{\Gamma[u(\tau), u(\tau)] + S(\tau)\Big\}d\tau, \qquad t \geq t_0. \tag{2.4.31}$$

Now, suppose that there exists a periodic solution $u^{per} = u^{per}(t), t \in \mathbb{R}$, with the period T_0. Then, it solves (2.4.30), and hence (2.4.31), with the particular initial data $u_0 = u^{per}(t_0)$ for each $t_0 \in \mathbb{R}$. Choose $t_0 = -kT_0$ for $k \in \mathbb{N}$. Clearly, $u_0 = u^{per}(-kT_0) = u^{per}(0)$ and (2.4.31) is written as

$$u^{per}(t) = e^{(t+kT_0)B}u^{per}(0) + \int_{-kT_0}^{t} e^{(t-\tau)B}\Big\{\Gamma[u^{per}(\tau), u^{per}(\tau)] + S(\tau)\Big\}d\tau. \tag{2.4.32}$$

Let $k \to \infty$. Then, the first term on the right hand side tends to 0 owing to Theorem 2.2.20. Therefore, we get

$$u^{per}(t) = \int_{-\infty}^{t} e^{(t-\tau)B}\Big\{\Gamma[u^{per}(\tau), u^{per}(\tau)] + S(\tau)\Big\}d\tau, \tag{2.4.33}$$

provided that the last integral converges. Admit this for the time being and define the nonlinear map,

$$N[u](t) = \int_{-\infty}^{t} e^{(t-\tau)B}\Big\{\Gamma[u(\tau), u(\tau)] + S(\tau)\Big\}d\tau. \tag{2.4.34}$$

Then, (2.4.33) indicates that u^{per} is a fixed point of N.

Conversely, suppose that N have a fixed point. It may not be time-periodic. But, let us suppose that the fixed point of N is unique. Then, we can claim that if S is periodic in t, so is this fixed point with the same period as S. For the proof, denote this unique fixed point by $\bar{u} = \bar{u}(t)$ and the period of S by T_0. Put $v(t) = \bar{u}(t + T_0)$. We have,

$$\begin{aligned} v(t) = N[\bar{u}](t + T_0) &= \int_{-\infty}^{t+T_0} e^{(t+T_0-\tau)B}\Big\{\Gamma[\bar{u}(\tau), \bar{u}(\tau)] + S(\tau)\Big\}d\tau \\ &= \int_{-\infty}^{t} e^{(t-\tau)B}\Big\{\Gamma[\bar{u}(\tau + T_0), \bar{u}(\tau + T_0)] + S(\tau + T_0)\Big\}d\tau \\ &= \int_{-\infty}^{t} e^{(t-\tau)B}\Big\{\Gamma[v(\tau), v(\tau)] + S(\tau)\Big\}d\tau \quad \text{(by periodicity of } S) \\ &= N[v](t). \end{aligned}$$

Thus, v is another fixed point but then, the uniqueness assumption says $v(t) = \bar{u}(t)$ for all $t \in \mathbb{R}$, proving the periodicity of $\bar{u}$ with the period T_0. It is evident that if this unique fixed point is differentiable with respect to t, it is a desired periodic solution to (2.4.17).

In order to substantiate this formal argument, first, we shall show that an integral similar to (2.3.18),

$$\Psi[h] = \int_{-\infty}^{t} e^{(t-s)B}(\nu h(s))ds, \tag{2.4.35}$$

converges. Define also the function

$$\phi(\sigma) = \begin{cases} \min(2\sigma, \frac{n}{4} + \frac{1}{2}), & \Omega = \mathbb{R}^n, \\ \min(2\sigma, \sigma_0), & \Omega = \mathbb{T}^n, \end{cases} \tag{2.4.36}$$

where σ_0 is that in (2.2.52).

Lemma 2.4.7 *Suppose* (2.4.2). *There is a positive constant* C_0 *and the following holds.*

(1) The case $\Omega = \mathbb{R}^n$. *Suppose that* $h \in Y^0_{\ell,\beta} \cap Z^1$ *and that one of the following three conditions is fulfilled.*

- **(a)** $n \geq 5$,
- **(b)** $\mathbf{P}(\nu h) = 0$, *or*
- **(c)** $(1 + |x|)\nu h \in Z^1$, $\quad \int_{\mathbb{R}^n} \mathbf{P}_0(\nu h) = 0.$

Then, the integral (2.4.35) *converges in the norm of* $Y^0_{\ell,\beta}$ *and the estimate*

$$|||\Psi[h]|||_{\ell,\beta} \le C_0(|||h|||_{\ell,\beta} + \|\nu h\|_{Z^1}) \tag{2.4.37}$$

holds for the cases (a) and (b), and

$$|||\Psi[h]|||_{\ell,\beta} \le C_0(|||h|||_{\ell,\beta} + \|(1+|x|)\nu h\|_{Z^1}). \tag{2.4.38}$$

for the case (c).

(2) The case $\Omega = \mathbb{T}^n$. *Suppose that* $h \in Y^0_{\ell,\beta}$ *and*

$$\int_{\mathbb{T}^n} \mathbf{P}(\nu h) = 0.$$

Then, the integral (2.4.35) *converges in the norm of* $Y^0_{\ell,\beta}$ *and the estimate*

$$|||\Psi[h]|||_{\ell,\beta} \le C_0|||h|||_{\ell,\beta}. \tag{2.4.39}$$

holds for some constant $C_0 > 0$.

Remark 2.4.8 The conditions (1)(b), (c) are introduced especially for the case $n = 3, 4$.

Proof of Lemma 2.4.7. First, set

$$\rho(t) = \begin{cases} (1+t)^{-\sigma}, & \Omega = \mathbb{R}^n, \\ e^{-\sigma_0 t}, & \Omega = \mathbb{T}^n, \end{cases}$$

where

$$\sigma = \frac{n}{4} \quad \text{for the case (a)}, \qquad = \frac{n}{4} + \frac{1}{2} \quad \text{for the cases (b), (c)},$$

and σ_0 is as in Theorem 2.2.20. This choice is made so that $\rho(t) \in L^1(0,\infty)$ holds for all cases in the lemma. Below, we will give the proof of the lemma only for the case (1)(a), since the proof of the other cases is the same.

First, define

$$\Psi_c[h] = \int_c^t e^{(t-s)B}(\nu h(s))ds, \tag{2.4.40}$$

for any $c \le t$. Let $c < c' \le t$. By virtue of Theorem 2.2.20,

$$\begin{aligned} \|\Psi_c[h] - \Psi_{c'}[h]\|_{\ell,\beta-1}(t) &\le C \int_c^{c'} \rho(t-s)ds(|||\nu h|||_{\ell,\beta-1} + ||\nu h||_{Z^1}) \\ &\le C \int_{t-c'}^{\infty} \rho(s)ds(|||\nu h|||_{\ell,\beta-1} + ||\nu h||_{Z^1}), \end{aligned}$$

which indicates that $\{\Psi_c[h]\}$ is Cauchy in $\dot{H}_{\ell,\beta-1}$ when $c \to -\infty$. Thus, the limit $\Psi[h]$ exists, and by a similar computation, the estimate

$$|||\Psi[h]|||_{\ell,\beta-1} \le C_0(|||h|||_{\ell,\beta} + \|\nu h\|_{Z^1})$$

follows. The weight loss $\beta - 1$ can be recovered again by appealing to Duhamel's formula and proceeding as in Lemma 2.3.18, to deduce (2.4.37). The detail is omitted.

Proof of Theorem 2.4.1. It is easy to see that Lemma 2.4.7 applies to $\nu h = \Gamma[u, v]$, which, together with the estimate (2.3.6), gives,

$$|||\Psi[\nu^{-1}\Gamma[u,v]]|||_{\ell,\beta} \leq C(|||\nu^{-1}\Gamma[u,v]|||_{\ell,\beta} + \|\Gamma[u,v]\|||_{Z^1}) \leq C_1||u|||_{\ell,\beta}||v|||_{\ell,\beta},$$

with some constants $C, C_1 > 0$. Similarly, under the assumption on S imposed in Theorem 2.4.1, Lemma 2.4.7 applies to $\nu h = S$, with the estimate,

$$|||\Psi[\nu^{-1}S]|||_{\ell,\beta} \leq C_0 S_0,$$

where $C_0 > 0$ is a constant and

$$S_0 = \begin{cases} |||S|||_{\ell,\beta-\gamma}, & \text{for the cases (1)(a)(b) and (2),} \\ |||S|||_{\ell,\beta-\gamma} + ||(1+|x|)S||_{Z^1}, & \text{for the cases (1)(c),} \end{cases} \tag{2.4.41}$$

where γ is as in (2.1.8).

Combining these two estimates yields

$$|||N[u]|||_{\ell,\beta} \leq C_0 S_0 + C_1|||u|||^2_{\ell,\beta},$$

and by a similar computation and the bilinear symmetry of Γ,

$$|||N[u] - N[v]|||_{\ell,\beta} \leq C_1|||u+v|||_{\ell,\beta}|||u-v|||_{\ell,\beta}.$$

Choose S_0 so small that

$$D \equiv 1 - 4C_0C_1S_0 > 0,$$

and set

$$a_1 = \frac{1}{2C_1}(1 - \sqrt{D}),$$

which is the smaller positive root of the quadratic equation

$$C_1 a^2 - a + C_0 S_0 = 0.$$

Set

$$W = \{u \in Y^0_{\ell,\beta} \mid |||u|||_{\ell,\beta} \leq a_1\},$$

which is a complete metric space with the distance induced by the norm $|||\cdot|||_{\ell,\beta}$. Now, for any $u, v \in W$, the above estimates yield,

$$\begin{aligned} |||N[u]|||_{\ell,\beta} &\leq C_0S_0 + C_1a_1^2 = a_1, \\ |||N[u] - N[v]|||_{\ell,\beta} &\leq \mu|||u-v|||_{\ell,\beta}, \qquad \mu = 2C_1a_1 = 1 - \sqrt{D} < 1, \end{aligned}$$

showing that N is a contraction on W. Thus we are done.

2.4.3 Proof of Theorem 2.4.3

This is almost the same as in §3.1. We shall put

$$v = u(t) - u^{per}(t), \tag{2.4.42}$$

and rewrite (2.4.17) as

$$\begin{cases} \dfrac{dv}{dt} = Bv + L^{per}(t)v + \Gamma[v, v] \qquad (t > t_0), \\ v(t_0) = v_0 \equiv u_0 - u^{per}(t_0). \end{cases} \tag{2.4.43}$$

Here, the inhomogeneous term does not appear, at the cost of the extra linear term

$$L^{per}(t)v = 2\Gamma[u^{per}(t), v].$$

Without loss of generality, it suffices to prove Theorem 2.4.3 for $t_0 = 0$. Then, the integral equation to solve is,

$$v(t) = e^{tB}v_0 + \int_0^t e^{(t-\tau)B}\Big\{L^{per}(\tau)v(\tau) + \Gamma[u(\tau), u(\tau)]\Big\}d\tau. \tag{2.4.44}$$

Recall the space and norm in (2.3.17).

$$\begin{aligned} X_{\ell,\beta,\sigma} &= \Big\{u \in BC^0([0,\infty); \dot{H}_{\ell,\beta}) \mid |||u|||_{\ell,\beta,\sigma} < +\infty\Big\}, \\ |||u|||_{\ell,\beta,\sigma} &= \sup_{t\geq 0}\Big(\rho_\sigma(t)^{-1}\|u(t)\|_{\ell,\beta}\Big). \end{aligned} \tag{2.4.45}$$

As in §3.1, we will show that N is a contraction map. Write

$$N[u] = e^{tB}v_0 + \Psi[\nu^{-1}L^{per}(t)u] + \Psi[\nu^{-1}\Gamma[u, u]].$$

The estimates of the first and last terms in N are derived in §3.1. The second term is estimated as follows.

$$\begin{aligned} |||\Psi[\nu^{-1}L^{per}(t)u]|||_{\ell,\beta,\sigma} &\leq C\int_0^t \rho_{\sigma_*}(t-s)\rho_\sigma(s)ds|||u^{per}|||_{\ell,\beta,0}|||u|||_{\ell,\beta,\sigma} \\ &\leq C_2S_0\rho_{\phi(\sigma)}(t)|||u|||_{\ell,\beta,\sigma}, \end{aligned}$$

where $\sigma_* = n/4 + 1/2$, $\phi(\sigma) = \min(\sigma_*, \sigma)$, and S_0 is as in (2.4.41).

Take $\sigma = n/4$. Then, $\phi(\sigma) = \sigma$. Now, we get,

$$|||N[u]|||_{\ell,\beta,\sigma} \leq C_0||v_0||_{\ell,\beta} + C_2S_0|||u|||_{\ell,\beta,\sigma} + C_1|||u|||^2_{\ell,\beta,\sigma},$$

and by a similar computation and the bilinear symmetry of Γ,

$$|||N[u] - N[v]|||_{\ell,\beta,\sigma} \leq C_2S_0|||u - v|||_{\ell,\beta} + C_1|||u + v|||_{\ell,\beta}|||u - v|||_{\ell,\beta}.$$

Choose S_0, $||v_0||_{\ell,\beta}$ so small that

$$1 - C_2 S_0 > 0, \qquad D \equiv (1 - C_2 S_0)^2 - 4C_0 C_1 ||v_0||_{\ell,\beta} > 0,$$

can hold. Then, set

$$a_1 = \frac{1}{2C_1}(1 - C_2 S_0 - \sqrt{D}),$$

which is the smaller positive root of the quadratic equation

$$C_1 a^2 - (1 - C_2 S_0)a + C_0 ||v_0||_{\ell,\beta} = 0.$$

Set

$$W = \{u \in Y^0_{\ell,\beta} \mid |||u|||_{\ell,\beta,\sigma} \leq a_1\},$$

which is a complete metric space with the distance induced by the norm $|||\cdot|||_{\ell,\beta}$. Now, for any $u, v \in W$, the above estimates yield,

$$|||N[u]|||_{\ell,\beta,\sigma} \leq C_0 ||v_0||_{\ell,\beta} + C_2 S_0 a_1 + C_1 a_1^2 = a_1,$$
$$|||N[u] - N[v]|||_{\ell,\beta,\sigma} \leq \mu |||u - v|||_{\ell,\beta,\sigma}, \qquad \mu = 2C_1 a_1 = 1 - C_2 S_0 - \sqrt{D} < 1,$$

showing that N is a contraction on W. Thus we are done.

Chapter 3

Solutions in L^2 Framework

This chapter concerns the solutions to the Boltzmann equation in the Sobolev space $\mathbf{H}^{l_0,l_1,l_2}_{t,x,\xi}$. One of the main reasons to study the solutions in this space is to apply the well-established theories on the conservation laws to the existence and stability analysis of the fluid dynamical wave patterns and solution profiles for the Boltzmann equation. In fact, this approach is now shown to be robust and recent progress has been made on various problems.

In what follows, we will start with the local existence of solutions in the above Sobolev space for the Boltzmann equation without forcing. Then the basic ideas of the Hilbert and Chapman-Enskog expansions are reviewed. In the same spirit of these two classical expansions, a new decomposition of the solution into the macroscopic and microscopic components by Maxwellian is introduced and the Boltzmann equation is reformulated into a system of conservation laws in the framework of the Navier-Stokes equations with a source term determined by the microscopic component, coupled with a time evolutional equation for the microscopic component. By rewriting the Boltzmann equation into this form, the time evolution of both the macroscopic and microscopic components are clearly presented. Moreover, there is no truncation in this reformulation which is unlike the Hilbert and Chapman-Enskog expansions. Hence, the analytic techniques from the study of the fluid dynamical systems can be fully used together with the dissipation on the microscopic component through the celebrated H-theorem. The stability of wave patterns for the Boltzmann equation by using energy method was initiated by the study of the shock profile in [53] through a rigorous analysis of the Chapman-Enskog expansion where the macro-micro decomposition is defined around the local Maxwellian given by the Chapman-Enskog expansion. The reformulation of the Boltzmann equation using the macro-micro decomposition with respect to the local Maxwellian defined by the solution itself was introduced in [35] which will be explained in §3.2.3. With the local existence given in [46], now the discussion on the existence and stability in various situations can be found in the references [54, 51, 46, 84], etc.

In the sections following the decomposition, we will present the recent results on the stability of the global Maxwellian and various wave patterns. Even though the above results are closely associated with the classical fluid dynamical systems, that is, the Euler equations and Navier-Stokes equations, there are some solution behaviors described by the Boltzmann equation which are not governed by the classical fluid dynamical systems. Therefore, in the last section, we will discuss some preliminary ideas on the non-classical fluid dynamical systems derived from the Boltzmann equation in some physical settings. The study of this kind of phenomena called "ghost effects" is so far limited to the numerical computations, asymptotic analysis and some linearized models. It will be interesting if some nonlinear theories on this subject can be established by energy method.

Throughout this chapter, we will concentrate on the Cauchy problem even though some of the analysis can be applied to the initial boundary value problem. The essential phenomena associated to the boundary is the boundary layer. Note that the existence of boundary layer has been proved in the L^∞ framework in some physical settings. To study the time evolution problem with boundary effect, it is natural to investigate the stability of the superposition of the boundary layer and the basic wave patterns or non-trivial solution profiles. However, the stability analysis given here on the wave patterns and solution profiles is in the L^2 framework. It is not clear, at least up to now, how to combine these two methods to study the problems associated with boundary layers.

3.1 Local Existence

Even though the solution to the Boltzmann equation does not fit in only the L^2 space because of the binary collision operator, it is well suited in L^2 space together with its derivatives. That is, one can consider the solution in some Sobolev space $H^N_{t,x,\xi}$ for some positive integer N. For illustration, we only consider the Boltzmann equation without forcing here. However, the local existence can be proved in more general situation, such as, the Boltzmann equation with an external force $F(t,x,\xi)$ and a source term $S(t,x,\xi)$:

$$f_t + \xi \cdot \nabla_x f + F \cdot \nabla_\xi f = Q(f,f) + S(t,x,\xi), \tag{3.1.1}$$

as long as $F(t,x,\xi)$ and $S(t,x,\xi)$ are in some suitable spaces and the bi-characteristics are smooth and one to one from $\mathbb{R}^3 \times \mathbb{R}^3$ to itself.

Consider the Boltzmann equation

$$f_t + \xi \cdot \nabla_x f = Q(f,f), \tag{3.1.2}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi). \tag{3.1.3}$$

In what follows, we will show that if the initial data $f_0(x,\xi)$ is a small perturbation of a global Maxwellian $\overline{\mathbf{M}}(\xi) = \mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}(\xi)$, then there exists a local classical solution to (3.1.2)-(3.1.3) in the space:

$$\overline{\mathbf{H}}^N_{x,\xi}([0,T]) = \left\{ g(t,x,\xi) \;\middle|\; \begin{array}{c} \frac{\partial_x^\alpha \partial_t^\beta g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left([0,T), L^2_{x,\xi}(\mathbb{R}^3\times\mathbb{R}^3)\right) \\ \|g\|_X \le M, \quad |\alpha|+|\beta| \le N \end{array} \right\}, \tag{3.1.4}$$

where N is an integer not less than 4 and $\mathbf{M}_- = \mathbf{M}_{[1,0,\theta_-]}$ is a fixed global Maxwellian. Here $g(t,x,\xi) = f(t,x,\xi) - \overline{\mathbf{M}}(\xi)$, M and T are some positive constants, and the norm $\|g\|_X$ is defined by

$$
\begin{aligned}
\|g\|_X = \quad & \sup_{0\le t\le T}\left\{\sum_{|\alpha|+|\beta|\le N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\partial_x^\alpha\partial_t^\beta g(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx\right\}\\
& + \sum_{|\alpha|+|\beta|\le N}\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nu_{\overline{\mathbf{M}}}(\xi)|\partial_x^\alpha\partial_t^\beta g(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx d\tau. \qquad (3.1.5)
\end{aligned}
$$

Without loss of generality, in the following discussion, we assume $N = 4$. The main idea of the proof is to rewrite the Boltzmann equation into an integral equation and then apply the contraction mapping theorem with the norm defined above. For this, we first define the backward bi-characteristic starting from a given point $(t_0, x_0, \xi_0) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$, denoted by $(X(t), \Xi(t)) \equiv (X, \Xi)(t; t_0, x_0, \xi_0)$:

$$
\begin{cases}
\frac{dX(t)}{dt} = \Xi(t),\\
\frac{d\Xi(t)}{dt} = 0,\\
(X(t),\Xi(t))|_{t=t_0} = (x_0,\xi_0),
\end{cases} \qquad (3.1.6)
$$

which simply gives $(X(t), \Xi(t)) = (x_0+\xi_0(t-t_0), \xi_0)$. For simplicity, we denote $(X, \Xi)(0; t_0, x_0, \xi_0)$ by (X_0, Ξ_0). Notice that

$$
\begin{cases}
\frac{\partial X_i(t;t_0,x_0,\xi_0)}{\partial x_{0j}} = \frac{\partial \Xi_i(t;t_0,x_0,\xi_0)}{\partial \xi_{0j}} = \delta_{ij}, \quad i,j=1,2,3,\\
\frac{\partial X_i(t;t_0,x_0,\xi_0)}{\partial \xi_{0j}} = (t-t_0)\delta_{ij}, \quad \frac{\partial \Xi_i(t;t_0,x_0,\xi_0)}{\partial x_{0j}} = 0, \quad i,j=1,2,3,
\end{cases} \qquad (3.1.7)
$$

which implies that

$$
\det\frac{\partial(X,\Xi)}{\partial(x_0,\xi_0)} = 1. \qquad (3.1.8)
$$

Now we turn to the equation for the perturbation $g(t, x, \xi)$ which solves

$$
\begin{cases}
g_t + \xi\cdot\nabla_x g = \mathbf{L}_{\overline{\mathbf{M}}}g + Q(g,g),\\
g(t,x,\xi)|_{t=0} = g_0(x,\xi),
\end{cases} \qquad (3.1.9)
$$

where $\mathbf{L}_{\bar{M}}$ is the linearized collision operator.

For the hard potentials with angular cut-off and the hard sphere model, the linearized collision operator takes the form:

$$
(\mathbf{L}_{\overline{\mathbf{M}}}h)(\xi) = -\nu_{\overline{\mathbf{M}}}(\xi)h(\xi) + \sqrt{\overline{\mathbf{M}}(\xi)}K_{\overline{\mathbf{M}}}\left(\left(\frac{h}{\sqrt{\overline{\mathbf{M}}}}\right)(\xi)\right), \qquad (3.1.10)
$$

where $K_{\overline{\mathbf{M}}}(\cdot) = -K_{1\overline{\mathbf{M}}}(\cdot) + K_{2\overline{\mathbf{M}}}(\cdot)$ is a symmetric compact L^2-operator. In particular, for the hard sphere model, the collision frequency $\nu_{\overline{\mathbf{M}}}(\xi)$ and $K_{i\overline{\mathbf{M}}}(\cdot)$ have the following expressions

$$\begin{cases} \nu_{\overline{\mathbf{M}}}(\xi) = \overline{\rho}(x) \quad \overline{\nu}(\xi), \overline{\nu}(\xi) = \overline{\nu}(|\xi|) = \frac{2}{\sqrt{2\pi}} \left\{ \left(\frac{1}{|\xi|} + |\xi| \right) \int_0^{|\xi|} \exp\left(-\frac{y^2}{2} \right) dy + \exp\left(-\frac{|\xi|^2}{2} \right) \right\}, \\ k_{1\overline{\mathbf{M}}}(\xi, \xi_*) \quad = \frac{\pi \overline{\rho}(x)}{\sqrt{(2\pi)^3}} |\xi - \xi_*| \exp\left(-\frac{|\xi|^2}{4} - \frac{|\xi_*|^2}{4} \right), \\ k_{2\overline{\mathbf{M}}}(\xi, \xi_*) \quad = \frac{2\overline{\rho}(x)}{\sqrt{2\pi}} |\xi - \xi_*|^{-1} \exp\left(-\frac{|\xi - \xi_*|^2}{8} - \frac{(|\xi|^2 - |\xi_*|^2)^2}{8|\xi - \xi_*|^2} \right), \end{cases} \tag{3.1.11}$$

where $k_{i\overline{\mathbf{M}}}(\xi, \xi_*)(i = 1, 2)$ is the kernel of the operator $K_{i\overline{\mathbf{M}}}(i = 1, 2)$ respectively.

Since

$$\Xi(t) = \Xi(s), \qquad \nu_{\overline{\mathbf{M}}}(\Xi(t)) \geq \nu_0(1 + |\Xi(t)|), \tag{3.1.12}$$

by using the explicit expressions of $k_{i\overline{\mathbf{M}}}(\xi, \xi_*)(i = 1, 2)$, straightforward calculation gives the following lemma.

Lemma 3.1.1 *If $0 < \frac{\overline{\theta}}{2} < \theta_-$, then for $i = 1, 2$*

$$\begin{cases} \sup\limits_{\xi \in \mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} |\mathcal{K}_i(\xi, \xi_*)| d\xi_* \right\} \leq O(1), \\ \sup\limits_{\xi_* \in \mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} |\mathcal{K}_i(\xi, \xi_*)| d\xi \right\} \leq O(1). \end{cases} \tag{3.1.13}$$

Here

$$\mathcal{K}_i(\xi, \xi_*) = \sqrt{\frac{\overline{\mathbf{M}}(\xi)}{\mathbf{M}_-(\xi_*)}} k_{i\overline{\mathbf{M}}}(\xi, \xi_*) \sqrt{\frac{\overline{\mathbf{M}}(\xi_*)}{\mathbf{M}_-(\xi)}}, \quad i = 1, 2. \tag{3.1.14}$$

Consequently

$$\left| \int_{\mathbb{R}^3} \frac{g(\xi) \left(\sqrt{\overline{\mathbf{M}}} K_{\overline{\mathbf{M}}} \left(\frac{h}{\sqrt{\overline{\mathbf{M}}}} \right) \right)(\xi)}{\mathbf{M}_-(\xi)} d\xi \right| \leq O(1) \left(\int_{\mathbb{R}^3} \frac{h^2}{\mathbf{M}_-} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \frac{g^2}{\mathbf{M}_-} d\xi \right)^{\frac{1}{2}}, \tag{3.1.15}$$

i.e.,

$$\int_{\mathbb{R}^3} \frac{\left| \left(\sqrt{\overline{\mathbf{M}}} K_{\overline{\mathbf{M}}} \left(\frac{h}{\sqrt{\overline{\mathbf{M}}}} \right) \right)(\xi) \right|^2}{\mathbf{M}_-(\xi)} d\xi \leq O(1) \int_{\mathbb{R}^3} \frac{h^2}{\mathbf{M}_-} d\xi. \tag{3.1.16}$$

With the above estimates, the local existence of solutions to the Cauchy problem (3.1.2), (3.1.3) in $\overline{\mathbf{H}}^4_{x,\xi}([0, T))$ can be proved by using the following iteration sequence $\{g^n(t, x, \xi)\}_{n=0}^{\infty}$

solving

$$\begin{cases} g^0(t,x,\xi) = g_0(x,\xi), \\ g_t^{n+1} + \xi\cdot\nabla_x g^{n+1} = \mathbf{L}_{\overline{\mathbf{M}}} g^{n+1} + Q(g^n,g^n) \\ \qquad\qquad = -\nu_{\overline{\mathbf{M}}}(\xi) g^{n+1} + \sqrt{\overline{\mathbf{M}}} K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right) + Q(g^n,g^n), \\ g^{n+1}(t,x,\xi)|_{t=0} = g_0(x,\xi). \end{cases} \tag{3.1.17}$$

Integrating $(3.1.17)_2$ along $(X,\Xi)(t;t_0,x_0,\xi_0)$ gives

$$\begin{aligned} & g^{n+1}(t_0,x_0,\xi_0) \\ = \; & \exp\left(-\int_0^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right) g_0(X_0,\Xi_0) \\ & +\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\eta,X(\eta),\Xi(\eta))d\eta \\ & +\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))d\eta. \end{aligned} \tag{3.1.18}$$

Now we need an estimate on the nonlinear collision operator $Q(f,f)$ given in Theorem 1.2.3.

Lemma 3.1.2 *For hard sphere model and hard potentials with angular cut-off defined by* (1.2.17)*, there exists a positive constant $C>0$ such that*

$$\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}Q(f,g)^2}{\tilde{\mathbf{M}}}d\xi \le \; C\left\{\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)f^2}{\tilde{\mathbf{M}}}d\xi\cdot\int_{\mathbb{R}^3}\frac{g^2}{\tilde{\mathbf{M}}}d\xi+\int_{\mathbb{R}^3}\frac{f^2}{\tilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)g^2}{\tilde{\mathbf{M}}}d\xi\right\}, \tag{3.1.19}$$

where $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are any Maxwellians such that the above integrals are well defined. Note that $\nu_{\mathbf{M}}(\xi) = O(|\xi|^\gamma)$ $(|\xi|\to\infty)$ for any Maxwellian $\mathbf{M}$.

It can be shown by induction that if $\|g_0\|_X \le \frac{M}{3C}$, then $\|g^n\|_X \le M$ for all n provided that M and T are sufficiently small. In fact, if $\|g^n\|_X\le M$, we have from (3.1.18) that

$$\begin{aligned} & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|g^{n+1}(t_0,x_0,\xi_0)|^2}{\mathbf{M}_-(\xi_0)}d\xi_0dx_0 \le 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|g_0(X(0),\Xi(0))|^2}{\mathbf{M}_-(\xi_0)}d\xi_0dx_0 \\ & +2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\eta,X(\eta),\Xi(\eta))d\eta\right|^2\frac{d\xi_0dx_0}{\mathbf{M}_-(\xi_0)} \\ & +2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))d\eta\right|^2\frac{d\xi_0dx_0}{\mathbf{M}_-(\xi_0)}. \end{aligned} \tag{3.1.20}$$

The estimates given in the above lemmas and the Cauchy-Schwarz inequality then give

$$\begin{aligned} \int_{\mathbb{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx \le \; & 4C\int_{\mathbb{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx + O(1)\left(M^4+T^2\right) \\ & +4C\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxdt. \end{aligned} \tag{3.1.21}$$

By choosing $T > 0$ sufficiently small such that $\exp(4CT) < 2$, the Gronwall inequality then implies

$$\int_{\mathbb{R}^3}\int_{\mathbf{R}^3} \frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx \leq 8C \int_{\mathbb{R}^3}\int_{\mathbf{R}^3} \frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx + O(1)\left(M^4 + T^2\right). \quad (3.1.22)$$

Similar argument gives

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{\nu_{\mathbf{M}}(\xi)|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx dt \leq \ & C^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx \\ & + O(1)\left(M^4 + T^2\right). \end{aligned} \quad (3.1.23)$$

Combining (3.1.22) with (3.1.23) yields

$$\begin{aligned} & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx + \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{\nu_{\mathbf{M}}(\xi)|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx dt \\ & \leq 4C^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx + O(1)\left(M^4 + T^2\right). \end{aligned} \quad (3.1.24)$$

For the estimates on $\partial_x^\alpha \partial_t^\beta g^{n+1}(t,x,\xi)$ with $|\alpha| + |\beta| = j, 1 \leq j \leq 4$, we write

$$\begin{cases} \left(\partial_x^\alpha \partial_t^\beta g^{n+1}\right)_t + & \xi \cdot \nabla_x \left(\partial_x^\alpha \partial_t^\beta g^{n+1}\right) \\ = & -\nu_{\overline{\mathbf{M}}}(\xi) \partial_x^\alpha \partial_t^\beta g^{n+1} + \sqrt{\overline{\mathbf{M}}} K_{\overline{\mathbf{M}}} \left(\frac{\partial_x^\alpha \partial_t^\beta g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right) \\ & + \sum_{|\alpha'|+|\beta'| \leq j} C_{\alpha,\beta}^{\alpha',\beta'} Q\left(\partial_x^{\alpha'} \partial_t^{\beta'} g^n, \partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} g^n\right). \\ \partial_x^\alpha \partial_t^\beta g^{n+1}\Big|_{t=0} = & \partial_x^\alpha g_0^{(\beta)}(x,\xi), \end{cases} \quad (3.1.25)$$

where the initial data $g_0^{(\beta)}(x,\xi)$ $(\beta \neq 0)$ are determined through g_0 and the compatibility conditions coming from the iteration equation (3.1.17). Similarly to the estimation on $g^{n+1}(t,x,\xi)$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{|\partial_x^\alpha \partial_t^\beta g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx + \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{\nu_{\mathbf{M}}(\xi)|\partial_x^\alpha \partial_t^\beta g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx dt \\ & \leq 4C^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{|\partial_x^\alpha \partial_t^\beta g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx + O(1)\left(M^4 + T^2\right). \end{aligned} \quad (3.1.26)$$

Thus, (3.1.24) together with (3.1.26) implies

$$\begin{aligned} \|g^{n+1}\|_X^2 \leq \ & 4C^2 \|g_0\|_X^2 + O(1)\left(M^4 + T^2\right) \\ \leq \ & \tfrac{4M^2}{9} + O(1)\left(M^4 + T^2\right) \leq M^2, \end{aligned} \quad (3.1.27)$$

provided that we choose $M > 0$ and $T > 0$ sufficiently small. Moreover, for each $|\alpha|+|\beta| \leq 4$, by using (3.1.27), the integral formula for $\partial_x^\alpha \partial_t^\beta g^{n+1}(t,x,\xi)$ along the bi characteristic can be solved in the Neumann series for small T_1. Hence, $\frac{\partial_x^\alpha \partial_t^\beta g^{n+1}(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left([0,T), L^2_{x,\xi}(\mathbb{R}^3 \times \mathbf{R}^3)\right)$ provided that $\frac{\partial_x^\alpha \partial_t^\beta g^{n}(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}}$ is in the same space. This, together with (3.1.27), implies that $g^{n+1}(t,x,\xi) \in \overline{\mathbf{H}}^4_{x,\xi}([0,T))$.

To show that $\{g^n(t,x,\xi)\}$ is a Cauchy sequence in $\overline{\mathbf{H}}^4_{x,\xi}([0,T))$, set

$$h^n(t,x,\xi) = g^{n+1}(t,x,\xi) - g^n(t,x,\xi), \quad n \geq 0. \tag{3.1.28}$$

Then $h^n(t,x,\xi)(n \geq 1)$ solves

$$\begin{cases} h^n_t + \xi \cdot \nabla_x h^n = -\nu_{\overline{\mathbf{M}}}(\xi) h^n + \sqrt{\overline{\mathbf{M}}} K_{\overline{\mathbf{M}}}\left(\frac{h^n}{\sqrt{\overline{\mathbf{M}}}}\right) + 2Q(g^n, h^{n-1}) + Q(h^{n-1}, h^{n-1}), \\ h^n(t,x,\xi)|_{t=0} = 0. \end{cases} \tag{3.1.29}$$

By (3.1.29), it can be deduced from (3.1.27) that

$$\|h^n\|_X \leq \frac{1}{2}\|h^{n-1}\|_X, \quad n \geq 1 \tag{3.1.30}$$

provided that $M > 0$ and $T > 0$ are sufficiently small. Thus $\{g^n(t,x,\xi)\}$ is a Cauchy sequence in $\overline{\mathbf{H}}^4_{x,\xi}([0,T))$ which converges to a unique solution locally in time. We can summarize it into the following theorem.

Theorem 3.1.3 (Local existence) *For any sufficiently small constant $M > 0$ and integer $N \geq 4$, there exists a positive constant $T^*(M) > 0$ such that if*

$$\mathcal{E}(f_0) = \sum_{|\alpha|+|\beta| \leq N} \left\| \frac{\partial_x^\alpha \partial_t^\beta (f_0(x,\xi) - \overline{\mathbf{M}}(\xi))}{\sqrt{\mathbf{M}_-(\xi)}} \right\|_{L^2_{x,\xi}(\mathbb{R}^3 \times \mathbf{R}^3)} \leq \frac{M}{3C}, \tag{3.1.31}$$

for some global Maxwllian satisfying the condition in Lemma 3.1.1., then the Cauchy problem (3.1.2), (3.1.3) admits a unique classical solution

$$f(t,x,\xi) \in \overline{\mathbf{H}}^N_{x,\xi}\left([0, T^*(M))\right), \tag{3.1.32}$$

such that $f(t,x,\xi) \geq 0$ and

$$\sup_{0 \leq t \leq T^*(M)} \sum_{|\alpha|+|\beta| \leq N} \int_{\mathbb{R}^3} \int_{\mathbf{R}^3} \frac{|\partial_x^\alpha \partial_t^\beta (f(t,x,\xi) - \overline{\mathbf{M}}(\xi))|^2}{\mathbf{M}_-(\xi)} d\xi dx \leq M. \tag{3.1.33}$$

3.2 Expansions and Decomposition

In this section, we will use the decomposition of the solution into the macroscopic and microscopic components to reformulate the Boltzmann equation as a system of conservation laws for the macroscopic components coupled with an equation for the microscopic component.

As mentioned earlier, this kind of thinking is similar to the Hilbert and Champman-Enskog expansions where the leading term is a local Maxwellian with its macroscopic components governed by the conservation laws, either the Euler equations or Navier-Stokes equations. The main difference between the reformulation introduced here and the classical expansions is that there is no approximation or truncation in the reformulation so that it is equivalent to the Boltzmann equation. Moreover, the system of conservation laws for the local Maxwellian has the framework of Navier-Stokes equations with a source term determined by the microscopic component.

For the completeness and the convenience of the readers, the Hilbert and Chapman-Enskog expansions are reviewed in the following two subsections before the decomposition and reformulation are given.

3.2.1 Hilbert Expansion

Consider the Boltzmann equation,

$$f_t + \xi \cdot \nabla_x f = \frac{1}{\kappa} Q(f, f), \tag{3.2.1}$$

where κ is the Knudsen number which is proportional to the mean free path. Here, we assume κ is a small constant and use it as the parameter for the expansion. In 1912, Hilbert [64] introduced the following famous expansion of the solution to the Boltzmann equation:

$$f = \sum_{n=0}^{\infty} \kappa^n f_n. \tag{3.2.2}$$

By putting this expansion into the Boltzmann equation (3.2.1) and comparing the terms by the order of κ, one has the following equations for f_n:

$$\begin{aligned} &Q_0 = 0, \\ &(f_{n-1})_t + \xi \cdot \nabla_x f_{n-1} = Q_n, \qquad n \geq 1, \end{aligned} \tag{3.2.3}$$

where

$$\begin{aligned} Q_0 &= Q(f_0, f_0), \\ Q_n &= 2Q(f_0, f_n) + \sum_{i=1}^{n-1} Q(f_k, f_{n-k}), \quad n \geq 1. \end{aligned} \tag{3.2.4}$$

Hence, thanks to the property $[Q2]$ in §1.2.2, the first equation in (3.2.4) implies that f_0 is a local Maxwellian, i.e.,

$$f_0 = \mathbf{M}_0 \equiv \mathbf{M}_{[\rho^0, u^0, \theta^0]} = \frac{\rho^0}{(2\pi R\theta^0)^{\frac{3}{2}}} \exp\{-\frac{|\xi - u^0|^2}{2R\theta^0}\}, \tag{3.2.5}$$

where ρ^0, u^0 and θ^0 are functions of (t, x). And f_0 satisfies

$$f_{0t} + \xi \cdot \nabla_x f_0 = Q_1. \tag{3.2.6}$$

Here, Q_1 is a microscopic component which is orthogonal to the five collision invariants $\psi_\alpha(\xi), \alpha = 0, 1, \cdots, 4$, given by $[Q1]$ in §1.2.2:

$$\begin{cases} \psi_0(\xi) \equiv & 1, \\ \psi_i(\xi) \equiv & \xi_i, \quad i = 1, 2, 3, \text{ or } \psi(\xi) = \xi, \\ \psi_4(\xi) \equiv & \frac{1}{2}|\xi|^2. \end{cases} \tag{3.2.7}$$

The solvability condition for (3.2.6) gives the system of conservation laws

$$\int_{\mathbb{R}^3} \psi_\alpha(f_{0t} + \xi \cdot \nabla_x f_0) d\xi = 0, \tag{3.2.8}$$

which are exactly the compressible Euler equations

$$\begin{cases} \rho^0_t + \nabla_x \cdot (\rho^0 u^0) = 0, \\ (\rho^0 u^0)_t + \nabla_x \cdot (\rho^0 u^0 \otimes u^0) + \nabla_x p^0 = 0, \\ [\rho^0(\frac{|u^0|^2}{2} + \mathcal{E}^0)]_t + \nabla_x \cdot \{[\rho^0(\frac{|u^0|^2}{2} + \mathcal{E}^0) + p^0]u\} = 0, \end{cases} \tag{3.2.9}$$

where the pressure function is given by $p^0 = R\rho^0\theta^0$ and the internal energy is $\mathcal{E}^0 = \frac{3}{2}R\theta^0$.

For $n \geq 1$, if we denote

$$S_n = \sum_{i=1}^{n-1} Q(f_k, f_{n-k}), \tag{3.2.10}$$

and

$$\mathbf{L}_{\mathbf{M}_0} h = 2Q(h, f_0), \tag{3.2.11}$$

which is the linearized collision operator with respect to the local Maxwellian $\mathbf{M}_0$, then under the solvability condition for $(3.2.3)_2$

$$\int_{\mathbb{R}^3} \psi_\alpha((f_{n-1})_t + \xi \cdot \nabla_x f_{n-1}) d\xi = 0, \tag{3.2.12}$$

f_n can be represented in terms of f_i for $i = 0, 1, \cdots, n-1$ by

$$f_n = \sum_{\alpha=0}^{4} c_\alpha \psi_\alpha \mathbf{M}_0 + \mathbf{L}^{-1}_{\mathbf{M}_0}\{(f_{n-1})_t + \xi \cdot \nabla_x f_{n-1} - S_n\}. \tag{3.2.13}$$

Thus, the conservation laws

$$\int_{\mathbb{R}^3} \psi_\alpha(f_{nt} + \xi \cdot \nabla_x f_n) d\xi = 0, \quad \alpha = 0, 1, \cdots, 4, \tag{3.2.14}$$

are the system of linearized Euler equations around the fluid variables (ρ^0, u^0, θ^0) for the macroscopic components in f_n.

Since to determine the value of f_n in the Hilbert expansion involves the differentiation of f_{n-1}, by induction, the convergence of this expansion can only be expected when the solution is infinitely differentiable and bounded with respect to the Knudsen number κ. Therefore, usually, the Hilbert expansion does not converge, especially in the present of initial layer, shock layer and boundary layer where the value of the differentiation grows when κ decreases.

3.2.2 Chapman-Enskog Expansion

The Chapman-Enskog expansion was introduced by Chapman and Enskog in 1916 and 1917 independently. The main idea in this expansion is to expand both the equation and the solution, but to keep the conservative quantities unexpanded. The advantage of this expansion is that the first order correction yields the Navier-Stokes equations for the macroscopic components so that the viscosity and heat conductivity are correctly represented. However, the drawback of the Chapman-Enskog expansion is that the higher order approximations give differential equations of higher order, such as the Burnett and super-Burnett equations for which there is no satisfactory mathematical theory. In other words, there is basically no established mathematical theory on this expansion beyond the Navier-Stokes level.

Formally, we can write

$$\frac{\partial f}{\partial t} = \sum_{n=0}^{\infty} \kappa^n \frac{\partial^{(n)} f_n}{\partial t}. \tag{3.2.15}$$

Since the conserved quantities are unexpanded, the consistency requires that for $n \geq 1$,

$$\int_{\mathbb{R}^3} \psi_\alpha f_n d\xi = 0, \qquad \alpha = 0, 1, 2, 3, 4, \tag{3.2.16}$$

which implies that all the function f_n for $n \geq 1$ are microscopic. By substituting (3.2.2) and (3.2.15) into the Boltzmann equation, we have

$$Q_0 = Q(f_0, f_0) = 0, \tag{3.2.17}$$

and for $n \geq 1$,

$$\sum_{i=0}^{n-1} \frac{\partial^{(i)} f_{n-i-1}}{\partial t} + \xi \cdot \nabla_x f_{n-1} = 2Q(f_0, f_n) + S_n, \tag{3.2.18}$$

where the notation has the same meaning as in the last subsection. However, one should notice that here each f_n is a functional of the conserved quantities which are not expanded. Again, the equation (3.2.17) implies that f_0 must be a local Maxwellian, i.e.,

$$f_0 = \mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]} = \frac{\rho}{(2\pi R\theta)^{\frac{3}{2}}} \exp\{-\frac{|\xi - u|^2}{2R\theta}\}. \tag{3.2.19}$$

Therefore, the equation (3.2.18) for $n = 1$ can be written as

$$\frac{\partial^{(0)} f_0}{\partial t} + \xi \cdot \nabla_x f_0 = \mathbf{L_M} f_1. \tag{3.2.20}$$

The solvability condition for (3.2.20) immediately gives the following Euler equations

$$\begin{cases} \dfrac{\partial^{(0)} \rho}{\partial t} = -\dfrac{\partial}{\partial x_i}(\rho u_i), \\ \dfrac{\partial^{(0)} u_i}{\partial t} = -u_j \dfrac{\partial u_i}{\partial x_j} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i}, \quad i = 1, 2, 3, \\ \dfrac{\partial^{(0)} \theta}{\partial t} = -u_i \dfrac{\partial \theta}{\partial x_i} - \dfrac{2}{3}\theta \dfrac{\partial u_i}{\partial x_i}, \end{cases} \tag{3.2.21}$$

where $p = R\rho\theta$, here and in what follows, the summation is over any repeated indices. By plugging the expression of the local Maxwellian of f_0 into the equation (3.2.20), we have

$$\frac{1}{\rho}\mathbf{M}\mathbf{B}^0\rho + \frac{1}{\theta}\left(\frac{c^2}{2R\theta} - \frac{3}{2}\right)\mathbf{M}\mathbf{B}^0\theta + \frac{1}{R\theta}c_j\mathbf{B}^0u_j = \mathbf{L_M}f_1, \tag{3.2.22}$$

where $c = \xi - u$ is the random velocity and $\mathbf{B}^0$ is the following linear operator

$$\mathbf{B}^0 = \frac{\partial^{(0)}}{\partial t} + \xi\cdot\nabla_x.$$

Now we can substitute the time derivative $\frac{\partial^{(0)}}{\partial t}$ of (3.2.21) into the equation (3.2.22) to have

$$\left(\frac{c^2}{2R\theta} - \frac{5}{2}\right)\mathbf{M}\frac{c_i}{\theta}\frac{\partial\theta}{\partial x_i} + \frac{1}{R\theta}\left(c_ic_j - \frac{1}{3}c^2\delta_{ij}\right)\mathbf{M}\frac{\partial u_i}{\partial x_j} = \mathbf{L_M}f_1. \tag{3.2.23}$$

By using the Burnett functions defined by

$$\begin{cases} A_j(\xi) = \dfrac{|\xi|^2 - 5}{2}\xi^j, \qquad j = 1, 2, 3, \\[2ex] B_{ij}(\xi) = \xi^i\xi^j - \dfrac{1}{3}\delta_{ij}|\xi|^2, \quad i, j = 1, 2, 3, \end{cases} \tag{3.2.24}$$

we have

$$f_1 = \mathbf{L}_{\mathbf{M}}^{-1}\left(\sqrt{R}A_i\left(\frac{c}{\sqrt{R\theta}}\right)\mathbf{M}\frac{\partial\theta}{\partial x_i} + B_{ij}\left(\frac{c}{\sqrt{R\theta}}\right)\mathbf{M}\frac{\partial u_i}{\partial x_j}\right). \tag{3.2.25}$$

Notice that we have used the fact that the operator $\mathbf{L_M}$ is invertible in the microscopic space which is the space orthogonal to the null space of $\mathbf{L_M}$.

Before going further, let's review the properties of the Burnett functions.

Proposition 3.2.1 *Denote*

$$A' = \mathbf{L}_{\mathbf{M}}^{-1}A, \qquad B' = \mathbf{L_M}B. \tag{3.2.26}$$

Then there exist positive functions $a(r)$ and $b(r)$ defined on $[0, \infty)$ such that

$$A'(\xi) = -a(|\xi|)A(\xi), \qquad B'(\xi) = -b(|\xi|)B(\xi). \tag{3.2.27}$$

And the following properties hold, where $(\cdot,\cdot)$ denotes the inner product of $L^2(\mathbb{R}^3)$.

- $(-A_i, A_i')$ *is positive and independent of i.*
- $(A_i, A_j') = 0$ *for any $i \neq j$.*
- $(A_i, B_{jk}') = 0$ *for any i, j, k.*
- $(B_{ij}, B_{kl}') = (B_{kl}, B_{ij}') = (B_{ji}, B_{kl}')$ *holds and is independent of i, j for any fixed k, l.*
- $-(B_{ij}, B_{ij}')$ *is positive and independent of i, j when $i \neq j$.*
- $-(B_{ii}, B_{jj}')$ *is positive and independent of i, j when $i \neq j$.*
- $-(B_{ii}, B_{ii}')$ *is positive and independent of i.*
- $(B_{ij}, B_{kl}') = 0$ *unless either $(i, j) = (k, l)$ or (l, k), or $i = j$ and $k = l$.*
- $(B_{ii}, B_{ii}') - (B_{ii}, B_{jj}') = 2(B_{ij}, B_{ij}')$ *holds for any $i \neq j$.*

The proof of this proposition is quite technical and can be found in [56].

With the Burnett functions, the viscosity $\mu(\theta)$ and heat conductivity coefficient $\varkappa(\theta)$ can be represented by

$$\begin{cases} \mu(\theta) = -\kappa R\theta \int_{\mathbb{R}^3} B_{ij}\left(\frac{c}{\sqrt{R\theta}}\right) \mathbf{L}_{\mathbf{M}}^{-1}\left(B_{ij}\left(\frac{c}{\sqrt{R\theta}}\right)\mathbf{M}\right) d\xi > 0, \quad i \neq j, \\ \varkappa(\theta) = -\kappa R^2\theta \int_{\mathbb{R}^3} A_l\left(\frac{c}{\sqrt{R\theta}}\right) \mathbf{L}_{\mathbf{M}}^{-1}\left(A_l\left(\frac{c}{\sqrt{R\theta}}\right)\mathbf{M}\right) d\xi > 0. \end{cases} \tag{3.2.28}$$

Note that these coefficients are independent of the density function ρ.

Now if we put f_1 into the conservation laws to include the first order approximation, then the conservation laws take the form

$$\int_{\mathbb{R}^3} \psi_\alpha((f_0)_t + \xi \cdot \nabla_x(f_0 + \kappa f_1)) d\xi = 0. \tag{3.2.29}$$

Since f_1 is microscopic, its contribution to the conservation of mass is zero. And its contribution to the equations of conservation of momentum and energy is represented by the stress tensor and heat flux:

$$p_{ij}^{(1)} = \kappa \int_{\mathbb{R}^3} c_i c_j f_1 d\xi, \qquad q_i^{(1)} = \frac{\kappa}{2}\int_{\mathbb{R}^3} c_i c^2 f_1 d\xi. \tag{3.2.30}$$

With Proposition 3.2.1, it is straightforward to calculate the stress tensor and heat flux in terms of the fluid variables:

$$\begin{cases} p_{ij}^{(1)} = -\mu(\theta)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2}{3}\mu(\theta)\frac{\partial u_k}{\partial x_k}\delta_{ij}, \\ q_i^{(1)} = -\varkappa(\theta)\frac{\partial \theta}{\partial x_i}. \end{cases} \tag{3.2.31}$$

In summary, the first order approximation in the Chapman-Enskog expansion is the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \nabla_x \cdot (\rho u) = 0, \\ (\rho u^i)_t + \nabla_x \cdot (\rho u^i u) + p_{x_i} = [\mu(\theta)(u^i_{x_j} + u^j_{x_i} - \frac{2}{3}\delta_{ij}\nabla_x \cdot u)]_{x_j}, \quad i = 1, 2, 3, \\ \left[\rho\left(\frac{1}{2}|u|^2 + \mathcal{E}\right)\right]_t + \nabla_x \cdot \left(\left[\rho\left(\frac{1}{2}|u|^2 + \mathcal{E}\right) + p\right] u\right) \\ \qquad = \mu(\theta)u^i\left(u^i_{x_j} + u^j_{x_i} - \frac{2}{3}\delta_{ij}\nabla_x \cdot u\right) + (\varkappa(\theta)\theta_{x_j})_{x_j}. \end{cases} \tag{3.2.32}$$

Again, similar but tedious calculation can be used to find the next terms, $f_2, f_3, \cdots$, in the Chapman-Enskog expansion, however, without good mathematical theory. In the next section, we will give a decomposition and reformulation of the Boltzmann equation without any expansion so that the structure of the systems for fluid dynamics together with the effects from the microscopic component become clear. In fact, one can compare it with the Hilbert and Chapman-Enskog expansions so that some similarities and subtle difference can be found as explained in the next subsection.

3.2.3 Macro-Micro Decomposition

In this subsection, based on the decomposition of the solution into its macroscopic (fluid dynamic) and microscopic (kinetic) component, we reformulate the Boltzmann equation into a system of conservation laws for the time evolution of the macroscopic variables and an equation for the time evolution of the microscopic variable. The main idea is not to have any approximation, but a complete description of the solutions to the Boltzmann equation so that the analytic techniques from the theory of conservation laws can be applied in the study of the Boltzmann equation.

To be precise, let $f(t,x,\xi)$ be the solution to the Boltzmann equation. We decompose it into the macroscopic component in the form of the local Mawellian $\mathbf{M}=\mathbf{M}(x,t,\xi)=\mathbf{M}_{[\rho,u,\theta]}(\xi)$, and the microscopic component $\mathbf{G}=\mathbf{G}(x,t,\xi)$:

$$f(t,x,\xi)=\mathbf{M}(t,x,\xi)+\mathbf{G}(t,x,\xi). \tag{3.2.33}$$

Here, $\mathbf{M}(t,x,\xi)$ is the local Maxwellian with its five fluid parameters (ρ,u,θ) defined by the five conserved quantities, the mass density $\rho(t,x)$, momentum $m(t,x)=\rho(t,x)u(t,x)$ and energy density $\mathcal{E}(t,x)+|u(t,x)|^2/2$:

$$\begin{cases} \rho(t,x)\equiv\displaystyle\int_{\mathbb{R}^3} f(t,x,\xi)d\xi, \\ m^i(t,x)=\rho u\equiv\displaystyle\int_{\mathbb{R}^3}\psi_i(\xi)f(t,x,\xi)d\xi, \quad i=1,2,3, \\ \left[\rho\left(\mathcal{E}+\frac{1}{2}|u|^2\right)\right](t,x)\equiv\displaystyle\int_{\mathbf{R}^3}\psi_4(\xi)f(t,x,\xi)d\xi. \end{cases} \tag{3.2.34}$$

To have an orthogonal basis for the subspace of the macroscopic components, we first define an inner product in the space $\mathbf{L}^2_\xi$ with a weight. For this, let $\tilde{\mathbf{M}}=\tilde{\mathbf{M}}_{[\tilde\rho,\tilde u,\tilde\theta]}$ be any given Maxwellian. Define

$$\langle h,g\rangle_{\tilde{\mathbf{M}}}\equiv\int_{\mathbb{R}^3}\frac{h(\xi)g(\xi)}{\tilde{\mathbf{M}}}d\xi, \tag{3.2.35}$$

for functions h and g of ξ such that the integral is well defined. Using this inner product, the subspace spanned by the collision invariants has the following set of orthogonal basis:

$$\begin{cases} \chi_0^{\tilde{\mathbf{M}}}=\chi_0(\xi;\tilde\rho,\tilde u,\tilde\theta)\equiv\frac{1}{\sqrt{\tilde\rho}}\tilde{\mathbf{M}}, \\ \chi_i^{\tilde{\mathbf{M}}}=\chi_i(\xi;\tilde\rho,\tilde u,\tilde\theta)\equiv\frac{\xi^i-\tilde u^i}{\sqrt{R\tilde\theta\tilde\rho}}\tilde{\mathbf{M}}, \quad i=1,2,3, \\ \chi_4^{\tilde{\mathbf{M}}}=\chi_4(\xi;\tilde\rho,\tilde u,\tilde\theta)\equiv\frac{1}{\sqrt{6\tilde\rho}}\left(\frac{|\xi-\tilde u|^2}{R\tilde\theta}-3\right)\tilde{\mathbf{M}}, \\ \langle\chi_\alpha^{\tilde{\mathbf{M}}},\chi_\beta^{\tilde{\mathbf{M}}}\rangle_{\tilde{\mathbf{M}}}=\delta_{\alpha\beta}, \quad \text{for } \alpha,\ \beta=0,1,2,3,4. \end{cases} \tag{3.2.36}$$

With this basis, define the macroscopic projection $\mathbf{P}_0^{\tilde{\mathbf{M}}}$ and microscopic projection $\mathbf{P}_1^{\tilde{\mathbf{M}}}$ by:

$$\begin{cases} \mathbf{P}_0^{\tilde{\mathbf{M}}}h\equiv\sum_{\alpha=0}^{4}\left\langle h,\chi_\alpha^{\tilde{\mathbf{M}}}\right\rangle_{\tilde{\mathbf{M}}}\chi_\alpha^{\tilde{\mathbf{M}}}, \\ \mathbf{P}_1^{\tilde{\mathbf{M}}}h\equiv h-\mathbf{P}_0^{\tilde{\mathbf{M}}}h. \end{cases} \tag{3.2.37}$$

Notice that the operators $\mathbf{P}_0^{\tilde{\mathbf{M}}}$ and $\mathbf{P}_1^{\tilde{\mathbf{M}}}$ are projections, that is,

$$\mathbf{P}_0^{\tilde{\mathbf{M}}}\mathbf{P}_0^{\tilde{\mathbf{M}}} = \mathbf{P}_0^{\tilde{\mathbf{M}}}, \quad \mathbf{P}_1^{\tilde{\mathbf{M}}}\mathbf{P}_1^{\tilde{\mathbf{M}}} = \mathbf{P}_1^{\tilde{\mathbf{M}}}, \quad \mathbf{P}_0^{\tilde{\mathbf{M}}}\mathbf{P}_1^{\tilde{\mathbf{M}}} = \mathbf{P}_1^{\tilde{\mathbf{M}}}\mathbf{P}_0^{\tilde{\mathbf{M}}} = 0.$$

Now, the system of conservation laws

$$\int_{\mathbb{R}^3} \psi_\alpha(f_t + \xi \cdot \nabla_x f)d\xi = 0, \quad \alpha = 0, 1, \cdots, 4, \tag{3.2.38}$$

takes the following form

$$\begin{cases} \rho_t + \mathrm{div}_x\, m = 0, \\ m_t^i + \left(\sum\limits_{j=1}^{3} u^j m^i\right)_{x^j} + p_{x^i} + \int_{\mathbb{R}^3} \psi_i(\xi)\Big(\xi \cdot \nabla_x \mathbf{G}\Big)d\xi = 0, \quad i = 1, 2, 3, \\ \left[\rho\left(\frac{|u|^2}{2} + \mathcal{E}\right)\right]_t + \sum\limits_{j=1}^{3}\left\{u^j\left[\rho\left(\frac{|u|^2}{2} + \mathcal{E}\right) + p\right]\right\}_{x^j} + \int_{\mathbb{R}^3} \psi_4(\xi)\Big(\xi \cdot \nabla_x \mathbf{G}\Big)d\xi = 0, \end{cases} \tag{3.2.39}$$

The equation of the state is for the monatomic gas, with the gas constant R chosen to be $\frac{2}{3}$ without loss of generality, given by

$$p = \frac{2}{3}\rho e.$$

And the macroscopic entropy S can be defined as:

$$S = -\frac{2}{3}\ln\rho + \ln\left(\frac{4}{3}\pi\theta\right) + 1.$$

The microscopic equation is obtained by applying the microscopic projection $\mathbf{P}_1^{\mathbf{M}}$ to the Boltzmann equation (3.2.1):

$$\mathbf{G}_t + \mathbf{P}_1^{\mathbf{M}}\Big(\xi \cdot \nabla_x \mathbf{G} + \xi \cdot \nabla_x \mathbf{M}\Big) = \frac{1}{\kappa}\mathbf{L}_{\mathbf{M}}\mathbf{G} + \frac{1}{\kappa}Q(\mathbf{G}, \mathbf{G}), \tag{3.2.40}$$

where $\mathbf{L}_{\mathbf{M}}$ is the linearized collision operator around the local Maxwellian $\mathbf{M}$.

From (3.2.40), we have

$$\begin{aligned} \mathbf{G} = {} & \kappa\mathbf{L}_{\mathbf{M}}^{-1}\left(\mathbf{P}_1^{\mathbf{M}}(\xi \cdot \nabla_x \mathbf{M})\right) + \mathbf{L}_{\mathbf{M}}^{-1}\left(\kappa\left(\partial_t \mathbf{G} + \mathbf{P}_1^{\mathbf{M}}\xi \cdot (\nabla_x \mathbf{G})\right) - Q(\mathbf{G}, \mathbf{G})\right) \\ = {} & \kappa\mathbf{L}_{\mathbf{M}}^{-1}\left(\mathbf{P}_1^{\mathbf{M}}(\xi \cdot \nabla_x \mathbf{M})\right) + \Theta. \end{aligned} \tag{3.2.41}$$

Substituting (3.2.41) into (3.2.39) yields the following fluid-type system for the macroscopic components:

$$\begin{cases} \rho_t + \mathrm{div}_x\, m = 0, \\ m_t^i + \left(\sum_{j=1}^3 u^j m^i\right)_{x^j} + p_{x^i} + \kappa \int_{\mathbf{R}^3} \psi_i(\xi)\left(\xi \cdot \nabla_x \mathbf{L}_{\mathbf{M}}^{-1}\left(\mathbf{P}_1^{\mathbf{M}}(\xi \cdot \nabla_x \mathbf{M})\right)\right) d\xi \\ \qquad + \int_{\mathbb{R}^3} \psi_i(\xi)\Big(\xi \cdot \nabla_x \Theta\Big) d\xi = 0, \quad i = 1,2,3, \\ \left[\rho\left(\frac{|u|^2}{2} + \mathcal{E}\right)\right]_t + \sum_{j=1}^3 \left\{u^j\left[\rho\left(\frac{|u|^2}{2} + \mathcal{E}\right) + p\right]\right\}_{x^j} + \kappa \int_{\mathbb{R}^3} \psi_4(\xi)\left(\xi \cdot \nabla_x \mathbf{L}_{\mathbf{M}}^{-1}\left(\mathbf{P}_1^{\mathbf{M}}(\xi \cdot \nabla_x \mathbf{M})\right)\right) d\xi \\ \qquad + \int_{\mathbb{R}^3} \psi_4(\xi)\Big(\xi \cdot \nabla_x \Theta\Big) d\xi = 0. \end{cases} \tag{3.2.42}$$

A straightforward calculation by using the Burnett functions as in the previous section, the fluid-type system (3.2.42) becomes

$$\begin{cases} \rho_t + \mathrm{div}_x m = 0, \\ m_t^i + \sum_{j=1}^3 (u^j m^i)_{x^j} + p_{x^i} = \sum_{j=1}^3 \left[\mu(\theta)\left(u^i_{x^j} + u^j_{x^i} - \frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right]_{x^j} \\ \qquad\qquad - \int_{\mathbb{R}^3} \psi_i(\xi)\left(\xi \cdot \nabla_x \Theta\right) d\xi, \quad i = 1,2,3, \\ \left[\rho(\frac{1}{2}|u|^2 + \mathcal{E})\right]_t + \sum_{j=1}^3 \left(u^j\left(\rho\left(\frac{1}{2}|u|^2 + \mathcal{E}\right) + p\right)\right)_{x^j} \\ \qquad = \sum_{i,j=1}^3 \left\{\mu(\theta) u^i\left(u^i_{x^j} + u^j_{x^i} - \frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right\}_{x^j} \\ \qquad\quad + \sum_{j=1}^3 (\kappa(\theta)\theta_{x^j})_{x^j} - \int_{\mathbb{R}^3} \psi_4(\xi)\left(\xi \cdot \nabla_x \Theta\right) d\xi. \end{cases} \tag{3.2.43}$$

From this fluid-type system, one can easily see the structure of the compressible Euler and the compressible Navier-Stokes equations. For instance, when the Knudsen number κ and Θ are set zero, the system (3.2.43) becomes the compressible Euler equations. On the other hand, when Θ is set to be zero in (3.2.43), it becomes the compressible Navier-Stokes equations. These fluid equations as derived through the Hilbert and Chapman-Enskog expansions are approximations to the Boltzmann equation. However, the above system is part of the Boltzmann equation. Nevertheless, this reformulation is consistent in spirit with the Chapman-Enskog expansion in that the higher order terms beyond zeroth order in the expansions must be microscopic. Therefore, it is interesting to notice that the first order approximation in Chapman-Enskog expansion f_1 is just the leading term in the microscopic component expression (3.2.41), that is,

$$f_1 = \mathbf{L}_{\mathbf{M}}^{-1}(\mathbf{P}_1^{\mathbf{M}}(\xi \cdot \nabla_x \mathbf{M})).$$

Throughout the rest of this chapter, the system consisting of the equation (3.2.41) and the conservation laws (3.2.43) will be used to study the behavior of the solutions to the Boltzmann equation. It is shown that the forthcoming energy method based on this decomposition is robust in the stability and the convergence rate analysis on the non-trivial solution profiles to the Boltzmann equation.

3.3 Perturbation of Global Maxwellian

With the reformulation of the Boltzmann equation, the energy method which is useful in the study of nonlinear partial differential equations can then be applied to the study of the Boltzmann equation directly. As the first step in this direction, we will illustrate this approach by considering the case of a perturbation of a given global Maxwellian. It will be shown that the stability argument mainly depends on the analyzing conservation laws (3.2.43) by treating the extra terms beyond the Navier-Stokes equations as source terms. Moreover, the estimation on the microscopic component is obtained by using the H-theorem.

We should point out that for the perturbation of a global Maxwellian, the analysis can also be carried out by using the decomposition around the global Maxwellian. However, for perturbation of a non-trivial solution profile such as a wave pattern, the decomposition around the local Maxwellian is more useful and the description of the time evolution of the macroscopic and microscopic components in the solution is clearer.

Since the solutions considered here is a small perturbation of a given global Maxwellian $\overline{\mathbf{M}}(\xi)$, in the following discussion, the Knudsen number κ is chosen to be 1 for simplicity. And for later use, we recall some Sobolev inequalities in the following lemma.

Lemma 3.3.1 *For $g(x) \in H^1(\mathbb{R}^3)$, we have*

$$\|g(x)\|_{L^6(\mathbb{R}^3)} \leq C_0 \|\nabla_x g(x)\|, \tag{3.3.1}$$

where C_0 is a positive constant independent of $g(x)$. Consequently, for $g(x) \in H^2(\mathbb{R}^3)$, there exists a positive constant C_1 independent of $g(x)$ such that

$$\begin{cases} \|g(x)\|_{L^\infty(\mathbb{R}^3)} \leq C_1 \|\nabla_x g(x)\|_1, \\ \\ \|g(x)\|_{L^4(\mathbb{R}^3)} \leq C_1 \|\nabla_x g(x)\|^{\frac{3}{4}} \|g(x)\|^{\frac{1}{4}}. \end{cases} \tag{3.3.2}$$

Here and in the sequel, $\|\cdot\|$ and $\|\cdot\|_s$ denote the standard $L^2(\mathbb{R}^3)-$norm and $H^s(\mathbb{R}^3)-$norm respectively.

As pointed out before, to perform the energy method for the Boltzmann equation (3.2.1), for $\mathbf{P}_1^{\mathbf{M}_0} f$, the microscopic projection of its solution $f(t, x, \xi)$ with respect to a given Maxwellian $\mathbf{M}_0$, the dissipative effect through the microscopic H-theorem should be used. In short, the microscopic H-theorem states that the linearized collision operator $\mathbf{L}_{\mathbf{M}_0}$ around a fixed Mawellian state $\mathbf{M}_0$ is negative definite on the non-fluid element $\mathbf{P}_1^{\mathbf{M}_0} f$, i.e.,

$$-\int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\mathbf{M}_0} f \mathbf{L}_{\mathbf{M}_0} \left(\mathbf{P}_1^{\mathbf{M}_0} f\right)}{\mathbf{M}_0} d\xi \geq \sigma \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}_0}(\xi) \left|\mathbf{P}_1^{\mathbf{M}_0} f\right|^2}{\mathbf{M}_0} d\xi,$$

for a positive constant σ. Furthermore, one can vary the background Maxwellians for linearization and the weight function. That is, we also have the following dissipative estimate coming from Lemma 3.1.2.

Lemma 3.3.2 *If $\frac{\theta}{2} < \tilde{\theta}$ and the assumptions in Lemma 3.1.2 are satisfied, then there exist two positive constants $\overline{\sigma} = \overline{\sigma}(u, \theta; \tilde{u}, \tilde{\theta})$ and $\eta_0 = \eta_0(u, \theta; \tilde{u}, \tilde{\theta})$ such that if $|u - \tilde{u}| + |\theta - \tilde{\theta}| < \eta_0$, we have for $h(\xi) \in \mathcal{N}^{\perp}$,*

$$-\int_{\mathbb{R}^3} \frac{h\mathbf{L_M} h}{\tilde{\mathbf{M}}} d\xi \geq \overline{\sigma} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi) h^2}{\tilde{\mathbf{M}}} d\xi. \tag{3.3.3}$$

Here $\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(\xi)$, $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}(\xi)$ and

$$\mathcal{N}^{\perp} = \left\{ f(\xi) : \quad \int_{\mathbf{R}^3} \psi_j(\xi) f(\xi) d\xi = 0, \; j = 0, 1, 2, 3, 4. \right\}.$$

Remark 3.3.3 *The constant η_0 is some positive constant depending on the first non-zero eigenvalue of the linearized operator $\mathbf{L_M}$. Note that η_0 is not necessary to be small.*

A direct consequence of Lemma 3.3.2 and the Cauchy-Schwartz inequality is the following corollary.

Corollary 3.3.4 *Under the assumptions in Lemma 3.3.2, we have for $h(\xi) \in \mathcal{N}^{\perp}$,*

$$\int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)}{\tilde{\mathbf{M}}} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leq \overline{\sigma}^{-2} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)^{-1} h^2(\xi)}{\tilde{\mathbf{M}}} d\xi. \tag{3.3.4}$$

To view the H-theorem in the fluid dynamical variables, we first set

$$-\frac{3}{2}\rho S \equiv \int_{\mathbb{R}^3} \boldsymbol{M} \log \boldsymbol{M} \; d\xi. \tag{3.3.5}$$

It is easy to see that $\log \boldsymbol{M}$ and $\frac{\partial_t \boldsymbol{M}}{\boldsymbol{M}}$ are collision invariants and so:

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} \boldsymbol{G} \; \log \boldsymbol{M} d\xi = \int_{\mathbb{R}^3} \boldsymbol{G} \; \frac{\partial_t \boldsymbol{M}}{\boldsymbol{M}} d\xi = 0, \\ \displaystyle\int_{\mathbb{R}^3} Q(f, f) \log \boldsymbol{M} d\xi = 0. \end{cases} \tag{3.3.6}$$

Multiply the Boltzmann equation by $\log \boldsymbol{M}$ and integrate in ξ:

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{\mathbb{R}^3} (\boldsymbol{M} + \boldsymbol{G}) \; \log \boldsymbol{M} \; d\xi + \nabla_x \cdot \int_{\mathbb{R}^3} \xi (\boldsymbol{M} + \boldsymbol{G}) \; \log \boldsymbol{M} d\xi \\ &\quad - \int_{\mathbb{R}^3} \frac{(\boldsymbol{M} + \boldsymbol{G}) \boldsymbol{M}_t}{\boldsymbol{M}} d\xi - \int_{\mathbb{R}^3} \frac{(\boldsymbol{M} + \boldsymbol{G}) \xi \cdot \nabla_x \boldsymbol{M}}{\boldsymbol{M}} d\xi = \int_{\mathbb{R}^3} Q(f, f) \; \log \boldsymbol{M} d\xi. \end{aligned} \tag{3.3.7}$$

Use (3.3.6) to simplify this into:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\boldsymbol{M}\log\boldsymbol{M}d\xi+\nabla_x\cdot\int_{\mathbb{R}^3}\xi\boldsymbol{M}\log\boldsymbol{M}d\xi+\nabla_x\cdot\int_{\mathbb{R}^3}\xi\boldsymbol{G}\log\boldsymbol{M}d\xi$$
$$-\int_{\mathbb{R}^3}\boldsymbol{M}_t d\xi-\nabla_x\cdot\int_{\mathbb{R}^3}\xi\boldsymbol{M}d\xi=\int_{\mathbb{R}^3}\frac{\boldsymbol{G}\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}d\xi. \quad (3.3.8)$$

Note that, from the continuity equation,

$$-\int_{\mathbb{R}^3}\boldsymbol{M}_t d\xi-\nabla_x\cdot\int_{\mathbb{R}^3}\xi\boldsymbol{M}d\xi=-\rho_t-\nabla_x\cdot(\rho u)=0. \quad (3.3.9)$$

Also from (3.3.5),

$$\int_{\mathbb{R}^3}\xi\boldsymbol{M}\log\boldsymbol{M}d\xi=u\int_{\mathbb{R}^3}\boldsymbol{M}\log\boldsymbol{M}d\xi+\int_{\mathbb{R}^3}(\xi-u)\boldsymbol{M}\log\boldsymbol{M}d\xi=-\frac{3}{2}u\rho S+0, \quad (3.3.10)$$

we have

$$-\frac{3}{2}(\rho S)_t-\frac{3}{2}\nabla_x\cdot(\rho Su)+\nabla_x\cdot\left(\int_{\mathbb{R}^3}\xi\boldsymbol{G}\log\boldsymbol{M}d\xi\right)=\int_{\mathbb{R}^3}\frac{\boldsymbol{G}\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}d\xi. \quad (3.3.11)$$

Plugging (3.2.41) into the right-hand side of (3.3.11) gives,

$$-\frac{3}{2}(\rho S)_t-\frac{3}{2}\nabla_x\cdot(\rho Su)+\nabla_x\cdot\left(\int_{\mathbb{R}^3}\xi\boldsymbol{G}\log\boldsymbol{M}d\xi\right)$$
$$=\int_{\mathbb{R}^3}\frac{\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L_{\mathbf{M}}^{-1}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})d\xi$$
$$+\int_{\mathbb{R}^3}\frac{\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L_{\mathbf{M}}^{-1}\left(\boldsymbol{G}_t+\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{G}-Q(\boldsymbol{G},\boldsymbol{G})\right)d\xi, \quad (3.3.12)$$

that is,

$$-\frac{3}{2}(\rho S)_t-\frac{3}{2}\nabla_x\cdot(\rho Su)+\nabla_x\cdot\left(\int_{\mathbb{R}^3}\xi\boldsymbol{G}\log\boldsymbol{M}d\xi\right)$$
$$=\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L_{\mathbf{M}}^{-1}(\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M})d\xi$$
$$+\int_{\mathbb{R}^3}\frac{\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{M}}{\boldsymbol{M}}L_{\mathbf{M}}^{-1}\left(\boldsymbol{G}_t+\boldsymbol{P}_1\xi\cdot\nabla_x\boldsymbol{G}-Q(\boldsymbol{G},\boldsymbol{G})\right)d\xi. \quad (3.3.13)$$

Note that from (3.3.5), we have

$$-\frac{3}{2}\rho S=\rho\log\rho-\frac{3}{2}\rho\log(2\pi R\theta)-\frac{1}{2}\rho. \quad (3.3.14)$$

Before performing the energy estimates for the Boltzmann equation (3.2.1), we first give the function space for the solutions considered in this section

$$\mathbf{H}_{x,\xi}^N([0,T])=\left\{g(t,x,\xi)\;\middle|\;\begin{array}{c}\frac{\partial_t^\beta\partial_x^\alpha g(t,x,\xi)}{\sqrt{\overline{\mathbf{M}}(\xi)}}\in\mathbf{BC}_t\left([0,T],L_{x,\xi}^2(\mathbb{R}^3\times\mathbb{R}^3)\right)\\ \frac{\sqrt{\nu_{\overline{\mathbf{M}}}(\xi)}\partial_t^\beta\partial_x^\alpha g(t,x,\xi)}{\sqrt{\overline{\mathbf{M}}(\xi)}}\in L_{t,x,\xi}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3),\ \text{for } |\alpha|+|\beta|>0\\ 0\le t\le T,\quad |\alpha|+|\beta|\le N\end{array}\right\}.$$

Here $g(t,x,\xi) = f(t,x,\xi) - \overline{\mathbf{M}}(\xi)$.

The estimate on the conserved quantities

$$m(t,x) \equiv \left(\rho, \rho(t,x)u(t,x), \rho(t,x)(\frac{1}{2}u^2(t,x) + \mathcal{E}(t,x))\right)$$

can be obtained by using the analytic techniques for the systems of conservation laws cf.[97, 102], based on the following a priori assumption

$$\begin{aligned} N(t)^2 \equiv & \sup_{0\le\tau\le t} \sum_{|\alpha|+|\beta|\le 4} \int_{\mathbf{R}^3} \left| \partial_x^\alpha \partial_t^\beta \left(\rho(\tau,x) - \overline{\rho}, u(\tau,x), \theta(\tau,x) - \overline{\theta}\right)\right|^2 dx \\ & + \sup_{0\le\tau\le t} \sum_{|\alpha|+|\beta|\le 4} \int_{\mathbb{R}^3} \int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta (f(\tau,x,\xi) - \overline{\mathbf{M}}(\xi))\right|^2}{\overline{\mathbf{M}}} d\xi dx \\ \le & \ \varepsilon^2. \end{aligned} \tag{3.3.15}$$

Here the discussion is in $\mathbf{H}^4_{x,\xi}([0,T])$ space which can be readily generalized to the $\mathbf{H}^s_{x,\xi}([0,T])$ space for $s > 4$.

The a priori estimate (3.3.15) and the conservation laws imply that

$$N(0) \le O(1)\mathcal{E}(f_0) \tag{3.3.16}$$

with

$$\mathcal{E}(f_0)^2 \equiv \sum_{|\alpha|+|\beta|\le 4} \int_{\mathbb{R}^3} \int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta (f_0(x,\xi) - \overline{\mathbf{M}}(\xi))\right|^2}{\overline{\mathbf{M}}} d\xi dx$$

and

$$\begin{aligned} & \sup_{0\le\tau\le t,\ x\in\mathbf{R}^3} \left(\sum_{|\alpha|+|\beta|\le 2} \left| \partial_x^\alpha \partial_t^\beta \left(\rho(\tau,x) - \overline{\rho}, u(\tau,x), \theta(\tau,x) - \overline{\theta}\right)\right|^2 \right) \\ & \quad + \sup_{0\le\tau\le t,\ x\in\mathbf{R}^3} \left(\sum_{|\alpha|+|\beta|\le 2} \int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta (f(\tau,x,\xi) - \overline{\mathbf{M}}(\xi))\right|^2}{\overline{\mathbf{M}}} d\xi \right) \\ & \le O(1)\varepsilon^2. \end{aligned} \tag{3.3.17}$$

The lower order estimate on the macroscopic components can be given by a convex entropy functional defined as follows. Let $\mathbf{m} \equiv (m^0, m, m^4)^t = (m^0, m^1, m^2, m^3, m^4)^t = \left(\rho, m, \rho\left(\frac{1}{2}u^2 + \mathcal{E}\right)\right)^t$, we now construct a pair of convex entropy-entropy flux pair (η, q) around the global Maxwellian $\overline{\mathbf{M}}$. First, we denote the conservation laws (3.2.42) as:

$$\mathbf{m}_t + \nabla_x \cdot \mathbf{n} = 0. \tag{3.3.18}$$

Set the entropy pair as:

$$\begin{cases} \eta = -\frac{3}{2}\rho S + \frac{3}{2}\bar{\rho}\bar{S} + \frac{3}{2}\ \nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\bar{\mathbf{m}}} \cdot (\mathbf{m} - \bar{\mathbf{m}}), \\ q = -\frac{3}{2}\rho S u + \frac{3}{2}\bar{\rho}\bar{S}\bar{u} + \frac{3}{2}\ \nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\bar{\mathbf{m}}} \cdot (\mathbf{n} - \bar{\mathbf{n}}). \end{cases} \tag{3.3.19}$$

That is,

$$\begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \overline{\theta}\rho S + \rho\left[\left(\overline{S} - \frac{5}{3}\right)\overline{\theta} + \frac{|u|^2}{2}\right] + \frac{2}{3}\overline{\rho}\overline{\theta}\right\}, \\ q_j = u_j\eta + u_j\left(\rho\theta - \overline{\rho}\overline{\theta}\right), j = 1, 2, 3. \end{cases} \tag{3.3.20}$$

It is straightforward to check that $\bar{\eta} = \eta(\bar{\mathbf{m}}) = 0$, $\nabla_{\mathbf{m}}\bar{\eta} = \nabla_{\mathbf{m}}\eta(\bar{\mathbf{m}}) = 0$, and the Hessian $\frac{\partial^2\eta}{\partial m_i \partial m_j}$ equals to

$$\frac{3\overline{\theta}}{2\rho\theta^2}\begin{pmatrix} \frac{5}{3}\theta^2 + \frac{1}{4}|u|^4 & -\frac{1}{2}u_1|u|^2 & -\frac{1}{2}u_2|u|^2 & -\frac{1}{2}u_3|u|^2 & \frac{1}{2}|u|^2 - \theta \\ -\frac{1}{2}u_1|u|^2 & \theta + u_1^2 & u_1u_2 & u_1u_3 & -u_1 \\ -\frac{1}{2}u_2|u|^2 & u_1u_2 & \theta + u_2^2 & u_2u_3 & -u_2 \\ -\frac{1}{2}u_3|u|^2 & u_1u_3 & u_2u_3 & \theta + u_3^2 & -u_3 \\ \frac{1}{2}|u|^2 - \theta & -u_1 & -u_2 & -u_3 & 1 \end{pmatrix},$$

which is positive definite for any $\mathbf{m}$ satisfying ρ, $\theta > 0$. Thus, in any closed bounded region $\mathcal{D} \subset \Sigma = \{\mathbf{m} : \rho > 0, \theta > 0\}$, there exists a positive constant c depending on $\mathcal{D}$ such that

$$c^{-1}\left|\mathbf{m} - \overline{\mathbf{m}}\right|^2 \le \eta \le c\left|\mathbf{m} - \overline{\mathbf{m}}\right|^2 \tag{3.3.21}$$

The equation for the entropy can be derived as follows. From (3.3.13) and (3.3.18),

$$\begin{aligned} \eta_t + \nabla_x \cdot q + \nabla_x \cdot \left(\int_{\mathbb{R}^3} \xi \boldsymbol{G} \log \boldsymbol{M} d\xi\right) \\ = \int_{\mathbb{R}^3} \frac{\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}}{\boldsymbol{M}} L_{\mathbf{M}}^{-1}(\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}) d\xi \\ + \int_{\mathbb{R}^3} \frac{\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}}{\boldsymbol{M}} L_{\mathbf{M}}^{-1}\left(\boldsymbol{G}_t + \boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{G} - Q(\boldsymbol{G}, \boldsymbol{G})\right) d\xi. \end{aligned} \tag{3.3.22}$$

In the above identity, the term

$$\int_{\mathbb{R}^3} \frac{\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}}{\boldsymbol{M}} L_{\mathbf{M}}^{-1}(\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}) d\xi = \langle \boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}, L_{\mathbf{M}}^{-1}(\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M})\rangle_{\mathbf{M}},$$

represents the entropy dissipation. Since the non-fluid functions $\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}$ belong to a finite dimensional space in the ξ variables, we have from the microscopic version of the H-theorem that the term satisfies, for some positive constants σ_1 and σ_2,

$$\sigma_1 \int_{\mathbb{R}^3} \frac{\left|\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}\right|^2}{\boldsymbol{M}} \le -\int_{\mathbb{R}^3} \frac{\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}}{\boldsymbol{M}} L_{\mathbf{M}}^{-1}(\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}) d\xi \le \sigma_2 \int_{\mathbb{R}^3} \frac{\left|\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}\right|^2}{\boldsymbol{M}} d\xi. \tag{3.3.23}$$

Simple calculation shows that

$$\int_{\mathbb{R}^3} \frac{\left|\boldsymbol{P}_1\xi \cdot \nabla_x \boldsymbol{M}\right|^2}{\boldsymbol{M}} d\xi = O(1)\sum_{j=1}^{3}[|\partial_{x^j} u|^2 + |\nabla_x \theta|^2], \tag{3.3.24}$$

for some positive function $O(1)$. Notice that in the macroscopic version (3.3.22) of H-theorem, the dominant term on the right hand side is the first integral, which, as we have just seen, represents the dissipation, and the second integral consists of only higher order derivatives and the quadratic term of microscopic component G. Therefore, it captures the dissipative effect of the fluid components in the solution of Boltzmann equation, and this is useful for the energy estimates.

Precisely, by the conservation laws (3.2.43), (η, q_1, q_2, q_3) satisfies:

$$\begin{aligned}\eta_t + \mathrm{div}_x q = & \sum_{i,j=1}^{3} \eta_{m^i}(\mathbf{m}) \left[\mu(\theta)\left(u^i_{x^j} + u^j_{x^i} - \tfrac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right]_{x^j} \\ & + \sum_{i,j=1}^{3} \eta_{m^4}(\mathbf{m}) \left[\mu(\theta)u^i\left(u^i_{x^j} + u^j_{x^i} - \tfrac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right]_{x^j} + \sum_{j=1}^{3} \eta_{m^4}(\mathbf{m})\Big(\kappa(\theta)\theta_{x^j}\Big)_{x^j} \\ & - \int_{\mathbb{R}^3} \nabla_{\mathbf{m}}\eta(\mathbf{m}) \cdot \Big(0, \psi_1(\xi), \psi_2(\xi), \psi_3(\xi), \psi_4(\xi)\Big)\Big(\xi \cdot \nabla_x \Theta\Big) d\xi. \end{aligned} \tag{3.3.25}$$

Since

$$\nabla_{\mathbf{m}}\eta(\mathbf{m}) = -\frac{3}{2}\overline{\theta}\left(S + \frac{|u|^2}{2\theta} - \overline{S}, -\frac{u^1}{\theta}, -\frac{u^2}{\theta}, -\frac{u^3}{\theta}, -\frac{\theta - \overline{\theta}}{\theta\overline{\theta}}\right), \tag{3.3.26}$$

integrating (3.3.25) with respect to t and x over $[0,t] \times \mathbb{R}^3$, and using the Cauchy-Schwarz inequality and (3.3.21) give

$$\begin{aligned} & \left\|(\rho - \overline{\rho}, u, \theta - \overline{\theta})\right\|^2(t) + \int_0^t \|\nabla_x(u,\theta)\|^2(\tau)d\tau \\ \le\ & O(1)\left\|(\rho_0(x) - \overline{\rho}, u_0(x), \theta_0(x) - \overline{\theta})\right\|^2 + O(1)\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\xi|^4|\Theta|^2 d\xi dx d\tau. \end{aligned} \tag{3.3.27}$$

When ε is chosen sufficiently small such that

$$\varepsilon < \min\left\{\frac{\eta_0}{2},\ \overline{\theta}\right\}, \tag{3.3.28}$$

then for any θ_- satisfying

$$\overline{\theta} < \theta_- < \overline{\theta} + \varepsilon, \tag{3.3.29}$$

we have

$$\begin{cases} \theta \le \overline{\theta} + |\theta - \overline{\theta}| < \overline{\theta} + \varepsilon < 2\overline{\theta} < 2\theta_-, \\ |u| + |\theta - \theta_-| \le \Big(|u| + |\theta - \overline{\theta}|\Big) + \theta_- - \overline{\theta} < 2\varepsilon < \eta_0. \end{cases} \tag{3.3.30}$$

Denote $\mathbf{M}_- = \mathbf{M}_{[\overline{\rho},0,\theta_-]}$, we have from Lemma 3.1.2, Corollary 3.3.4, (3.1.19), (3.3.21), (3.3.15), (3.3.17) and (3.3.30) that

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbf{R}^3} |\xi|^4|\Theta|^2 d\xi dx \le\ & O(1)\int_{\mathbf{R}^3}\int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|L^{-1}_{\mathbf{M}}\left(\mathbf{G}_t + \mathbf{P}^{\mathbf{M}}_1(\xi\cdot\nabla_x\mathbf{G}) - Q(\mathbf{G},\mathbf{G})\right)\right|^2}{\mathbf{M}_-} d\xi dx \\ \le\ & O(1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tfrac{1}{\overline{\mathbf{M}}}\left(\mathbf{G}_t^2 + |\nabla_x \mathbf{G}|^2 + \varepsilon^2\mathbf{G}^2\right) d\xi dx. \end{aligned} \tag{3.3.31}$$

Putting (3.3.27) and (3.3.31) together gives

$$\begin{aligned}
&\left\|(\rho-\overline{\rho},u,\theta-\overline{\theta})\right\|^2(t)+\int_0^t\|\nabla_x(u,\theta)\|^2(\tau)d\tau\\
\leq\ & O(1)\left\|(\rho_0(x)-\overline{\rho},u_0(x),\theta_0(x)-\overline{\theta})\right\|^2\\
&+O(1)\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tfrac{1}{\overline{\mathbf{M}}}\left(\mathbf{G}_t^2+|\nabla_x\mathbf{G}|^2+\varepsilon^2\mathbf{G}^2\right)d\xi dxd\tau. \qquad (3.3.32)
\end{aligned}$$

To obtain the higher order estimates on the conserved quantities, we first note that the system (3.3.27) can be rewritten as

$$\begin{cases}
\rho_t=-(\rho-\overline{\rho})\text{div}_x u-\nabla_x(\rho-\overline{\rho})\cdot u-\overline{\rho}\text{div}_x u,\\
u_t^i+\sum\limits_{j=1}^3 u^j u^i_{x^j}+\frac{2}{3\rho}\left(\rho\theta-\overline{\rho}\overline{\theta}\right)_{x^i}=-\int_{\mathbb{R}^3}\frac{\psi_i(\xi\cdot\nabla_x\Theta)}{\rho}d\xi\\
\qquad\qquad +\frac{1}{\rho}\sum\limits_{j=1}^3\left\{\mu(\theta)\left(u^i_{x^j}+u^j_{x^i}-\frac{2}{3}\delta_{ij}\text{div}_x u\right)\right\}_{x^j},\quad i=1,2,3,\\
\theta_t+\sum\limits_{j=1}^3\left(u^j\theta_{x^j}+\frac{2}{3}\theta u^j_{x^j}\right)=-\int_{\mathbb{R}^3}\frac{\psi_4-\xi\cdot u}{\rho}(\xi\cdot\nabla_x\Theta)d\xi\\
\qquad\qquad +\frac{1}{\rho}\left\{\sum\limits_{j=1}^3(\kappa(\theta)\theta_{x^j})_{x^j}+\frac{1}{2}\mu(\theta)\sum\limits_{i,j=1}^3\left(u^i_{x^j}+u^j_{x^i}\right)^2-\frac{2}{3}\mu(\theta)(\text{div}_x u)^2\right\}.
\end{cases}\qquad (3.3.33)$$

Similar to the analysis for the compressible Navier-Stokes equations, by applying $\partial^\gamma=\partial_x^\alpha\partial_t^\beta(1\leq|\alpha|+|\beta|\leq 3)$ to $(3.3.33)_2$ and $(3.3.33)_3$, multiplying the resulting identities by $\rho\partial^\gamma u_i$ and $\frac{\overline{\rho}}{\theta}\partial^\gamma\theta$, taking the summation with respect to i from 1 to 3, and integrating the resulting equations with respect to t and x over $[0,t]\times\mathbb{R}^3$, for $j=1,2,3$, we have

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|=j}\int_{\mathbb{R}^3}\left|\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho},u,\theta)\right|^2dx+\sum_{|\alpha|+|\beta|=j}\int_0^t\int_{\mathbb{R}^3}\left|\nabla_x\partial_x^\alpha\partial_t^\beta(u,\theta)\right|^2dxd\tau\\
\leq\ & O(1)\mathcal{E}(f_0)^2+O(1)\sum_{|\gamma|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left(\left|\partial_x^\alpha\partial_t^\beta\mathbf{G}_t\right|^2+\left|\nabla_x\partial_x^\alpha\partial_t^\beta\mathbf{G}\right|^2\right)}{\overline{\mathbf{M}}}d\xi dxd\tau\\
&+O(1)\varepsilon\sum_{1\leq|\alpha|+|\beta|\leq j+1}\int_0^t\int_{\mathbb{R}^3}\left|\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho},u,\theta)\right|^2dxd\tau\\
&+O(1)\varepsilon\sum_{|\alpha|+|\beta|\leq j}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\mathbf{G}\right|^2}{\overline{\mathbf{M}}}d\xi dxd\tau. \qquad (3.3.34)
\end{aligned}$$

It is worthy to pointing out that, compared with the estimates for the Navier-Stokes equations, the only difference comes from the terms containing Θ. This can be estimated suitably as in the proof of (3.3.31) by using Lemma 3.1.2, Corollary 3.3.4, (3.1.19), (3.3.21), (3.3.15), (3.3.17), (3.3.30) and the following basic estimate on the collision operator $Q(f,g)$

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}|\partial_x^\alpha\partial_t^\beta Q(\mathbf{G},\mathbf{G})|^2}{\mathbf{M}_-}d\xi dxd\tau\\ \le\ & O(1)\varepsilon\sum_{|\alpha'|+|\beta'|\le 4}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{G}|^2}{\overline{\mathbf{M}}}d\xi dxd\tau. \qquad (3.3.35)\end{aligned}$$

Here $|\alpha|+|\beta|\le 4$.

To get the $L^2_{t,x}$ estimates on $\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho})$ for $1\le|\alpha|+|\beta|\le 4$, we use the conservation laws (3.3.27) as in the study of Navier-Stokes equations to deduce that

$$\begin{aligned}&\sum_{|\alpha|+|\beta|=j+1}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho})|^2dxd\tau\\ \le\ & O(1)\mathcal{E}(f_0)^2+O(1)\int_{\mathbb{R}^3}\left(\sum_{|\alpha|+|\beta|=j+1}|\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho})|^2+\sum_{|\alpha|+|\beta|=j}|\partial_x^\alpha\partial_t^\beta u|^2\right)dx\\ &+O(1)\sum_{1\le|\alpha|+|\beta|\le j+1}\int_0^t\int_{\mathbb{R}^3}|\partial_x^\alpha\partial_t^\beta(u,\theta)|^2dxd\tau\\ &+O(1)\varepsilon\sum_{1\le|\alpha|+|\beta|\le j}\int_0^t\int_{\mathbb{R}^3}|\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho})|^2dxd\tau\\ &+O(1)\sum_{|\alpha|+|\beta|\le j+1}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\partial_t^\beta\mathbf{G}|^2}{\overline{\mathbf{M}}}d\xi dxd\tau,\quad j=0,1,2,3. \qquad (3.3.36)\end{aligned}$$

A suitable linear combination of (3.3.32), (3.3.34) and (3.3.36) yields an estimate on the conserved quantities $\left(\rho,m,\rho\left(\frac{1}{2}|u|^2+\mathcal{E}\right)\right)$ which is controlled by $\mathbf{G}$ besides the initial data.

Lemma 3.3.5 *Under the a priori assumption (3.3.15), we have*

$$\begin{aligned}&\sum_{|\alpha|+|\beta|\le 3}\left\|\partial_x^\alpha\partial_t^\beta(\rho-\overline{\rho},u,\theta-\overline{\theta})\right\|^2(t)+\sum_{1\le|\alpha|+|\beta|\le 4}\int_0^t\left\|\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2(\tau)d\tau\\ \le\ & O(1)\mathcal{E}(f_0)^2+O(1)\sum_{|\alpha|+|\beta|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\mathbf{G}\right|^2}{\overline{\mathbf{M}}}d\xi dxd\tau. \qquad (3.3.37)\end{aligned}$$

Since we want to close the energy estimates for the solution $f(t,x,\xi)$ of the Boltzmann equation by performing the energy estimates on the original equation (3.2.1) with respect to the global Maxwellian $\overline{\mathbf{M}}$, we need to transform the estimates on $\mathbf{G}$, the microscopic projection of $f(t,x,\xi)$ with respect to the local Maxwellian $\mathbf{M}$ into $\mathbf{P}_1^{\overline{\mathbf{M}}}f$, the microscopic projection of $f(t,x,\xi)$ with respect to the global Maxwellian $\overline{\mathbf{M}}$. For this purpose, by noticing

$$\mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{G}=\mathbf{G},\quad \mathbf{P}_1^{\overline{\mathbf{M}}}f=\mathbf{G}+\mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{M}, \qquad (3.3.38)$$

we need to obtain an estimate on $\mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{M}$ which is presented in the following lemma.

Lemma 3.3.6 *Under the assumptions of Lemma 3.3.5, we can deduce*

$$\int_{\mathbb{R}^3} \frac{(1+|\xi|)^k \left|\mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{M}\right|^2}{\mathbf{M}_0} d\xi \leq O(1) \left|\left(\rho-\overline{\rho}, u, \theta-\overline{\theta}\right)\right|^4. \tag{3.3.39}$$

Here $k>0$ is any positive constant and $\mathbf{M}_0 = \mathbf{M}_{[\rho_0,u_0,\theta_0]}$ can be any Maxwellian satisfying $\theta_0 > \frac{1}{2}\max\left\{\theta, \overline{\theta}\right\}$.

Consequently, we have for all $|\alpha|+|\beta| \leq 4$ that

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \mathbf{G}\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \leq & \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} f\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ & + O(1)\varepsilon \sum_{1\leq|\alpha|+|\beta|\leq 4} \int_0^t \left\|\partial_x^\alpha \partial_t^\beta (\rho, u, \theta)\right\|^2 (\tau) d\tau. \end{aligned} \tag{3.3.40}$$

Proof. We only prove (3.3.39) since (3.3.40) follows immediately from (3.3.38), (3.3.39), Lemma 3.3.1 and the a priori assumption (3.3.15).

To prove (3.3.39), first notice that $\mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{M}$ is a smooth function of $\rho,\ u,\ \theta$ and

$$\nabla_{(\rho,u,\theta)} \mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{M} = \mathbf{P}_1^{\overline{\mathbf{M}}}\left(\nabla_{(\rho,u,\theta)}\mathbf{M}\right).$$

Since

$$\mathbf{P}_1^{\overline{\mathbf{M}}}\left(\nabla_{(\rho,u,\theta)}\mathbf{M}\right)\Big|_{(\rho,u,\theta)=(\overline{\rho},0,\overline{\theta})} = 0,$$

we can easily deduce that $\mathbf{P}_1^{\overline{\mathbf{M}}}\mathbf{M}$ is quadratic with respect to $\left(\rho-\overline{\rho},\ u,\ \theta-\overline{\theta}\right)$ and (3.3.39) follows immediately. This completes the proof of Lemma 3.3.6.

The following corollary is a direct consequence of Lemma 3.3.5 and Lemma 3.3.6.

Corollary 3.3.7 *Under the assumptions in Lemma 3.3.2, we have*

$$\begin{aligned} & \sum_{|\alpha|+|\beta|\leq 3} \left\|\partial_x^\alpha \partial_t^\beta (\rho-\overline{\rho}, u, \theta-\overline{\theta})\right\|^2 (t) + \sum_{1\leq|\alpha|+|\beta|\leq 4} \int_0^t \left\|\partial_x^\alpha \partial_t^\beta (\rho, u, \theta)\right\|^2 (\tau) d\tau \\ \leq\ & O(1)\mathcal{E}(f_0)^2 + O(1) \sum_{|\alpha|+|\beta|\leq 4} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} f\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau. \end{aligned} \tag{3.3.41}$$

Now we can finalize the energy estimates on the solutions $f(t,x,\xi)$ of the Boltzmann equation. To this end, since $g(t,x,\xi) = f(t,x,\xi) - \overline{\mathbf{M}}(\xi)$ solves

$$\begin{aligned} g_t + \xi\cdot\nabla_x g = & \mathbf{L}_{\overline{\mathbf{M}}}\left(\mathbf{P}_1^{\overline{\mathbf{M}}} g\right) + Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}} g, \mathbf{P}_1^{\overline{\mathbf{M}}} g\right) \\ & + 2Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}} g, \mathbf{P}_0^{\overline{\mathbf{M}}}\left(\mathbf{M}-\overline{\mathbf{M}}\right)\right) + Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}}\left(\mathbf{M}-\overline{\mathbf{M}}\right), \mathbf{P}_0^{\overline{\mathbf{M}}}\left(\mathbf{M}-\overline{\mathbf{M}}\right)\right), \end{aligned} \tag{3.3.42}$$

by applying $\partial_x^\alpha \partial_t^\beta (|\alpha|+|\beta|\le 4)$ to (3.3.42) and integrating its product with $\frac{\partial_x^\alpha\partial_t^\beta g}{\overline{\mathbf{M}}}$ over $[0,t]\times\mathbb{R}^3\times\mathbb{R}^3$, we have that

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta g\right|^2}{\overline{\mathbf{M}}}d\xi dx\Big|_0^t = & \int_0^t\int_{\mathbf{R}^3}\int_{\mathbb{R}^3}\frac{\partial_x^\alpha\partial_t^\beta g\cdot\partial_x^\alpha\partial_t^\beta \mathbf{L}_{\overline{\mathbf{M}}}\left(\mathbf{P}_1^{\overline{\mathbf{M}}}g\right)}{\overline{\mathbf{M}}}d\xi dx d\tau\\
&+\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_x^\alpha\partial_t^\beta g\cdot\partial_x^\alpha\partial_t^\beta Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}}g,\mathbf{P}_1^{\overline{\mathbf{M}}}g\right)}{\overline{\mathbf{M}}}d\xi dx d\tau\\
&+2\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_x^\alpha\partial_t^\beta g\cdot\partial_x^\alpha\partial_t^\beta Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}}g,\mathbf{P}_0^{\overline{\mathbf{M}}}(\mathbf{M}-\overline{\mathbf{M}})\right)}{\overline{\mathbf{M}}}d\xi dx d\tau\\
&+\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_x^\alpha\partial_t^\beta g\cdot\partial_x^\alpha\partial_t^\beta Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}}(\mathbf{M}-\overline{\mathbf{M}}),\mathbf{P}_0^{\overline{\mathbf{M}}}(\mathbf{M}-\overline{\mathbf{M}})\right)}{\overline{\mathbf{M}}}d\xi dx d\tau\\
:= & \sum_{j=1}^4 I_j. \qquad (3.3.43)
\end{aligned}$$

Here $I_i (i=1,2,3,4)$ are the corresponding terms in the above equation without any ambiguity.

Since $\overline{\mathbf{M}}$ is independent of t and x, we have

$$\begin{cases}
\mathbf{P}_1^{\overline{\mathbf{M}}}\left(\partial_x^\alpha\partial_t^\beta g\right)=\partial_x^\alpha\partial_t^\beta g\mathbf{P}_1^{\overline{\mathbf{M}}}g,\\
\partial_x^\alpha\partial_t^\beta\mathbf{L}_{\overline{\mathbf{M}}}\left(\mathbf{P}_1^{\overline{\mathbf{M}}}g\right)=\mathbf{L}_{\overline{\mathbf{M}}}\left(\partial_x^\alpha\partial_t^\beta g\mathbf{P}_1^{\overline{\mathbf{M}}}g\right).
\end{cases} \qquad (3.3.44)$$

Thus from Lemma 3.1.2, Lemma 3.3.2 and (3.3.15), I_1 and I_2 satisfy

$$\begin{aligned}
I_1 = & \int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\overline{\mathbf{M}}}g\cdot\mathbf{L}_{\overline{\mathbf{M}}}\left(\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\overline{\mathbf{M}}}g\right)}{\overline{\mathbf{M}}}d\xi dx d\tau\\
\le & -\sigma\int_0^t\int_{\mathbb{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\overline{\mathbf{M}}}g\right|^2}{\overline{\mathbf{M}}}d\xi dx d\tau, \qquad (3.3.45)
\end{aligned}$$

$$\begin{aligned}
I_2 = & \int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\overline{\mathbf{M}}}g\cdot\partial_x^\alpha\partial_t^\beta Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}}g,\mathbf{P}_1^{\overline{\mathbf{M}}}g\right)}{\overline{\mathbf{M}}}d\xi dx d\tau\\
\le & \frac{\sigma}{4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\overline{\mathbf{M}}}g\right|^2}{\overline{\mathbf{M}}}d\xi dx d\tau\\
&+O(1)\varepsilon\sum_{|\alpha\|+|\beta'|\le 4}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\overline{\mathbf{M}}}g\right|^2}{\overline{\mathbf{M}}}d\xi dx d\tau. \qquad (3.3.46)
\end{aligned}$$

Here we have used the inequality

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|\partial_x^\alpha\partial_t^\beta Q\left(\mathbf{P}_1^{\overline{\mathbf{M}}}g,\mathbf{P}_1^{\overline{\mathbf{M}}}g\right)\right|^2}{\overline{\mathbf{M}}}d\xi dx d\tau\\
\le & O(1)\varepsilon\sum_{|\alpha\|+|\beta'|\le 4}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\overline{\mathbf{M}}}g\right|^2}{\overline{\mathbf{M}}}d\xi dx d\tau. \qquad (3.3.47)
\end{aligned}$$

For I_3 and I_4, we have

$$\begin{aligned} I_3 \leq\ & \frac{\sigma}{4}\int_0^t \int_{\mathbf{R}^3} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} g\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ & + O(1)\varepsilon \sum_{|\alpha'|+|\beta'|\leq 4} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^{\alpha'} \partial_t^{\beta'} \mathbf{P}_1^{\overline{\mathbf{M}}} g\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ & + O(1)\varepsilon \sum_{1\leq|\alpha'|+|\beta'|\leq 4} \int_0^t \int_{\mathbb{R}^3} \left|\partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right|^2 dx d\tau, \end{aligned} \tag{3.3.48}$$

and

$$\begin{aligned} I_4 \leq\ & \frac{\sigma}{4}\int_0^t \int_{\mathbf{R}^3} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} g\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ & + O(1)\varepsilon \sum_{1\leq|\alpha'|+|\beta'|\leq 4} \int_0^t \int_{\mathbb{R}^3} \left|\partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right|^2 dx d\tau. \end{aligned} \tag{3.3.49}$$

Now substituting (3.3.45), (3.3.46), (3.3.48) and (3.3.49) into (3.3.43) yields

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta g\right|^2}{\overline{\mathbf{M}}} d\xi dx + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} g\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ \leq\ & O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \sum_{|\alpha'|+|\beta'|\leq 4} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^{\alpha'} \partial_t^{\beta'} \mathbf{P}_1^{\overline{\mathbf{M}}} g\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ & + O(1)\varepsilon \sum_{1\leq|\alpha'|+|\beta'|\leq 4} \int_0^t \int_{\mathbb{R}^3} \left|\partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right|^2 dx d\tau. \end{aligned} \tag{3.3.50}$$

Thus

$$\begin{aligned} & \sum_{|\alpha|+|\beta|\leq 4} \int_{\mathbf{R}^3} \int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta g\right|^2}{\overline{\mathbf{M}}} d\xi dx + \sum_{|\alpha|+|\beta|\leq 4} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} g\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ \leq\ & O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \sum_{1\leq|\alpha|+|\beta|\leq 4} \int_0^t \int_{\mathbf{R}^3} \left|\partial_x^{\alpha} \partial_t^{\beta} (\rho, u, \theta)\right|^2 dx d\tau. \end{aligned} \tag{3.3.51}$$

Multiplying (3.3.51) by a suitably large positive constant C_4 and adding the result to (3.3.41) yield

$$\begin{aligned} & \sum_{|\alpha|+|\beta|\leq 4} \int_{\mathbf{R}^3} \int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta (f-\overline{\mathbf{M}})\right|^2}{\overline{\mathbf{M}}} d\xi dx + \sum_{|\alpha|+|\beta|\leq 4} \int_{\mathbf{R}^3} \left|\partial_x^\alpha \partial_t^\beta \left(\rho-\overline{\rho}, u, \theta-\overline{\theta}\right)\right|^2 dx \\ & + \sum_{|\alpha|+|\beta|\leq 4} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\overline{\mathbf{M}}} f\right|^2}{\overline{\mathbf{M}}} d\xi dx d\tau \\ & + \sum_{1\leq|\alpha|+|\beta|\leq 4} \int_0^t \int_{\mathbb{R}^3} \left|\partial_x^{\alpha} \partial_t^{\beta} (\rho, u, \theta)\right|^2 dx d\tau \\ & \leq O(1)\mathcal{E}(f_0)^2. \end{aligned} \tag{3.3.52}$$

This closes the a priori assumption (3.3.15) provided that we choose $\delta_0 > 0$ sufficiently small such that

$$\begin{cases} \mathcal{E}(f_0) < \delta_0, \\ O(1)\delta_0^2 < \varepsilon^2. \end{cases} \tag{3.3.53}$$

The above analysis yields the following energy estimates for the solution $f(t,x,\xi)$ of the Boltzmann equation with initial data $f_0(x,\xi)$.

Lemma 3.3.8 (Energy estimates) *Assume that $f(t,x,\xi) \in \mathbf{H}^4_{x,\xi}([0,T])$ is a solution of the Cauchy problem (3.2.1) and (2.3) for some constant $T > 0$. Then there exist two sufficiently small positive constants ε, δ_0 such that if $\mathcal{E}(f_0) < \delta_0$, we have*

$$N(T) < \varepsilon. \tag{3.3.54}$$

By combining Lemma 3.3.8 and the local existence theorem, the global existence theorem can be stated as follows.

Theorem 3.3.9 *Let $N \geq 4$ be an integer and $\overline{\mathbf{M}}(\xi)$ be any given global Maxwellian, then there exist two sufficiently small positive constants δ_0 and ε such that if*

$$\mathcal{E}(f_0) \equiv \sum_{|\alpha|+|\beta|\leq N} \int_{\mathbb{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta (f_0(x,\xi) - \overline{\mathbf{M}}(\xi))\right|^2}{\overline{\mathbf{M}}} d\xi dx < \delta_0, \tag{3.3.55}$$

the Cauchy problem (3.2.1) and (2.3) admits a unique global classical solution $f(t,x,\xi) \in \mathbf{H}^N_{x,\xi}(\mathbb{R}^+)$ satisfying $f(t,x,\xi) \geq 0$ and

$$\begin{cases} \sum\limits_{|\alpha|+|\beta|\leq N} \sup\limits_{t\in\mathbf{R}^+} \int_{\mathbb{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta (f(t,x,\xi) - \overline{\mathbf{M}}(\xi))\right|^2}{\overline{\mathbf{M}}} d\xi dx \leq \varepsilon, \\ \lim\limits_{t\to\infty} \sup\limits_{x\in\mathbf{R}^3} \sum\limits_{|\alpha|+|\beta|\leq N-3} \int_{\mathbb{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \left(f(t,x,\xi) - \overline{\mathbf{M}}(\xi)\right)\right|^2}{\overline{\mathbf{M}}(\xi)} d\xi = 0. \end{cases} \tag{3.3.56}$$

3.4 Stability of Wave Patterns

Through the Hilbert and Chapman-Enskog expansions or the decomposition, it is clear that the Boltzmann equation has close relation to the systems of fluid dynamics, such as the Euler and Navier-Stokes equations. For these systems of fluid dynamics, it is well known that the solution contains three basic wave patterns. Hence, it is natural to study the corresponding wave phenomena in the solutions to the Boltzmann equation. For this purpose, we will first review some basic concepts in the fluid dynamics, especially the systems of conservation laws.

The study of hyperbolic conservation laws in the form of

$$U_t + F(U)_x = 0, \tag{3.4.1}$$

has a long history and the earliest mathematical work can be traced back to Euler in 1755 on the study of acoustic waves. And the pioneer nonlinear formulation on the fluid dynamics was done by Riemann through the consideration of two stationary gases separated by a membrane when the membrane was suddenly removed. This fundamental work born the name of the Riemann problem is so essential that it plays an important role on the existence and stability theories. As a typical example of hyperbolic conservation laws, the system of compressible Euler equations has three basic wave patterns in the solution to the Riemann problem. They are two nonlinear waves, called shock and rarefaction waves, and one linearly degenerate wave called contact discontinuity. These dilation invariant solutions and their linear superposition in the increasing order of characteristic speeds, called Riemann solutions, govern both the local and large time asymptotic behavior of general solutions to the Euler system. Since the inviscid system is an idealization when the dissipative effects are neglected, thus it is of great importance to study the large time asymptotic behavior of solutions to the corresponding viscous systems in the form of

$$U_t + F(U)_x = (B(U)U_x)_x, \tag{3.4.2}$$

toward the viscous versions of these basic wave patterns. As a basic system for the viscous fluid, the compressible Navier-Stokes equations which include the effects of viscosity and heat conductivity, have the above wave phenomena which are smoothed out by the dissipative effect. Furthermore, coming from statistics physics for rarefied gas, the Boltzmann equation which describes the macroscopic and microscopic aspects in the non-equilibrium gas motion, has similar wave phenomena in the macroscopic level.

In this section, the stability of the above three wave patterns for the Boltzmann equation will be illustrated. It is worthy to pointing out that even though the stability of each wave pattern is now well understood, the stability of the wave pattern to the Riemann problem consisting of these basic wave patterns is still not known. It is somehow due to the differences in the analytic techniques used for different wave patterns and the different properties of the basic wave patterns in terms of monotonicity and decay rates. Notice also that these three wave patterns are the basic components in the wave patterns for general systems (3.4.1) and (3.4.2).

In the level of the compressible Navier-Stokes equations, there have been intensive studies in the respect of wave phenomena in the development of the mathematical theory for viscous systems of conservation laws since 1980's, started with studies on the nonlinear stability of viscous shock profiles. Deeper understanding has been achieved on the asymptotic stability toward nonlinear waves, viscous shock profiles and viscous rarefaction waves; and the linearly degenerate wave, contact discontinuities. They are shown to be nonlinearly stable with quite general perturbations for the compressible Navier-Stokes system and more general system of viscous strictly hyperbolic conservation laws (3.4.2). Moreover, some new phenomena have been discovered and new techniques, such as weighted characteristic energy methods and uniform approximate Green's functions, have been developed based on the intrinsic properties of the underlying wave patterns. Precisely, when the solution to the corresponding Riemann problem of the compressible Euler equations consists of only shock waves, the smooth solution profile to the compressible Navier-Stokes equations is the so called shock profile satisfying a system of differential equations with two given end states. Since shock wave is a compression wave, the monotone decreasing property of the characteristic speed in the shock profile plays a crucial role in the stability analysis. In different settings, the nonlinear stability of the shock profiles with smallness assumption on its wave strength has also been established.

When the solution to the corresponding Riemann problem consists of only rarefaction waves, the corresponding nonlinear stability results are also obtained in different settings. Notice that the rarefaction wave is an expansion wave and the monotone increasing property of the characteristics is also crucially used in the stability analysis. In particular, the stability of strong rarefaction wave can be studied. Moreover, it shows that, for the general gas, a global stability result holds for the non-isentropic ideal polytropic gas provided that the adiabatic exponent γ is close to 1. Furthermore, for the isentropic compressible Navier-Stokes equations, the corresponding global stability result holds provided that the resulting compressible Euler equations is strictly hyperbolic and both characteristic fields are genuinely nonlinear. Here, global stability means that the initial perturbation can be large. Since it does not require the strength of the rarefaction waves to be small, these results give the nonlinear stability of strong rarefaction waves for the one-dimensional compressible Navier-Stokes equations.

The problem of stability of contact discontinuities is more subtle because of its degeneracy. The contact waves for the systems of viscous conservation laws with uniform viscosity was shown to be metastabe. Moreover, the point wise asymptotic behavior toward viscous contact wave by approximate fundamental solutions leads to the nonlinear stability of the viscous contact wave in L_p-norms for all $p \geq 1$.

For the compressible Navier-Stokes equations, the nonlinear stability of a viscous contact wave to the free boundary value problem was proved in the sup-norm and then to the Cauchy problem with zero excessive mass condition which excludes the possible presence of diffusion waves in the sound wave families. The rigorous mathematical proof of the stability of contact wave for general perturbation was obtained recently which gives a satisfactory answer to the stability of this linearly degenerate wave.

Notice also that for the contact wave, a convergence rate of the order of $(1+t)^{-\frac{1}{4}}$ in sup-norm is by-product of the stability analysis. However, there is no convergence rates obtained so far for the two nonlinear wave, i.e., shock and rarefaction wave.

Based on the knowledge on the Navier-Stokes equations, the stability of wave patterns for the Boltzmann equation can be studied accordingly.

For the Boltzmann equation with a non-trivial solution profile connecting two different global Maxwellians at $x = \pm\infty$, it is reasonable and better to decompose the Boltzmann equation and its solution with respect to the local Maxwellian. As presented in the last section, the governing system for the fluid components is of fluid-type so that the techniques for the compressible Navier-Stokes equations can be applied with some extra terms coming from the non- fluid component. Moreover, the dissipative effect of the linearized operator on the non-fluid component helps to close the energy estimate for the Boltzmann equation. Similar to the compressible Navier-Stokes equations, the dissipative effect in the Boltzmann equation also spreads out the basic wave patterns so that the energy method can be applied.

3.4.1 Basic Wave Patterns

In this subsection, we first review some concepts and definitions of basic wave patterns to the system of fluid dynamics, that is, the systems of Euler equations and Navier-Stokes equations.

To have the picture of basic wave patterns, one can consider the Riemann problem for the one dimensional compressible Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (up)_x = 0, \end{cases} \tag{3.4.3}$$

with Riemann data

$$\left(v, u, e + \frac{u^2}{2}\right)(0, x) = \left(v_0^r, u_0^r, e_0^r + \frac{(u_0^r)^2}{2}\right)(x) = \begin{cases} \left(v_-, u_-, e_- + \frac{u_-^2}{2}\right), & x < 0, \\ \left(v_+, u_+, e_+ + \frac{u_+^2}{2}\right), & x > 0. \end{cases} \tag{3.4.4}$$

For simplicity of presentation and the consistency with the Botlzmann equation for monatomic gas, the gas is assumed to be ideal and polytropic so that the pressure p and the internal energy e have the following constitutive relation:

$$p(v, \theta) = \frac{R\theta}{v} = Av^{-\gamma} \exp\left(\frac{\gamma - 1}{r} s\right), \quad e(v, \theta) = \frac{R\theta}{\gamma - 1}, \tag{3.4.5}$$

where $R > 0$ is the gas constant, $\gamma > 1$ the adiabatic exponent and A a positive constant. In thermodynamics, by giving any two of the five thermodynamic variables, v, p, e, θ, and s, the remaining three are determined.

Denote the conserved quantities by

$$m(t, x) = \left(v, u, \theta + \frac{\gamma - 1}{2R} u^2\right)^t. \tag{3.4.6}$$

By finding the eigenvalues of the Jacobi matrix of the flux function with respect to the conservative quantities, the characteristic speeds of the Euler equations are

$$\lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\frac{\gamma p}{v}}, \tag{3.4.7}$$

with the corresponding right eigenvectors

$$r_2(m) = \begin{pmatrix} R \\ 0 \\ p \end{pmatrix}, \qquad r_i(m) = \begin{pmatrix} -1 \\ \lambda_i \\ \frac{(\gamma-1)p}{R} + \frac{(\gamma-1)u}{R} \lambda_i \end{pmatrix}, \quad i = 1, 3. \tag{3.4.8}$$

The first and third characteristic fields are genuinely nonlinear which give rise to rarefaction and shock waves, while the second characteristic field is linear degenerate which gives rise to contact discontinuities. We will assume in the following that the solution $m^r(t,x)$ of the above Riemann problem consists of only one single type of waves. More precisely, we assume that m_- and m_+ can be connected either by a contact discontinuity, i.e., $m_+ \in \mathcal{CD}(m_-) = \{m \in \mathbf{R}^3 : \quad p = p_-, \ u = u_-\}$; or by one 1−rarefaction wave and one 3−rarefaction wave, i.e., there exists a unique $\overline{m} \in \mathbb{R}^3$ such that $m_+ \in \mathcal{R}_1(m_-) \cup \mathcal{R}_3(\overline{m})$, where

$$\mathcal{R}_1(m_-) = \left\{ m \in \mathbb{R}^3 : \quad s = s_-, \ u - \int^v \sqrt{-p_v(z,s_-)}dz = u_- - \int^{v_-} \sqrt{-p_v(z,s_-)}dz, \ v \le v_- \right\},$$

and

$$\mathcal{R}_3(\overline{m}) = \left\{ m \in \mathbb{R}^3 : \quad s = \overline{s}, \ u + \int^v \sqrt{-p_v(z,s_-)}dz = \overline{u} + \int^{\overline{v}} \sqrt{-p_v(z,s_-)}dz, \ v \ge \overline{v} \right\};$$

or by one 1−shock wave and one 3−shock wave, i.e., there exists a unique $\overline{m} \in \mathbb{R}^3$ such that $m_+ \in \mathcal{S}_1(m_-) \cup \mathcal{S}_3(\overline{m})$, where

$$\mathcal{S}_1(m_-) = \left\{ m \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} \sigma(m_-,m)[v] = -[u], \quad \sigma[u] = [p] \\ \sigma(m_-,m)\left[e + \frac{u^2}{2}\right] = [up] \\ \lambda_1(m) < \sigma(m_-,m) < \lambda_1(m_-) \end{array} \right\},$$

and

$$\mathcal{S}_3(\overline{m}) = \left\{ m \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} \sigma(\overline{m},m_+)[v] = -[u], \quad \sigma(\overline{m},m_+)[u] = [p] \\ \sigma(\overline{m},m_+)\left[e + \frac{u^2}{2}\right] = [up] \\ \lambda_3(m) < \sigma(\overline{m},m_+) < \lambda_3(\overline{m}) \end{array} \right\}.$$

Corresponding to the above three cases, we simply write the Riemann solution $m^r(t,x)$ to (3.4.3) and (3.4.4) as $m^{\mathcal{CD}}(t,x)$, $m^{\mathcal{R}}(t,x) = m^{\mathcal{R}_1}(t,x) + m^{\mathcal{R}_3}(t,x) - \overline{m}$, and $m^{\mathcal{S}}(t,x) = m^{\mathcal{S}_1}(t,x) + m^{\mathcal{S}_3}(t,x) - \overline{m}$ respectively. It is worthy to pointing out that to guarantee the solvability of the Riemann problem (3.4.3) and (3.4.4), $|m_+ - m_-|$ is not necessary to be small.

In the Navier-Stokes level, the dissipation from the viscosity and heat conductivity spread out the discontinuity so that the shock and contact discontinuities become smooth shock profile and contact wave profile. Without any ambiguity, in the sequel, we sometimes still call such waves in the Navier-Stokes or Boltzmann level as shock and contact discontinuity to adhere to its original property in the Euler level.

The one-dimensional compressible Navier-Stokes equations in the Lagrangian coordinates takes the form

$$\begin{cases} v_t - u_x &= 0, \\ u_t + p_x &= \left(\mu \frac{u_x}{v}\right)_x, \\ \left(e + \frac{u^2}{2}\right)_t + (up)_x &= \left(\kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v}\right)_x, \end{cases} \tag{3.4.9}$$

where μ and κ are the coefficients of viscosity and heat-conductivity.

Now we assume that the initial data satisfy

$$\left(v,u,e+\frac{u^2}{2}\right)(0,x)=\left(v_0,u_0,e_0+\frac{u_0^2}{2}\right)(x)\to\left(v_\pm,u_\pm,e_\pm+\frac{u_\pm^2}{2}\right)\text{ as } x\to\pm\infty. \quad (3.4.10)$$

Here $v_\pm>0, e_\pm, u_\pm$ are constants. Depending on the values of these two states, the solution profile has the above three basic wave structures as illustrated below.

For the case when $m_+\in\mathcal{S}_1(m)\cup\mathcal{S}_3(\overline{m})$, (3.4.9) admits viscous shock profiles $M^{S_1}(x-\delta_1(m_-,\overline{m})t)$ and $M^{\mathcal{S}_3}(x-\sigma_3(\overline{m},m_+)t)$ which are unique up to a shift and satisfy $M^{\mathcal{S}_1}(-\infty)=m_-, M^{\mathcal{S}_1}(+\infty)=\overline{m}, M^{\mathcal{S}_3}(-\infty)=\overline{m}, M^{\mathcal{S}_3}(+\infty)=m_+$. Here the j-th characteristic speed $\lambda_j(m)$ is monotone decreasing along the j-th viscous shock profile $M^{S_j}(x)$, ie.

$$\frac{\partial\lambda_j\left(M^{S_j}(x)\right)}{\partial x}<0,\quad j=1,3. \quad (3.4.11)$$

Notice that the above monotone property of the characteristic speeds along viscous shock profiles is crucially used in the nonlinear stability analysis of viscous shock profiles.

Since the shock profile is orbital stable, for any constant vector $(\alpha,\beta)\in\mathbb{R}^2$, the shock profile with two shock waves of different families takes the form

$$M^S(t,x;\alpha,\beta)=M^{S_1}\Big(x-\sigma_1(m_-,\overline{m})t+\alpha\Big)+M^{S_3}\Big(x-\sigma_3(\overline{m},m_+)t+\beta\Big)-\overline{m}. \quad (3.4.12)$$

Both for Navier-Stokes equations and the Boltzmann equation, the rarefaction wave profile is not defined as an exact solution, instead, it is an approximate profile with monotonicity property. One way to construct the smooth approximate rarefaction profiles is to use the inviscid Burgers equation for the characteristic functions. That is, when $m_+\in\mathcal{R}_1(m_-)\cup\mathcal{R}_3(\overline{m})$, let $w_j(t,x)(j=1,3)$ be the global smooth solutions to the following Cauchy problem

$$\begin{cases}\lambda_{jt}+\lambda_j\lambda_{jx}= & 0,\\ \lambda_j(0,x)= & \lambda_{j0}(x)=\frac{\lambda_{j-}+\lambda_{j+}}{2}+\frac{\lambda_{j+}-\lambda_{j-}}{2}\tanh(\varepsilon x),\quad j=1,3,\end{cases} \quad (3.4.13)$$

where $\lambda_1=\lambda_1(m_-),\lambda_{1+}=\lambda_1(\overline{m}),\lambda_{3-}=\lambda_3(\overline{m}),\lambda_{3+}=\lambda_3(m_+)$, and $\varepsilon>0$ is a suitably small but fixed constant which is introduced here to control both the possible growth of the solution caused by the nonlinearity and the interactions of rarefaction waves from different families.

Since $\lambda_{j0}(x)$ is increasing, (3.4.13) admits a unique global smooth solution $w_j(t,x)$. By using $w_j(t,x)$, the smooth j-th approximate rarefaction profile $(V^{R_j}(t,x),U^{R_j}(t,x),S^{R_j}(t,x))$ can be constructed as follows:

$$\begin{cases} S^{R_1}(t,x)= & S^{R_3}(t,x)=s_-=s_+,\\ \lambda_j\left(V^{R_j}(t,x),s_-\right)= & w_j(t,x),\quad j=1,3,\\ U^{R_1}(t,x)-\displaystyle\int^{V^{R_1}(t,x)}\sqrt{-p_v(z,s_-)}dz= & u_--\displaystyle\int^{v_-}\sqrt{-p_v(z,s_-)}dz,\\ U^{R_3}(t,x)+\displaystyle\int^{V^{R_3}(t,x)}\sqrt{-p_v(z,s_-)}dz= & u_++\displaystyle\int^{v_+}\sqrt{-p_v(z,s_-)}dz.\end{cases} \quad (3.4.14)$$

Then the smooth functions $(V^{R_j}(t,x), U^{R_j}(t,x), S^{R_j}(t,x))$ are globally well-defined and solve the Euler equations (3.4.3) exactly. Moreover, they satisfy that the j-th characteristic speed $\lambda_j(m)$ is monotone increasing along the j-th approximate rarefaction profile $(V^{R_j}(t,x), U^{R_j}(t,x), S^{R_j}(t,x))$, i.e.

$$\frac{\partial \lambda_j\left(V^{R_j}(t,x), s_-\right)}{\partial x} > 0, \quad j = 1, 3, \tag{3.4.15}$$

and

$$\lim_{t\to+\infty} \lim_{x\in\mathbb{R}} \left|\left(V^{R_j}(t,x), U^{R_j}(t,x)\right) - \left(v^{\mathcal{R}_j}(t,x), u^{\mathcal{R}_j}(t,x)\right)\right| = 0, \quad j = 1, 3. \tag{3.4.16}$$

Finally, the contact wave corresponding to the contact discontinuity can be defined as a nonlinear diffusion wave approximately as follows. When $m_+ \in m^{\mathcal{CD}}(m_-)$, the contact wave profile $(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x))$ can be constructed as

$$V^{CD}(t,x) = \frac{R\Theta(t,x)}{p_+}, \; U^{CD}(t,x) = \frac{aR\Theta_x(t,x)}{p_+\Theta(t,x)}, \; \Theta^{CD}(t,x) = \Theta(t,x) - \frac{\gamma-1}{2R}\left(U^{CD}(t,x)\right)^2. \tag{3.4.17}$$

Here $\Theta(t,x) = \Theta\left(\frac{x}{\sqrt{t+1}}\right)$ is the unique self-similar solution to the following diffusion equation

$$\begin{cases} \theta_t = \left(a\frac{\theta_x}{\theta}\right)_x, & a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0, \\ \theta(-\infty) = \theta_-, & \theta(+\infty) = \theta_+. \end{cases}$$

It is straightforward to check that $(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x))$ satisfies

$$\begin{aligned} &\left\|\left(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x)\right) - \left(v^{\mathcal{CD}}(t,x), u^{\mathcal{CD}}(t,x), \theta^{\mathcal{CD}}(t,x)\right)\right\|_{L^p(\mathbb{R})} \\ &\quad = O\left(\kappa^{\frac{1}{2p}}\right)(1+t)^{\frac{1}{2p}}, \end{aligned} \tag{3.4.18}$$

which implies that the nonlinear diffusion wave $(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x))$ converges to the contact discontinuity $(v^{\mathcal{CD}}(t,x), u^{\mathcal{CD}}(t,x), \theta^{\mathcal{CD}}(t,x))$ to the compressible Euler equation (3.4.3) in $L^p(\mathbb{R})$ norm for each $p \geq 1$ on any finite time interval as the heat conductivity coefficient κ tends to zero.

With the above preparation, in the following subsections, we will discuss the stability of the above three basic wave patterns to the Boltzmann equation respectively.

3.4.2 Basic Ideas in Stability Analysis

Before discussing the stability of basic wave patterns in the Boltzmann equation, we now review some basic ideas on the stability analysis of the wave patterns in fluid dynamics by considering the viscous Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx}, \tag{3.4.19}$$

where $u(t, x) \in \mathbb{R}$ and $\epsilon > 0$ is a constant.

Given a background solution $\phi(t, x)$ with definite sign on $\phi_x(t, x)$, the following simple calculation shows that anti-derivative should be taken when the sign of $\phi_x(t, x)$ is negative, cf. [92].

When $\phi_x(t, x) > 0$ which corresponds to the case of rarefaction wave, set

$$u = \phi + v.$$

The equation for v becomes

$$v_t + (\frac{v^2}{2})_x + (\phi v)_x = \epsilon v_{xx}. \tag{3.4.20}$$

By assuming v vanishes at infinity, direct energy estimate yields

$$\int_{\mathbb{R}} v^2 dx + \int_0^t \int_{\mathbb{R}} (\phi_x v^2 + 2\epsilon v_x^2) dx dt = \int_{\mathbb{R}} v_0^2 dx,$$

which is the lower order estimate. Here, v_0 is the initial data. And the higher order estimates follow straightforwardly for this simple example.

However, when $\phi_x(t, x) < 0$ corresponding to shock wave, the above calculation does not give any desired energy estimate. To cope with the sign of ϕ_x appropriately, one should consider the anti-derivative of the perturbation. Since the shock wave is orbital stable, up to a shift, we can assume

$$\int_{\mathbb{R}} (u - \phi) dx = 0.$$

Then by defining $V = \int_{-\infty}^x (u - \phi) dx$, the equation for V becomes

$$V_t + \phi V_x + \frac{V_x^2}{2} = \epsilon V_{xx}. \tag{3.4.21}$$

Then direct calculation yields the lower order estimate

$$\int_{\mathbb{R}} V^2 dx + \int_0^t \int_{\mathbb{R}} (-\phi_x V^2 + 2\epsilon V_x^2) dx dt = \int_{\mathbb{R}} V_0^2 dx + h.o.t.,$$

where V_0 is the initial data and $h.o.t.$ represents the higher order terms which can be closed by higher order estimates.

Finally, for contact wave profile, there is no definite sign of $\phi_x(t, x)$, the above analysis does not apply. For this, let's assume $\phi = \frac{\delta_1}{\sqrt{4\pi(1+t)}} e^{-\frac{x^2}{4(1+t)}}$ is a linear diffusion profile satisfying

$$\phi_t = \phi_{xx},$$

where $\delta_1 = \int_{\mathbb{R}} u(0, x) dx$. Then, since $\int_{\mathbb{R}} (u - \phi) dx = 0$, we can set $V = \int_{-\infty}^x (u - \phi) dx$ which satisfies

$$V_t + \frac{1}{2} V_x^2 + \phi V_x + \frac{1}{2} \phi^2 = V_{xx}. \tag{3.4.22}$$

Under the a priori estimate $\|V\|_{L_x^\infty} + \|V_x\|_{L_x^\infty} \leq \delta_2$ with $\delta = |\delta_1| + \delta_2 << \epsilon$, we have the following lower order estimate by direct calculation

$$\frac{d}{dt}\int_{\mathbb{R}} V^2 dx + \epsilon \int_{\mathbb{R}} V_x^2 dx \leq c\delta(1+t)^{-1}\int_{\mathbb{R}} V^2 dx + c\delta(1+t)^{-\frac{1}{2}}, \tag{3.4.23}$$

where $c > 0$ is a generic constant. This immediately implies that the L^2 norm of V satisfies $\|V\| \leq c\delta(1+t)^{\frac{1}{4}}$. To close the L_x^∞ estimate, one can combine this with the estimate on the derivatives. That is, by differentiating the equation for V with respect to x and multiplying it by V_x, we have

$$\frac{d}{dt}\int_{\mathbb{R}} V_x^2 dx + \epsilon \int_{\mathbb{R}} V_{xx}^2 dx \leq c\delta(1+t)^{-1}\int_{\mathbb{R}} V_x^2 dx + c\delta(1+t)^{-\frac{3}{2}}. \tag{3.4.24}$$

This gives $\|V_x\| \leq c\delta(1+t)^{-\frac{1}{4}}$. Hence, the $\|V\|_{L_x^\infty}$ can be closed. In fact, a priori estimate on the H^l norm on V_x together with $\|V\|_{L_x^\infty}$ can be closed by similar estimates on higher derivatives. The convergence rate for contact wave is a by-product of the stability analysis even though it is an almost open problem for shock wave and rarefaction wave.

3.4.3 Stability of Shock Profile

Now we come to the stability of basic wave patterns for the Botlzmann equation. Since the wave patterns considered here are one dimensional profiles, we consider the Botlzmann equation with slab symmetry,

$$f_t + \xi_1 f_x = Q(f,f), \quad (f,t,x,\xi) \in \mathbb{R}_+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^3. \tag{3.4.25}$$

According to the decomposition and reformulation in Section 1.2.3, the system of conservation laws governs the macroscopic components becomes

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4}{3}\Big(\mu(\theta)u_{1x}\Big)_x - \Big(\int_{\mathbf{R}^3} \xi_1^2 \Theta_1 d\xi\Big)_x, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = \Big(\mu(\theta)u_{2x}\Big)_x - \Big(\int_{\mathbf{R}^3} \xi_1\xi_2 \Theta_1 d\xi\Big)_x, \\ (\rho u_3)_t + (\rho u_1 u_3)_x = \Big(\mu(\theta)u_{3x}\Big)_x - \Big(\int_{\mathbf{R}^3} \xi_1\xi_3 \Theta_1 d\xi\Big)_x, \\ \big[\rho(\frac{1}{2}|u|^2 + \mathcal{E})\big]_t + \Big(u_1\Big(\rho\left(\frac{1}{2}|u|^2 + \mathcal{E}\right) + p\Big)\Big)_x = \Big(\kappa(\theta)\theta_x\Big)_x + \frac{4}{3}\Big(\mu(\theta)u_1 u_{1x}\Big)_x \\ \qquad + \sum_{i=2}^{3}\Big(\mu(\theta)u_i u_{ix}\Big)_x - \frac{1}{2}\Big(\int_{\mathbb{R}^3} \xi_1|\xi|^2\Theta_1 d\xi\Big)_x. \end{cases} \tag{3.4.26}$$

where

$$\Theta_1 = L_M^{-1}(\mathbf{G}_t + \mathbf{P}_1(\xi_1\mathbf{G}_x) - Q(\mathbf{G},\mathbf{G})). \tag{3.4.27}$$

And the equation for the microscopic component is

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \mathbf{L_M G} + Q(\mathbf{G}, \mathbf{G}). \tag{3.4.28}$$

Since the problem considered here is one-dimensional, it is more convenient to use the *Lagrangian* coordinates as in the study of the conservation laws. That is, consider the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(y,t)dy, \qquad t \Rightarrow t.$$

We will still denote the *Lagrangian* coordinates by (x,t) for simplicity of notation. Then the equations (3.4.25), (3.4.26), (3.4.27), (3.4.28) and the initial condition can be rewritten in the Lagrangian coordinates as

$$\begin{cases} f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = Q(f,f), \\ f(0,x,\xi) = f_0(x,\xi) \to \mathbf{M}_{[v_\pm, u_\pm, \theta_\pm]}(\xi), \quad \text{as } x \to \pm\infty. \end{cases} \tag{3.4.29}$$

$$\mathbf{G}_t - \frac{u_1}{v}\mathbf{G}_x + \frac{1}{v}\mathbf{P}_1(\xi_1 \mathbf{M}_x) + \frac{1}{v}\mathbf{P}_1(\xi_1 \mathbf{G}_x) = \mathbf{L_M G} + Q(\mathbf{G}, \mathbf{G}), \tag{3.4.30}$$

with

$$\mathbf{G} = L_{\mathbf{M}}^{-1}\left(\frac{1}{v}\mathbf{P}_1(\xi_1 \mathbf{M}_x)\right) + \Theta_2,$$

and

$$\Theta_2 = \mathbf{L}_{\mathbf{M}}^{-1}\left(\mathbf{G}_t - \frac{u_1}{v}\mathbf{G}_x + \frac{1}{v}\mathbf{P}_1(\xi_1 \mathbf{G}_x) - Q(\mathbf{G}, \mathbf{G})\right), \tag{3.4.31}$$

and

$$\begin{cases} v_t - u_{1x} = & 0, \\ u_{1t} + p_x = & \frac{4}{3}\left(\frac{\mu(\theta)}{v} u_{1x}\right)_x - \int_{\mathbb{R}^3} \xi_1^2 \Theta_{2x} d\xi, \\ u_{it} = & \left(\frac{\mu(\theta)}{v} u_{ix}\right)_x - \int_{\mathbf{R}^3} \xi_1 \xi_i \Theta_{2x} d\xi, \ i = 2,3, \\ \left(\mathcal{E} + \frac{|u|^2}{2}\right)_t + (pu_1)_x = & \left(\frac{\varkappa(\theta)}{v}\theta_x\right)_x + \frac{4}{3}\left(\frac{\mu(\theta)}{v} u_1 u_{1x}\right)_x \\ & + \sum_{i=2}^3 \left(\frac{\mu(\theta)}{v} u_i u_{ix}\right)_x - \int_{\mathbf{R}^3} \frac{1}{2}\xi_1 |\xi|^2 \Theta_{2x} d\xi. \end{cases} \tag{3.4.32}$$

When $(\rho_+, u_+, \theta_+) \in \mathcal{S}_1(\rho_-, u_-, \theta_-) \cup \mathcal{S}_3(\overline{\rho}, \overline{u}, \overline{\theta})$, as for the compressible Navier-Stokes equations, one first needs to construct the shock profile $\phi(x - st, \xi)$ to the Boltzmann equation (3.4.25). It is proved in [50] that (3.4.25) admits a travelling wave solution $\phi_1(x - \sigma_1 t, \xi)$ and $\phi_3(x - \sigma_3 t, \xi)$ which connect $\mathbf{M}_{[\rho_-, u_-, \theta_-]}$ and $\mathbf{M}_{[\overline{\rho}, \overline{u}, \overline{\theta}]}$, $\mathbf{M}_{[\overline{\rho}, \overline{u}, \overline{\theta}]}$ and $\mathbf{M}_{[\rho_+, u_+, \theta_+]}$ respectively. Here $\sigma_1 = \sigma_1(\rho_-, u_-, \theta_-; \overline{\rho}, \overline{u}, \overline{\theta})$ and $\sigma_3 = \sigma_3(\overline{\rho}, \overline{u}, \overline{\theta}; \rho_+, u_+, \theta_+)$.

Set

$$\phi(t,x,\xi) = \phi_1(x - \sigma_1 t, \xi) + \phi_3(x - \sigma_3 t, \xi) - \mathbf{M}_{[\overline{\rho}, \overline{u}, \overline{\theta}]}(\xi). \tag{3.4.33}$$

Assume that the perturbation of $f_0(x,\xi)$ around $\phi(0,x,\xi)$ has zero mass, i.e.,

$$\int_{\mathbb{R}}\int_{\mathbf{R}^3}\overline{\psi}_i(0,x,\xi)\Big(f_0(x,\xi)-\phi(0,x,\xi)\Big)d\xi dx=0,\quad i=0,1,2,3,4. \tag{3.4.34}$$

Then it is easy to verify

$$\int_{\mathbb{R}}\int_{\mathbf{R}^3}\overline{\psi}_i(t,x,\xi)\Big(f(t,x,\xi)-\phi(t,x,\xi)\Big)d\xi dx=0,\quad i=0,1,2,3,4,$$

so that we can introduce the anti-derivative of the macroscopic components

$$W_i(t,x)=\int_{-\infty}^{x}\int_{\mathbf{R}^3}\overline{\psi}_i(t,y,\xi)\Big(f(t,y,\xi)-\phi(t,y,\xi)\Big)d\xi dy,\quad i=0,1,2,3,4. \tag{3.4.35}$$

Here, $\overline{\psi}_i(t,x,\xi)(i=0,1,2,3,4)$ given by

$$\begin{cases}\overline{\psi}_0(t,x,\xi)= & -v^2(t,x),\\ \overline{\psi}_i(t,x,\xi)= & v(t,x)\xi_i,\quad i=1,2,3,\\ \overline{\psi}_4(t,x,\xi)= & \frac{1}{2}v(t,x)|\xi|^2,\end{cases}$$

are the collision invariants in the Lagrangian coordinates.

To estimate the macroscopic components $W_i(t,x)$, one can apply the techniques for the compressible Navier-Stokes equations. Notice that the compressibility property of the viscous shock profiles, i.e. the inequality (3.4.11) plays an essential role in the analysis. However, the profile $\phi(t,x,\xi)$ may not have this monotonicity property. Fortunately, $\phi(t,x,\xi)$ can be well-approximated by $\phi^{NS}(t,x,\xi)$. Here

$$\phi^{NS}(t,x,\xi)=\mathbf{M}_{[V^S,U^S,\Theta^S]}(t,x,\xi), \tag{3.4.36}$$

is defined by using the shock profile for the compressible Navier-Stokes equations with the corresponding fluid components equal to $V^S(t,x)$, $\big(U^S(t,x),0,0\big)$ and $\Theta^S(t,x)$. Recall that the functions $V^S(t,x),U^S(t,x)$, and $\Theta^S(t,x)$ are uniquely determined by $M^S(t,x;\alpha,\beta)$ defined by (3.4.12) through the relation

$$M^S(t,x)=\left(V^S(t,x),U^S(t,x),\Theta^S(t,x)+\frac{\gamma-1}{2R}\left|U^S(t,x)\right|^2\right).$$

Notice that in the present case, $\gamma=\frac{5}{3}$, $R=\frac{2}{3}$, $A=\frac{1}{2\pi e}$, and $\alpha=\beta=0$. And the monotonicity property of the characteristic fields in the present case implies that $U^s_x<0$ for $x\in\mathbb{R}$.

With the above notations, the nonlinear stability of the Boltzmann shock profile proved in [53] can be stated as follows.

Theorem 3.4.1 (Boltzmann shock profile) *Assume that*

$$(\rho_+,u_+,\theta_+)\in\mathcal{S}_1(\rho_-,u_-,\theta_-)\cup\mathcal{S}_3(\overline{\rho},\overline{u},\overline{\theta})$$

and (3.4.34) is satisfied, then there exist small positive constants δ_0 and ε_0, a global Maxwellian $\mathbf{M}_*$ such that if $|(\rho_+, u_+, \theta_+) - (\rho_-, u_-, \theta_-)| < \delta_0$ and

$$\sum_{i=0}^{4} \|W_i(0,x)\|_{H^2(\mathbb{R})} + \sum_{0\le i\le 2} \left\| \frac{\partial^i}{\partial x^i}\Big(f_0(x,\xi) - \phi(0,x,\xi)\Big) \right\|_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)\right)} \le \varepsilon_0, \qquad (3.4.37)$$

then the Cauchy problem (3.4.25) admits a unique global solution $f(t,x,\xi)$ satisfying $f \ge 0$ and

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \Big\| f(t,x,\xi) - \phi(t,x,\xi) \Big\|_{L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)} = 0. \qquad (3.4.38)$$

As a consequence, the positivity of the Boltzmann shock profile $\phi(t,x,\xi)$ can also be verified. Here and in the sequel, $f(\xi) \in L^2_\xi\left(\frac{1}{\sqrt{M_*}}\right)$ means that $\frac{f(\xi)}{\sqrt{M_*}} \in L^2_\xi\left(\mathbb{R}^3\right)$.

The proof of this theorem is based on the following a priori estimate

$$N(t)^2 = \sup_{0\le\tau\le t} \left\{ \sum_{i=0}^{4} \|W_i(\tau,x)\|_{H^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}^3} \left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_*} + \sum_{|\alpha|=1} \frac{(\partial^\alpha \mathbf{M})^2 + (\partial^\alpha \mathbf{G})^2}{\mathbf{M}_*} + \sum_{|\alpha|=2} \frac{(\partial^\alpha f)^2}{\mathbf{M}_*} \right)(\tau) d\xi dx \right\} \le \delta_1^2.$$

where $\delta_1 > 0$ is a sufficiently small constant, and $\partial^\alpha = \frac{\partial^\alpha}{\partial x^\alpha}$. Here, $\tilde{\mathbf{G}} = \mathbf{G} - \bar{\mathbf{G}}$ with

$$\overline{\mathbf{G}}(t,x,\xi) = \frac{L_{\mathbf{M}}^{-1}\left\{ \mathbf{P}_1\left[\xi_1 \left(\frac{|\xi - u(t,x)|^2}{2\theta(t,x)} \Theta_x^s(t,x) + \xi_1 \cdot U_x^s \right) \mathbf{M}(t,x) \right] \right\}}{Rv\theta(t,x)}. \qquad (3.4.39)$$

With this a priori assumption, the following estimate can be obtained

$$\begin{aligned} &\sum_{i=0}^{4} \|W_i(t,x)\|^2_{H^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}^3} \left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_*} + \sum_{|\alpha|=1} \frac{(\partial^\alpha \mathbf{M})^2 + (\partial^\alpha \mathbf{G})^2}{\mathbf{M}_*} + \sum_{|\alpha|=2} \frac{(\partial^\alpha f)^2}{\mathbf{M}_*} \right) d\xi dx \\ &+ \int_0^t \int_{\mathbb{R}} \Big((-U_x^s) \sum_{i=0}^{4} |W_i(t,x)|^2 + \sum_{i=0}^{4} \sum_{1\le|\alpha|\le 2} |\partial^\alpha W_i(t,x)|^2 \Big) dx dt \\ &+ \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}^3} \Big(\frac{\nu_{\mathbf{M}}(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}_*} + \sum_{1\le|\alpha|\le 2} \frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_*} \Big) d\xi dx dt \\ &\le c(\delta_1^2 + N(0)^2), \end{aligned}$$

which leads to the stability of the shock profile in the theorem.

3.4.4 Stability of Rarefaction Wave

For the other two cases, unlike the Boltzmann shock profile, we use the time-asymptotic wave patterns for the compressible Navier-Stokes equations to construct the time-asymptotic wave patterns for the Boltzmann equation. In fact, for the case when

$$(\rho_+, u_+, \theta_+) \in \mathcal{R}_1(\rho_-, u_-, \theta_-) \cup \mathcal{R}_3(\overline{\rho}, \overline{u}, \overline{\theta}),$$

let $V^R(t,x),\ U^R(t,x)$, and $S^R(t,x)$ be the functions defined by (3.4.14) and

$$\Theta^R(t,x) = \frac{A}{R}\exp\left(\frac{\gamma-1}{R}S^R(t,x)\right)(V^R(t,x))^{-(\gamma-1)},$$

and set

$$V(t,x) = V^R(t,x),\ U(t,x) = \left(U^R(t,x), 0, 0\right),\ \Theta(t,x) = \Theta^R(t,x).$$

By using the monotonic property of the characteristic fields, it is straightforward to check that $U^R_x > 0$ for $x \in \mathbb{R}$ which is used in the energy estimate. For the Boltzmann equation, we have $\gamma = \frac{5}{3}$, $R = \frac{2}{3}$ and $A = \frac{1}{2\pi e}$ as in the case for shock profile. Then, define

$$\overline{\mathbf{M}}(t,x,\xi) = \mathbf{M}_{[V^R, U^R, \Theta^R]}(t,x,\xi). \tag{3.4.40}$$

The following result from [54] is about the stability of this rarefaction wave.

Theorem 3.4.2 (Boltzmann rarefaction wave) *Assume that*

$$(\rho_+, u_+, \theta_+) \in \mathcal{R}_1(\rho_-, u_-, \theta_-) \cup \mathcal{R}_3(\overline{\rho}, \overline{u}, \overline{\theta}),$$

and $\delta = |u_- - u_+| + |\theta_- - \theta_+|$ *satisfying*

$$\delta < \eta_0, \quad \frac{1}{2}\sup_{(t,x)\in\mathbf{R}_+\times\mathbb{R}} \Theta^R(t,x) < \inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \Theta^R(t,x). \tag{3.4.41}$$

If the initial data $f_0(x,\xi)$ *is close to the local Maxwellian* $\overline{\mathbf{M}}$ *defined in (3.4.40):*

$$\left\|f_0(x,\xi) - \overline{\mathbf{M}}(0,x,\xi)\right\|_{H^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)\right)} \le \varepsilon_0, \tag{3.4.42}$$

then for sufficiently small positive constants ε_0 *and* ε, *there exists a global Maxwellian* $\mathbf{M}_*$ *such that the Cauchy problem (3.4.25) admits a unique global solution* $f(t,x,\xi)$ *which for some positive constant* $\delta_0 = O(1)(\varepsilon_0 + \varepsilon)$ *satisfies*

$$\left\|f(t,x,\xi) - \overline{\mathbf{M}}(t,x,\xi)\right\|_{H^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)\right)} \le \delta_0. \tag{3.4.43}$$

Moreover, the solution tends to the Euler rarefaction wave time asymptotically:

$$\lim_{t\to\infty}\left\|f(t,x,\xi) - \mathbf{M}_{[v^{\mathcal{R}}, u^{\mathcal{R}}, \theta^{\mathcal{R}}]}\right\|_{L^\infty_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)\right)} = 0. \tag{3.4.44}$$

The proof of this theorem is based on the following a priori estimate

$$\begin{aligned} N(t)^2 = \sup_{0\le\tau\le t} \Bigg\{ \int_{\mathbf{R}} (v - V^R, u - U^R, \theta - \Theta^R)^2(\tau, x)dx \\ + \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_*} + \sum_{|\alpha|=1} \frac{(\partial^\alpha \mathbf{M})^2 + (\partial^\alpha \mathbf{G})^2}{\mathbf{M}_*} + \sum_{|\alpha|=2} \frac{(\partial^\alpha f)^2}{\mathbf{M}_*} \right) (\tau, x, \xi)d\xi dx \Bigg\} \le \delta_1^2. \end{aligned} \tag{3.4.45}$$

Here $\delta_1 > 0$ is a suitably chosen small constant. Here, again $\tilde{\mathbf{G}} = \mathbf{G} - \bar{\mathbf{G}}$ with

$$\overline{\mathbf{G}}(t, x, \xi) = \frac{L_{\mathbf{M}}^{-1} \left\{ \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u(t,x)|^2}{2\theta(t,x)} \Theta_x^R(t,x) + \xi_1 \cdot U_x^R \right) \mathbf{M}(t,x) \right] \right\}}{Rv\theta(t,x)}. \tag{3.4.46}$$

The following energy estimate can be obtained by some technical calculations

$$\begin{aligned} &\int_{\mathbb{R}} (v - V^R, u - U^R, \theta - \Theta^R)^2 dx + \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_*} + \sum_{|\alpha|=1} \frac{(\partial^\alpha \mathbf{M})^2 + (\partial^\alpha \mathbf{G})^2}{\mathbf{M}_*} + \sum_{|\alpha|=2} \frac{(\partial^\alpha f)^2}{\mathbf{M}_*} \right) d\xi dx \\ &+ \int_0^t \int_{\mathbb{R}} \Big(U_x^R (v - V^R, u - U^R, \theta - \Theta^R)^2 + \sum_{1\le|\beta|\le 2} |\partial^\beta (v - V^R, u - U^R, \theta - \Theta^R)|^2 \Big) dxdt \\ &+ \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \Big(\frac{\nu_{\mathbf{M}}(\xi) \tilde{\mathbf{G}}^2}{\mathbf{M}_*} + \sum_{1\le|\alpha|\le 2} \frac{\nu_{\mathbf{M}}(\xi) |\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_*} \Big) d\xi dxdt \\ &\le c(\epsilon^{\frac{1}{8}} + N(0)^2), \end{aligned}$$

which gives the theorem.

3.4.5 Stability of Contact Wave

Finally, for the case

$$(\rho_+, u_+, \theta_+) \in \mathcal{CD}(\rho_-, u_-, \theta_-),$$

we first need to define a wave profile consisting the contact wave and two diffusion waves in the other two characteristic families so that the mass of the macroscopic components in the initial perturbation can be uniquely distributed. Precisely, let $\Theta(\frac{x}{\sqrt{1+t}})$ be the unique self-similar solution of the following nonlinear diffusion equation

$$\Theta_t = (a(\Theta)\Theta_x)_x, \quad \Theta(-\infty, t) = \theta_-, \quad \Theta(+\infty, t) = \theta_+,$$

where the function $a(s) = \frac{9p_+ k(s)}{10s} > 0$. Here we have used $\gamma = \frac{5}{3}$, $R = \frac{2}{3}$ and $\kappa = k(\theta)$. Then the contact wave profile defined in Section 1.4.1 gives

$$V^{CD} = \frac{2}{3p_+}\Theta, \quad U_1^{CD} = \frac{2a(\Theta)}{3p_+}\Theta_x, \quad U_i^{CD} = 0, \ i = 2, 3, \quad \Theta^{CD} = \Theta - \frac{1}{2}|U^{CD}|^2.$$

To introduce two diffusion waves in the sound wave families, let

$$A_- = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_-}{v_-} & 0 & \frac{2}{3v_-} \\ 0 & p_- & 0 \end{pmatrix}, \quad A_+ = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_+}{v_+} & 0 & \frac{2}{3v_+} \\ 0 & p_+ & 0 \end{pmatrix},$$

be the Jacobians of the flux function of the corresponding Navier-Stokes equations at the states $(v_-, 0, \theta_-)$ and $(v_+, 0, \theta_+)$ respectively. It is easy to check that $\lambda_1^- = -\sqrt{\frac{5p_-}{3v_-}}$ is the first eigenvalue of A_- with $r_1^- = (-1, \lambda_1^-, p_-)^t$ being the corresponding right eigenvector. And $\lambda_3^+ = \sqrt{\frac{5p_+}{3v_+}}$ and $r_3^+ = (-1, \lambda_3^+, p_+)^t$ are those values in the third family of A_+. Since r_1^-, $(v_+ - v_-, 0, \theta_+ - \theta_-)^t$ and r_3^+ are linearly independent in $\mathbb{R}^3$ by strict hyperbolicity, we have

$$\int_{-\infty}^{\infty} (\mathbf{m}(x,0) - \bar{\mathbf{m}}(x,0))dx = \bar{\theta}_1 r_1^- + \bar{\theta}_2(v_+ - v_-, 0, \theta_+ - \theta_-)^t + \bar{\theta}_3 r_3^+,$$

with unique constants $\bar{\theta}_i, i = 1, 2, 3$. Here $\mathbf{m}(x,t)$ and $\bar{\mathbf{m}}(x,t)$ are the conserved quantities in the solution to the Boltzmann equation and the Navier-Stoke contact wave profile $(V^{CD}, U^{CD}, \Theta^{CD})$ without the momentum in the y and z directions. The time asymptotic wave pattern in the conserved quantities can then be defined as

$$\tilde{\mathbf{m}}(x,t) = \bar{\mathbf{m}}(x + \bar{\theta}_2, t) + \bar{\theta}_1 \theta_1 r_1^- + \bar{\theta}_3 \theta_3 r_3^+,$$

where

$$\theta_1(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}}, \quad \theta_3(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}},$$

satisfying $\theta_{1t} + \lambda_1^- \theta_{1x} = \theta_{1xx}$, $\theta_{3t} + \lambda_3^+ \theta_{3x} = \theta_{3xx}$ and $\int_{-\infty}^{\infty} \theta_i(x,t)dx = 1$ for $i = 1, 3$ and all $t \geq 0$. Thus, we have $\int_{-\infty}^{\infty} (\mathbf{m}(x,0) - \tilde{\mathbf{m}}(x,0))dx = 0$.

Let $\overline{V}^{CD}(t,x)$, $\overline{U}^{CD}(t,x)$ and $\overline{\Theta}^{CD}(t,x)$ be the corresponding fluid components in the above time-asymptotic wave pattern. The whole time asymptotic contact wave pattern in the fluid components can then be defined as:

$$\begin{cases} V(t,x) = \overline{V}^{CD}(t,x), \\ U(t,x) = \left(\overline{U}^{CD}(t,x), U_2(t,x), U_3(t,x)\right), \\ \Theta(t,x) = \overline{\Theta}^{CD}(t,x), \end{cases} \tag{3.4.47}$$

with

$$\begin{cases} U_i(t,x) = \frac{\overline{\theta}_{i+2}}{\sqrt{4\pi(t+1)}} \exp\left(-\frac{x^2}{4(t+1)}\right), \quad i = 2, 3, \\ \overline{\theta}_{i+2} = \int_{\mathbf{R}} u_i(0,x)dx, \quad i = 2, 3. \end{cases} \tag{3.4.48}$$

The time asymptotic wave pattern for the Boltzmann equation is,

$$\overline{\mathbf{M}}(t,x,\xi) = \mathbf{M}_{[\overline{V}^{CD}, \overline{U}^{CD}, \overline{\Theta}^{CD}]}(t,x,\xi), \tag{3.4.49}$$

which satisfies

$$\int_{\mathbb{R}}\int_{\mathbf{R}^3} \overline{\psi}_i(t,x,\xi)\Big(f(t,x,\xi)-\overline{\mathbf{M}}(t,x,\xi)\Big)d\xi dx = 0, \quad i=0,1,2,3,4. \tag{3.4.50}$$

We can then define the anti-derivatives of the fluid components

$$W_i(t,x) = \int_{-\infty}^{x}\int_{\mathbf{R}^3} \overline{\psi}_i(t,y,\xi)\Big(f(t,y,\xi)-\overline{\mathbf{M}}(t,y,\xi)\Big)d\xi dy, \quad i=0,1,2,3,4. \tag{3.4.51}$$

The following stability result from [51] is on the stability of contact discontinuity.

Theorem 3.4.3 (Boltzmann contact discontinuity) *Assume that*

$$(\rho_+,u_+,\theta_+)\in \mathcal{CD}(\rho_-,u_-,\theta_-)$$

and let $\delta=|\theta_+-\theta_-|$. Then there exist two small positive constants δ_0, ε_0 and a global Maxwellian $\mathbf{M}_*=\mathbf{M}_{[\rho_*,u_*,\theta_*]}$, such that if $\delta\le\delta_0$ and the initial data $f_0(t,x,\xi)$ satisfies

$$\sum_{i=0}^{4}\Big\|W_i(0,x)\Big\|_{H^2(\mathbf{R})} + \sum_{0\le i\le 2}\left\|\frac{\partial^\alpha}{\partial x^i}\Big(f_0(x,\xi)-\overline{\mathbf{M}}(0,x,\xi)\Big)\right\|_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)\right)} \le \varepsilon_0, \tag{3.4.52}$$

then the Cauchy problem (3.4.14) admits a unique global solution $f(t,x,\xi)$ satisfying

$$\Big\|f(t,x,\xi)-\overline{\mathbf{M}}(t,x,\xi)\Big\|_{L^\infty_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)\right)} \le C\left(\varepsilon_0+\delta_0^{\frac{1}{2}}\right)(1+t)^{-\frac{1}{4}}. \tag{3.4.53}$$

For the contact wave, the a priori estimate can be set as:

$$\begin{aligned} N(T) = \sup_{0\le t\le T}\Big\{&\sum_{i=0}^{4}\left(\|W_i\|^2_{L^\infty}+\|W_{ix}\|^2_{H^2_x}\right)\\ &+\int_R\int_{R^3}\Big(\frac{\tilde{G}^2}{\mathbf{M}_*}+\sum_{|\alpha|=1}\frac{(\partial^\alpha G)^2}{\mathbf{M}_*}+\sum_{|\alpha|=2}\frac{(\partial^\alpha f)^2}{\mathbf{M}_*}\Big)d\xi dx\Big\}\le\delta_1^2. \end{aligned} \tag{3.4.54}$$

Here, $\tilde{\mathbf{G}}=\mathbf{G}-\bar{\mathbf{G}}$ with

$$\overline{\mathbf{G}}(t,x,\xi) = \frac{L_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u(t,x)|^2}{2\theta(t,x)}\overline{\Theta}^{CD}_x(t,x)+\xi_1\cdot\overline{U}^{CD}_x\right)\mathbf{M}(t,x)\right]\right\}}{Rv\theta(t,x)}. \tag{3.4.55}$$

Under this a priori assumption, one can have the following energy estimates. Denote

$$A=\sum_{i=0}^{4}\|W_i\|^2_{H^2_x}+\int_R\int_{R^3}\Big(\frac{\tilde{G}^2}{\mathbf{M}_*}+\sum_{|\alpha|=1}\frac{(\partial^\alpha G)^2}{\mathbf{M}_*}+\sum_{|\alpha|=2}\frac{(\partial^\alpha f)^2}{\mathbf{M}_*}\Big)d\xi dx,$$

$$B=\sum_{i=0}^{4}\|W_{ix}\|^2_{H^2_x}+\int_R\int_{R^3}\Big(\frac{\nu_{\mathbf{M}}(\xi)\tilde{G}^2}{\mathbf{M}_*}+\sum_{1\le|\alpha|\le2}\frac{\nu_{\mathbf{M}}(\xi)(\partial^\alpha G)^2}{\mathbf{M}_*}\Big)d\xi dx,$$

$$C=\sum_{i=0}^{4}\|W_{ix}\|^2_{H^1_x}+\int_R\int_{R^3}\Big(\frac{\tilde{G}^2}{\mathbf{M}_*}+\sum_{|\alpha|=1}\frac{(\partial^\alpha G)^2}{\mathbf{M}_*}+\sum_{|\alpha|=2}\frac{(\partial^\alpha f)^2}{\mathbf{M}_*}\Big)d\xi dx,$$

$$D=\sum_{i=0}^{4}\|W_{ixx}\|^2_{H^1_x}+\int_R\int_{R^3}\Big(\frac{\nu_{\mathbf{M}}(\xi)\tilde{G}^2}{\mathbf{M}_*}+\sum_{1\le|\alpha|\le2}\frac{\nu_{\mathbf{M}}(\xi)(\partial^\alpha G)^2}{\mathbf{M}_*}\Big)d\xi dx.$$

Basically, the energy estimates can be written as:

$$A_t + B \leq c(\delta_0 + \epsilon_0^2)(1+t)^{-1}A + c(\delta_0 + \epsilon_0^2)(1+t)^{-\frac{1}{2}},$$

$$C_t + D \leq c(\delta_0 + \epsilon_0^2)(1+t)^{-1}C + c(\delta_0 + \epsilon_0^2)(1+t)^{-\frac{3}{2}}.$$

These two inequalities imply

$$A + \int_0^t B dt \leq c(\delta_0 + \epsilon_0^2)(1+t)^{\frac{1}{2}},$$
$$C \leq c(\delta_0 + \epsilon_0^2)(1+t)^{-\frac{1}{2}},$$

which give the desired estimate in the theorem.

3.5 Discussion

Besides the results presented above, the energy method has also been applied to the study on the Boltzmann equation in different settings. For example, a lot of work has been done on the Boltzmann equation with forcing, such as the case with external force, Vlasov-Poisson-Boltzmann and Vlasov-Maxwell-Boltzmann systems.

However, so far the energy method is mainly used to describe the solution behavior in the Boltzmann equation which has counterparts in the classical fluid dynamical systems. For the solution to the Boltzmann equation, there is an interesting and important phenomenon called "ghost effect" which is captured by the Boltzmann equation, but not by the classical fluid dynamical systems. So far, it is not clear even though it is very hopeful that the combination of the analytic techniques from the Boltzmann equation and the systems of conservation laws can also be applied to the investigation of this interesting phenomenon.

To present the ghost effect in the Boltzmann equation, we write the Boltzmann equation in its non-dimensional form

$$Shf_t + \xi \cdot \nabla_x f + F \cdot \nabla_\xi \cdot f = \frac{1}{\kappa}Q(f,f), \quad (t,x,\xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{3.5.1}$$

Here F is the vector for the external force. There are two parameters Sh and k in the above equation which are called Strouhal and Knudsen numbers respectively. Their product $Sh \cdot \kappa$ is $2\sqrt{\pi}$ times the ratio of the mean free time to the reference time. And the non-dimensional parameters κ and $Sh \cdot \kappa$ not only characterize the different effects coming from the molecular collisions, but also give the weight of the spatial and temporal derivatives with respect to the collision operator.

In the previous sections, the relation between the Boltzmann equation and the classical systems of fluid dynamics was presented in the Hilbert and Chapman-Enskog expansions when $Sh = 1$ and κ is small. And the study on this kind of relations was raised as the Hilbert's sixth problem, i.e., the "Mathematical treatment of the axioms of physics ", in his famous lecture "Mathematical Problems" at ICM in 1900.

On the other hand, one can also establish some mathematical theories for the phenomena described by the Boltzmann equation where the time evolution of the macroscopic components are not captured by the classical fluid dynamical systems. This happens, for example, when the parameters Sh and κ as well as the macroscopic velocity are small while the density and temperature are of the order 1, such as in the thermal creep flow. Unlike the Poiseuille flow induced by the gradient of pressure and described by the Navier-Stokes equations, the thermal creep flow is induced by the gradient of the wall temperature and can not be modeled by the Navier-Stokes equations. There have been a lot of studies on this kind of phenomena which is called the "ghost effect" in the Boltzmann equation. However, most of the previous results are mainly built on the asymptotic expansions and numerical computations, or the stationary linearized problems. Therefore, the time evolutionary and nonlinear problems on this kind of phenomena provide a lot of challenging mathematical topics which have not been well solved.

For the "ghost-effect", since the fluid dynamic systems governing the time evolution of the macroscopic components in the solutions are not classical, the well-posedness theory for these systems by itself is already an interesting problem. In fact, we have found a new way to derive these non-classical systems systematically. That is, we can use the decomposition and new reformulation of the Boltzmann equation in Section 3.2.3 and the known and new analytic techniques to study the limit process when κ tends to zero. In other words, the non-classical systems for the time evolution of the leading order in the macroscopic variables provide a good description of the solution behavior when κ is sufficiently small. Furthermore, since most of the physical models have boundary and sometimes have external forces, one should also investigate the effects of the boundary and the external forces on the solution behavior.

For illustration, we now focus on the problems when the macroscopic velocity (i.e., flow velocity) is of the order of κ^γ for $\gamma > 0$ such as in the case for the thermal creep flow. The problems related to the geometry of the boundary will not be discussed here.

To give a clear presentation of the problem, let's rewrite the Boltzmann equation under the following scalings:

$$\kappa^\alpha f_t + \xi \cdot \nabla_x f + \kappa^\beta \nabla_\xi \cdot (Ff) = \frac{1}{\kappa} Q(f, f). \tag{3.5.2}$$

Here α and β are positive constants and the scalings just mean that the parameter Sh is of the order of κ^α and the strength of the external force is of order of κ^β. Furthermore, we assume that the solution to the Boltzmann equation has the following macroscopic and microscopic decomposition:

$$f = \mathbf{M}_{[\rho, \kappa^\gamma u, \theta]} + \kappa^\delta G, \tag{3.5.3}$$

for some positive constants γ and δ. Here $M_{[\rho,\kappa^\gamma u,\theta]}$ is the local Maxwellian and G is the microscopic component. Moreover, the local Maxwellian $\mathbf{M}_{[\rho,\kappa^\gamma u,\theta]}$ is again defined by the five conserved quantities, that is, the mass density $\rho(t, x)$, momentum density $m(t, x) =$

$\kappa^\gamma\rho(t,x)u(t,x)$ and energy density $\mathcal{E}(t,x)+\frac{\kappa^{2\gamma}}{2}|u(t,x)|^2$ given by:

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbb{R}^3} f(t,x,\xi)d\xi, \quad \kappa^\gamma \rho u^i(t,x) \equiv \int_{\mathbb{R}^3} \psi_i(\xi) f(t,x,\xi) d\xi \text{ for } i=1,2,3, \\ \left[\rho\left(\mathcal{E}+\frac{\kappa^{2\gamma}}{2}|u|^2\right)\right](t,x) \equiv \int_{\mathbb{R}^3} \psi_4(\xi) f(t,x,\xi)d\xi, \end{cases}$$

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,\kappa^\gamma u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\left(-\frac{|\xi-\kappa^\gamma u(t,x)|^2}{2R\theta(t,x)}\right).$$

As usual, $\theta(t,x)$ is the temperature related to the internal energy $\mathcal{E}$ by $\mathcal{E}=\frac{3}{2}R\theta$ with R being the gas constant, and $\kappa^\gamma u(t,x)$ is the flow velocity. Here u is the scaled flow velocity which appears in the equations for the macroscopic variables ρ and θ.

When $\kappa \to 0$, equation (3.5.3) implies that the solution converges to $\mathbf{M}_{[\rho,0,\theta]}$ formally. Mathematically, the "ghost effect" means that the equations governing the time evolution of the functions ρ and θ depend actually on the scaled velocity u even though the macroscopic velocity tends to zero. Therefore, the resulted system of equations for these macroscopic variables, ρ, u and θ are not given by either the classical Euler or Navier-Stokes equations. Instead, different systems of equations arise from different settings. Even though they are approximations, these systems provide the right description of the time evolution of the macroscopic variables of the solutions in different physical settings. However, these systems are not classical so that the known theories for Euler and Navier-Stokes equations can not be applied.

Let us consider the typical case considered by Sone through Hilbert expansion, when $\alpha=\gamma=\delta=1$ and $F\equiv 0$. In this case, the system for the time evolution of the leading terms of ρ, u and θ can be given as follows.

Firstly, we expand all the variables in the power of κ and let (ρ^0,u^0,θ^0) be the leading order of the variables (ρ,u,θ). (Note that the zeroth order of u here is the first order in Sone's description because we put a factor κ in front of u) Then formal expansion gives that the zeroth and first order of the pressure function, denoted by p^0 and p^1, should be functions of t only. And then the system for (ρ^0,u^0,θ^0) is:

$$\begin{cases} \rho^0_t + \nabla_x\cdot(\rho^0 u^0) = 0, \\ (\rho^0 u^0_i)_t + \nabla_x \cdot (\rho^0 u^0_i u^0) \\ \qquad = -\frac{1}{2}p^{2*}_{x_i} + \frac{1}{2}[\Gamma_1(\theta^0)((u^0_i)_{x_j} + (u^0_j)_{x_i} - \frac{2}{3}\nabla_x\cdot u^0\delta_{ij})]_{x_j} \\ \qquad\quad + \frac{1}{2p^0}[\Gamma_2(\theta)(\theta^0_{x_i}\theta^0_{x_j} - \frac{1}{3}|\nabla_x\theta^0|^2\delta_{ij}]_{x_j}, \quad i=1,2,3, \\ \frac{3}{2}(\rho^0\theta^0)_t + \frac{5}{2}\nabla_x\cdot(\rho^0\theta^0u^0) = \frac{5}{4}(\Gamma_3(\theta^0)\theta^0_{x_i})_{x_i}, \end{cases} \tag{3.5.4}$$

where

$$p^0 = \rho^0\theta^0, \quad p^{2*} = p^2 + \frac{2}{3p^0}(\Gamma_4(\theta^0)\theta^0_{x_k})_{x_k},$$

and p^2 is the second order term in the expansion for the pressure function, the summation is over all the repeated indices. Here, all $\Gamma_i(\theta^0)$, $i=1,2,3,4$, are positive smooth functions, and we do not give their explicit expressions here for brevity.

If p^{2*} is replaced by p^0, the system (3.5.4) is similar to the compressible Navier-Stokes equations even though there are some extra nonlinear terms in the momentum equation. However, the pressure function in (3.5.4) is p^2 which is not given only by ρ^0 and θ^0 in the system. Instead, the system is given under the constrain that the product $\rho^0\theta^0$ is a function of t only. Usually, the function p^0 is given by the boundary condition. Therefore, the well-posedness of the above fluid dynamic system does not follow from the classical theory and thus remains unsolved.

There is a systematic method to derive the leading order system for the macroscopic components under the assumption (3.5.3) by using the Hilbert expansion described in Sone's book. However, this method does not give a clear presentation of the truncated terms which need to be estimated in the limit process analysis. By using the macro-micro decomposition and the reformulation introduced in Section 1.2.3, the derivation of these systems could be more direct and useful when we study the limit process $\kappa \to 0$.

For the well-posedness theory of the non-classical fluid dynamic systems, let's use the system (3.5.4) to explain our ideas on solving these problems. As mentioned before, the main difficulty in solving system (3.5.4) is that the pressure function p^{2*} is not given by ρ^0 and θ^0 through the traditional equation of state. Instead, it should be solved by using the constrain that $p^0 = \rho^0\theta^0$ is a given function of t only. Notice that both ρ^0 and θ^0 are functions of x and t so that the constrain that their product should be a function of t only is very strong. One way to overcome this is to use the classical technique in the study of incompressible fluid dynamics. That is, we set $w^0 = \nabla_x \times (\rho^0 u^0)$, and then derive a closed system only for w^0 and θ^0. Notice that when one takes "$\nabla_x \times$" on both sides of the momentum equation in the system (3.5.4), the pressure term $\nabla_x P^{2*}$ vanishes. In fact, the system for w^0 and θ^0 can be obtained by firstly noticing that

$$\begin{aligned} \rho^0 u^0 &= \Delta_x^{-1}\nabla_x \times w^0 + \Delta_x^{-1}\nabla_x(\nabla_x \cdot (\rho^0 u^0)) \\ &= \Delta_x^{-1}\nabla_x \times w^0 - \Delta_x^{-1}\nabla_x\left(\frac{p_t^0}{\theta^0} - \rho^0\frac{\theta_t^0}{\theta^0}\right). \end{aligned} \tag{3.5.5}$$

By using (3.5.5) in the momentum equation after applying $\nabla_x\times$, together with the energy equation, the system for (w^0, θ^0) takes the form:

$$w_t^0 = M_1 w^0 + E_1, \qquad \theta_t^0 = M_2\theta^0 + E_2, \tag{3.5.6}$$

where M_1 and M_2 are some elliptic pseudo-differential operators when $\nabla_x\theta^0$ is small, and E_1 and E_2 are some nonlinear funtionals of (w^0, θ^0) and its spatial differentations together with the operator Δ_x^{-1}. Since the system (3.5.6) is very complicated, we will not give its explicit expression here for brevity. It is hopeful to solve this system because the main structure in (3.5.6) is parabolic. After solving (3.5.6), the values of ρ^0 and p^{2*} can be determined accordingly. Notice that the non-classical fluid dynamic system is different for different setting so that some other analytic techniques could be needed for other cases.

After solving the non-classical fluid dynamic systems, one can try to justify the limit process when $\kappa \to 0$. This justification can be based on the energy method to obtain the uniform bounds on the macroscopic components (ρ, u, θ) and microscopic component G. The enegy method is useful because the system has the dissipative structure similar to the Navier-Stokes equation so that the techniques from the theory of conservation laws can be applied. And the microscopic component G can be estimated through the celebrated H-theorem.

Finally, physical models are usually accompanied by boundaries and/or external forces. In fact, the "ghost-effect" usually comes from the effect of the boundary, such as in the thermal creep flow. And the effect of the external force also yields new phenomena in the solution behavior such as bifurcation. In summary, there are a lot of interesting mathematical problems for the non-classical fluid dynamic systems related to the "ghost-effect" of the Boltzmann equation, especially with boundaries and/or external forces. And it will be our next research project to establish some mathematical theories for this kind of phenomena, in particular, for the time evolutional and nonlinear problems.

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