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Linear Difference Equations with Transition Points

Z. Wang[†] and R. Wong[‡]

Abstract

Two linearly independent asymptotic solutions are constructed for the second-order linear difference equation

$$y_{n+1}(x) - (A_n x + B_n) y_n(x) + y_{n-1}(x) = 0,$$

where A_n and B_n have power series expansions of the form

$$A_n \sim \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}$$

with $\alpha_0 \neq 0$. Our results holds uniformly for x in an infinite interval containing the *transition point* x_+ given by $\alpha_0 x_+ + \beta_0 = 2$. As an illustration, we present an asymptotic expansion for the monic polynomials $\pi_n(x)$ which are orthogonal with respect to the modified Jacobi weight $w(x) = (1-x)^\alpha (1+x)^\beta h(x)$, $x \in (-1,1)$, where α, $\beta > -1$ and h is real analytic and strictly positive on $[-1,1]$.

Key words and phrases. Difference equation, transition points, three–term recurrence relation, orthogonal polynomials.

1 Introduction

Ever since Deift and Zhou [7] introduced the steepest descent method for Riemann-Hilbert problems, there has been a considerable amount of activities in the study of asymptotics of orthogonal polynomials by using this approach. For instance, in [5] Deift et al studied the asymptotics of orthogonal polynomials with respect to the weight $w(x) = e^{-Q(x)}$ on the real line, where $Q(x)$ is a polynomial of even degree with positive leading coefficient, and obtained uniform Plancherel-Rotach-type asymptotics in the entire complex plane. Also, in [11] Kuijlaars and McLaughlin used this method to investigate the asymptotic behavior of Laguerre polynomials $L_n^{(\alpha_n)}(x)$, where α_n is a sequence of negative numbers such that $-\alpha_n/n$ tends to a limit $A > 1$ as $n \to \infty$. Furthermore, Kuijlaars et al [12, 13] considered the asymptotics of the polynomials that are orthogonal with respect to the modified Jacobi weight

$$w(x) = (1-x)^\alpha (1+x)^\beta h(x), \qquad x \in (-1,1), \tag{1.1}$$

where $\alpha, \beta > -1$ and the extra factor h is real analytic and strictly positive on $[-1, 1]$. For other investigations of similar nature, we refer to [3, 4, 6, 10, 18]. An advantage of this new approach over the more classical asymptotic methods is that it is applicable to orthogonal polynomials which do not have an integral representation or satisfy a second-order differential equation. Examples of such cases are provided by the two sets of orthogonal polynomials mentioned above, namely those associated with (i) the exponential weight and (ii) the modified Jacobi weight. However, so far this new method has not been able to produce results as strong as those obtainable from the classical approaches when an integral representation is available, or when the differential equation theory can be applied. For instance, in the case of Meixner-Pollaczek polynomials $M_n(x; \delta, \eta)$, one can use a Cauchy integral representation to derive an infinite asymptotic expansion for $M_n(\alpha n; \delta, \eta)$, which holds uniformly for $-M \leq \alpha \leq M$, where M can be any positive number; see [14]. Also, when the polynomial $Q(x) = x^{2m} + \cdots$ in the weight function $w(x) = e^{-Q(x)}$ is even and convex, one can use the turning point theory for differential equations to obtain an asymptotic formula for the polynomials $p_n(x)$ orthogonal with respect to $w(x)$, which holds uniformly in the unbounded interval $0 \leq x \leq O(n^{1/2m})$; see [16].

In our view, a desirable approach to derive asymptotic expansions for orthogonal polynomials now is to develop an asymptotic theory for linear second-order difference equations, just like what Langer, Cherry, Olver and others have done for linear differential equations; see the definitive book by Olver [15]. Our view is based on the fact that any sequence of orthogonal polynomials satisfies the three-term recurrence relation

$$p_{n+1}(x) = (a_n x + b_n) p_n(x) - c_n p_{n-1}(x), \qquad n = 1, 2, \cdots, \tag{1.2}$$

where a_n, b_n and c_n are constants. If we define a sequence $\{K_n\}$ recursively by $K_{n+1}/K_{n-1} = c_n$, with K_0 and K_1 depending on the particular sequence of polynomials, and put $A_n \equiv a_n K_n / K_{n+1}$, $B_n \equiv b_n K_n / K_{n+1}$ and $P_n(x) \equiv p_n(x)/K_n$, then (1.2) can be written as

$$P_{n+1}(x) - (A_n x + B_n) P_n(x) + P_{n-1}(x) = 0. \tag{1.3}$$

We shall assume that the coefficients A_n and B_n are real, and have asymptotic expansions of the form

$$A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s} \qquad \text{and} \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}, \tag{1.4}$$

where θ is a real number and $\alpha_0 \neq 0$. If x is a fixed number, then asymptotic solutions to (1.3) can be obtained from existing results in the literature; see, e.g., the papers by Birkhoff [1] and Birkhoff - Trjitzinsky [2]. These papers, however, have been considered far too complicated and even impenetrable. For a more accessible account of the asymptotic behavior of the solutions to equation (1.3), we refer to the two more recent papers by Wong and Li [21, 22]. When x is a parameter and allowed to vary, then not much work has been done in this area until just recently. In [20], we have studied a case in which the exponent θ in (1.4) is *not* zero. This case corresponds to the turning-point problem for second-order linear differential equations; and the asymptotic expansions derived for the solutions involve the Airy functions $\mathrm{Ai}(\cdot)$, $\mathrm{Bi}(\cdot)$ and their derivatives; see also [19]. In this paper, we shall consider the case $\theta = 0$ in (1.3). The approximants of the asymptotic solutions turn out to be Bessel functions or modified Bessel functions. As an illustration, we shall present an infinite asymptotic expansion for the monic polynomials which are orthogonal with respect to the modified Jacobi weight given in (1.1). Our expansion will hold uniformly for x in $[-1+\delta,\infty)$, $\delta>0$. Although we will make use of a result of Kuijlaars et al [12] on the coefficients of a relevant recurrence relation, our result for the modified Jacobi polynomials is stronger than those given in their papers [12, 13].

The presentation of this paper is arranged as follows. In Sec. 2, we show how the Bessel functions arise in the asymptotic solutions. In Sec. 3, we give a preliminary lemma which is crucial to the derivation of the asymptotic expansions. The construction of the formal solutions is presented in Sec. 4. In Sec. 5, we establish the asymptotic nature of the expansions. The final section is devoted to a study of the orthogonal polynomials with the modified Jacobi weight.

2 Motivation Leading to the Expansion

Returning to (1.3), we try a solution of the form $P_n(x)=\lambda^n$ and replace the coefficients A_n and B_n by their respective asymptotic expansions given in (1.4) with $\theta=0$. Upon letting $n\to\infty$, this yields the characteristic equation

$$\lambda^2-(\alpha_0x+\beta_0)\lambda+1=0. \tag{2.1}$$

The roots of this equation are

$$\lambda_\pm=\frac{1}{2}\left[\alpha_0x+\beta_0\pm\sqrt{(\alpha_0x+\beta_0)^2-4}\right], \tag{2.2}$$

and they coincide when $x=x_\pm$, where $x_\pm$ satisfy

$$\alpha_0x_\pm+\beta_0=\pm2. \tag{2.3}$$

The points x_+ and x_- are called *transition points* in [20]. Throughout this paper, we shall assume that

$$\alpha_1=\beta_1=0 \tag{2.4}$$

This assumption naturally implies the condition

$$\alpha_1x_++\beta_1=0, \tag{2.5}$$

which was used in our previous paper on turning point theory; see equation (2.7) in [20]. It is interesting to note that in a paper on WKB methods for difference equations, Dingle and Morgan [8, 9] also assumed this condition. In fact, they assumed the stronger condition that all coefficients α_s and β_s in (1.4) with odd indices vanish. (We believe that under assumption (1.4), condition (2.4) for the three–term recurrence relation (1.3) is probably satisfied by all orthogonal polynomials in the Szegö class. This possibility is currently under investigation.) Note that since $\widetilde{P}_n(x)\equiv(-1)^nP_n(x)$ satisfies the recurrence relation $\widetilde{P}_{n+1}(x)+(A_nx+B_n)\widetilde{P}_n(x)+\widetilde{P}_{n-1}(x)=0$, we may, without loss of generality, assume $\alpha_0>0$.

Now let $\tau_0 := -(\alpha_3 x_+ + \beta_3)/2(\alpha_2 x_+ + \beta_2)$; cf. (4.20) below. Define $N := n + \tau_0$ and recast the expansions in (1.4), with $\theta = 0$, in terms of N. This gives

$$A_n \sim \sum_{s=0}^{\infty} \frac{\alpha'_s}{N^s} \qquad \text{and} \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{N^s} \tag{2.6}$$

with $\alpha'_0 = \alpha_0 > 0$ and $\alpha'_1 = \beta'_1 = 0$. Since the transition points x_+ and x_- in (2.3) are distinct, we may restrict ourselves to just the case $x = x_+$. As in [20], we try a formal solution of the form

$$P_n(x) = \sum_{s=0}^{\infty} X_s(\xi) N^{-s} \tag{2.7}$$

for x near x_+, where ξ depends on x and N. In this paper, we choose $\xi = N\zeta^{1/2}(x)$, where $\zeta(x)$ is an increasing function with $\zeta(x_+) = 0$. Clearly,

$$\begin{aligned} P_{n+1}(x) &= \sum_{s=0}^{\infty} X_s[(N+1)\zeta^{1/2}(x)](N+1)^{-s} \\ &= \sum_{s=0}^{\infty} X_s\Big[\big(1+\frac{1}{N}\big)\xi\Big] N^{-s}\big(1+\frac{1}{N}\big)^{-s}. \end{aligned}$$

By expanding $X_s\big(\xi + \frac{\xi}{N}\big)$ into a Taylor series and using the binomial expansion, we obtain

$$P_{n+1}(x) = \sum_{l=0}^{\infty} \frac{1}{N^l} \sum_{s=0}^{l} \Big[\sum_{j=0}^{l-s} X_s^{(j)}(\xi)\frac{\xi^j}{j!}\binom{-s}{l-j-s}\Big]. \tag{2.8}$$

Similarly,

$$P_{n-1}(x) = \sum_{l=0}^{\infty} \frac{1}{N^l} \sum_{s=0}^{l} \Big[(-1)^{l-s}\sum_{j=0}^{l-s} X_s^{(j)}(\xi)\frac{\xi^j}{j!}\binom{-s}{l-j-s}\Big]. \tag{2.9}$$

Since $\xi = N\zeta^{1/2}(x)$, we also have

$$x = \zeta^{-1}(\xi^2/N^2) = \sum_{j=0}^{\infty} \frac{1}{j!}(D^j\zeta^{-1})(0)\frac{\xi^{2j}}{N^{2j}}, \tag{2.10}$$

where D^j denotes the j–th derivative of ζ^{-1}. From (2.6) and (2.10), it follows that

$$A_n x + B_n = \sum_{s=0}^{\infty} \frac{Q_s(\xi)}{N^s}, \tag{2.11}$$

where $Q_0(\xi) = \alpha'_0 x_+ + \beta'_0 = 2$, $Q_1(\xi) = \alpha'_1 x_+ + \beta'_1 = 0$ by (2.5), and

$$Q_s(\xi) = \beta'_s + \sum_{2j \le s} \frac{1}{j!}(D^j\zeta^{-1})(0)\alpha'_{s-2j}\,\xi^{2j}, \qquad s = 2, 3, \cdots. \tag{2.12}$$

Upon substituting (2.7), (2.8), (2.9) and (2.12) into (1.3), we find that $X_0(\xi)$ satisfies the Bessel equation

$$\frac{d^2X_0}{d\xi^2} = \left(\frac{\alpha_0'}{\zeta'(0)} + \frac{\alpha_2'x_+ + \beta_2'}{\xi^2}\right)X_0. \tag{2.13}$$

Thus, it follows that $X_0(\xi)$ can be expressed in terms of either the Bessel functions $J_\nu(\xi)$ and $Y_\nu(\xi)$, or the modified Bessel functions $I_\nu(\xi)$ and $K_\nu(\xi)$. That is, there are constants C_1 and C_2 such that

$$X_0(\xi) = C_1\xi^{1/2}J_\nu(\xi) + C_2\,\xi^{1/2}Y_\nu(\xi) \qquad \text{if} \quad \alpha_0' < 0$$

and

$$X_0(\xi) = C_1\xi^{1/2}I_\nu(\xi) + C_2\,\xi^{1/2}K_\nu(\xi) \qquad \text{if} \quad \alpha_0' > 0,$$

where

$$\nu = \pm\left(\alpha_2'x_+ + \beta_2' + \frac{1}{4}\right)^{1/2}. \tag{2.14}$$

Under the assumption in (2.4), it is easily verified that $\alpha_2' = \alpha_2$ and $\beta_2' = \beta_2$. In view of the wellknown identities

$$J_{-\nu}(z) = \cos\nu\pi J_\nu(z) - \sin\nu\pi Y_\nu(z), \qquad I_{-\nu}(z) = I_\nu(z) + \frac{2\sin\nu\pi}{\pi}K_\nu(z),$$

$$Y_{-\nu}(z) = \sin\nu\pi J_\nu(z) + \cos\nu\pi Y_\nu(z) \qquad \text{and} \qquad K_{-\nu}(z) = K_\nu(z),$$

we may, without loss of generality, take the square root with the + sign in (2.14). Moreover, each of the subsequent coefficient functions $X_s(\xi), s = 1, 2, \cdots$, in (2.7) satisfies an inhomogeneous Bessel equation. This suggests that instead of (2.7), we might as well try the formal series solution

$$P_n(x) = Z_\nu(N\zeta^{1/2})\sum_{s=0}^{\infty}\frac{\overline{A}_s(\zeta)}{N^{s-\frac{1}{2}}} + Z_{\nu-1}(N\zeta^{1/2})\sum_{s=0}^{\infty}\frac{\overline{B}_s(\zeta)}{N^{s-\frac{1}{2}}} \tag{2.15}$$

motivated from the differential equation theory. In (2.15), $Z_\nu(\xi)$ can be any solution of the modified Bessel equation

$$y'' + \frac{1}{x}y' - \left(1 + \frac{\nu^2}{x^2}\right)y = 0. \tag{2.16}$$

The main result of this paper is given in the following theorem.

THEOREM 1. *Assume that the coefficients A_n and B_n in the recurrence relation* (1.3) *are real, and have asymptotic expansions given in* (1.4) *with $\theta = 0$. Let $x_\pm$ be the transition points defined in* (2.3)*, $\zeta^{1/2} = \cosh^{-1}(\alpha_0 x + \beta_0)/2$ and $\tau_0 = -(\alpha_3 x_+ + \beta_3)/2(\alpha_2 x_+ + \beta_2)$. Then, for each nonnegative integer p, equation* (1.3) *has a pair of linearly independent solutions*

$$P_n(x) = \left(\frac{4\zeta}{(\alpha_0 x + \beta_0)^2 - 4}\right)^{\frac{1}{4}}\left[N^{\frac{1}{2}}I_\nu(N\zeta^{\frac{1}{2}})\sum_{s=0}^{p}\frac{A_s(\zeta)}{N^s} + N^{\frac{1}{2}}\zeta^{\frac{1}{2}}I_{\nu-1}(N\zeta^{\frac{1}{2}})\sum_{s=0}^{p}\frac{B_s(\zeta)}{N^s} + \varepsilon_p(N,x)\right] \tag{2.17}$$

and

$$Q_n(x) = \left(\frac{4\zeta}{(\alpha_0 x+\beta_0)^2-4}\right)^{\frac{1}{4}}\Bigg[N^{\frac{1}{2}}K_\nu(N\zeta^{\frac{1}{2}})\sum_{s=0}^{p}\frac{A_s(\zeta)}{N^s} - N^{\frac{1}{2}}\zeta^{\frac{1}{2}}K_{\nu-1}(N\zeta^{\frac{1}{2}})\sum_{s=0}^{p}\frac{B_s(\zeta)}{N^s}+\delta_p(N,x)\Bigg], \tag{2.18}$$

where $N = n+\tau_0$ *and* ν *is given in* (2.14)*. The error terms satisfy*

$$|\varepsilon_p(N,x)| \le \frac{M_p}{N^{p+\frac{1}{2}}}\left[|I_\nu(N\zeta^{\frac{1}{2}})|+|I_{\nu-1}(N\zeta^{\frac{1}{2}})|\right] \tag{2.19}$$

and

$$|\delta_p(N,x)| \le \frac{M_p}{N^{p+\frac{1}{2}}}\left[|K_\nu(N\zeta^{\frac{1}{2}})|+|K_{\nu-1}(N\zeta^{\frac{1}{2}})|\right] \tag{2.20}$$

for $x_-+\delta \le x < \infty$*, where* M_p *is a positive constant. The coefficients* $A_s(\zeta)$ *and* $B_s(\zeta)$ *can be determined successively for any given* $A_0(\zeta)$ *and* $B_0(\zeta)$*; see* (4.11) *and* (4.12)*.*

To see how the function $\zeta(x)$ and the constant τ_0 in the above theorem are chosen, we refer to § 4; see, in particular, (4.8) and (4.20).

3 A Preliminary Lemma

When we replace n by $n+1$, the two functions $Z_\nu(N\zeta^{1/2})$ and $Z_{\nu-1}(N\zeta^{1/2})$ in (2.15) become $Z_\nu[(N+1)\zeta^{1/2}]$ and $Z_{\nu-1}[(N+1)\zeta^{1/2}]$. An important connection between the second two functions and the first two is given in the following lemma, which plays a crucial role in the derivation of the formal series solution (2.15).

LEMMA 1. *Let* $Z_\nu(x)$ *be any solution of the modified Bessel equation* (2.16)*, which satisfies*

$$Z_{\nu-1}(x) = Z'_\nu(x)+\left(\frac{\nu}{x}\right)Z_\nu(x). \tag{3.1}$$

We have

$$\left(1+\frac{\theta}{N}\right)^{1/2} Z_\nu[(N+\theta)\zeta^{1/2}] = Z_\nu(N\zeta^{1/2})G(N;\theta,\zeta)+\zeta^{1/2}Z_{\nu-1}(N\zeta^{1/2})H(N;\theta,\zeta) \tag{3.2}$$

and

$$\left(1+\frac{\theta}{N}\right)^{1/2} Z_{\nu-1}[(N+\theta)\zeta^{1/2}] = \zeta^{1/2}Z_\nu(N\zeta^{1/2})L(N;\theta,\zeta)+Z_{\nu-1}(N\zeta^{1/2})K(N;\theta,\zeta), \tag{3.3}$$

where

$$G(N;\theta,\zeta)\sim\sum_{s=0}^{\infty}\frac{G_s(\theta,\zeta)}{N^s}, \qquad H(N;\theta,\zeta)\sim\sum_{s=0}^{\infty}\frac{H_s(\theta,\zeta)}{N^s}, \tag{3.4}$$

$$L(N;\theta,\zeta) \sim \sum_{s=0}^{\infty} \frac{L_s(\theta,\zeta)}{N^s}, \qquad K(N;\theta,\zeta) \sim \sum_{s=0}^{\infty} \frac{K_s(\theta,\zeta)}{N^s}, \tag{3.5}$$

the expansions being uniformly valid with respect to bounded θ and all real ζ.

Proof. Let $w(\theta,\zeta) = (1+\theta/N)^{1/2} Z_\nu[(N+\theta)\zeta^{1/2}]$. Straightforward calculation gives

$$\frac{\partial w}{\partial \theta} = \frac{\frac{1}{2}-\nu}{N}\left(1+\frac{\theta}{N}\right)^{-\frac{1}{2}} Z_\nu + \left(1+\frac{\theta}{N}\right)^{\frac{1}{2}} \zeta^{\frac{1}{2}} Z_{\nu-1} \tag{3.6}$$

and

$$\frac{\partial^2 w}{\partial \theta^2} = \left(\zeta + \frac{\nu^2-\frac{1}{4}}{(N+\theta)^2}\right) w. \tag{3.7}$$

The last equation can also be obtained directly from (2.16) by eliminating the term involving the first derivative. From Taylor's expansion,

$$w(\theta,\zeta) = w(0,\zeta) + \frac{\partial w}{\partial \theta}(0,\zeta)\theta + \frac{1}{2!}\frac{\partial^2 w}{\partial \theta^2}(0,\zeta)\theta^2 + \cdots\cdots .$$

In view of (3.6) and (3.7), this series can be rearranged as

$$\left(1+\frac{\theta}{N}\right)^{\frac{1}{2}} Z_\nu[(N+\theta)\zeta^{\frac{1}{2}}] = Z_\nu(N\zeta^{\frac{1}{2}})G(N;\theta,\zeta) + \zeta^{\frac{1}{2}} Z_{\nu-1}(N\zeta^{\frac{1}{2}})H(N;\theta,\zeta), \tag{3.8}$$

where $G(N;0,\zeta) = 1, H(N;0,\zeta) = 0$,

$$\frac{\partial G}{\partial \theta}(N;0,\zeta) = \frac{\frac{1}{2}-\nu}{N} \qquad \text{and} \qquad \frac{\partial H}{\partial \theta}(N;0,\zeta) = 1.$$

Differentiating (3.8) with respect to θ yields

$$\frac{\partial^2 G}{\partial \theta^2} = \left(\zeta + \frac{\nu^2-\frac{1}{4}}{(N+\theta)^2}\right) G, \qquad G\Big|_{\theta=0} = 1, \qquad \frac{\partial G}{\partial \theta}\Big|_{\theta=0} = \frac{\frac{1}{2}-\nu}{N} \tag{3.9}$$

and

$$\frac{\partial^2 H}{\partial \theta^2} = \left(\zeta + \frac{\nu^2-\frac{1}{4}}{(N+\theta)^2}\right) H, \qquad H\Big|_{\theta=0} = 0, \qquad \frac{\partial H}{\partial \theta}\Big|_{\theta=0} = 1. \tag{3.10}$$

The solutions to these two differential equations have formal asymptotic solutions

$$G = \sum_{s=0}^{\infty} \frac{G_s(\theta,\zeta)}{N^s} \qquad \text{and} \qquad H = \sum_{s=0}^{\infty} \frac{H_s(\theta,\zeta)}{N^s}, \tag{3.11}$$

where the coefficients can be determined recursively by the equations

$$\begin{cases} \dfrac{\partial^2 G_0}{\partial \theta^2} - \zeta G_0 = 0, & G_0\Big|_{\theta=0} = 1, \quad \dfrac{\partial G_0}{\partial \theta}\Big|_{\theta=0} = 0, \\[2ex] \dfrac{\partial^2 H_0}{\partial \theta^2} - \zeta H_0 = 0, & H_0\Big|_{\theta=0} = 0, \quad \dfrac{\partial H_0}{\partial \theta}\Big|_{\theta=0} = 1, \end{cases} \tag{3.12}$$

$$\begin{cases} \dfrac{\partial^2 G_1}{\partial\theta^2} - \zeta G_1 = 0, & G_1\Big|_{\theta=0} = 0, & \dfrac{\partial G_1}{\partial\theta}\Big|_{\theta=0} = \dfrac{1}{2} - \nu, \\ \dfrac{\partial^2 H_1}{\partial\theta^2} - \zeta H_1 = 0, & H_1\Big|_{\theta=0} = 0, & \dfrac{\partial H_1}{\partial\theta}\Big|_{\theta=0} = 0, \end{cases} \tag{3.13}$$

$$\begin{cases} \dfrac{\partial^2 G_s}{\partial\theta^2} - \zeta G_s = \left(\nu^2 - \dfrac{1}{4}\right)\displaystyle\sum_{j=2}^{s}(-1)^j(j-1)\theta^{j-2}G_{s-j}, \\ G_s\Big|_{\theta=0} = \dfrac{\partial G_s}{\partial\theta}\Big|_{\theta=0} = 0, \end{cases} \tag{3.14}$$

and

$$\begin{cases} \dfrac{\partial^2 H_s}{\partial\theta^2} - \zeta H_s = \left(\nu^2 - \dfrac{1}{4}\right)\displaystyle\sum_{j=2}^{s}(-1)^j(j-1)\theta^{j-2}H_{s-j}, \\ H_s\Big|_{\theta=0} = \dfrac{\partial H_s}{\partial\theta}\Big|_{\theta=0} = 0 \end{cases} \tag{3.15}$$

for $s \geq 2$. These equations can be solved explicitly, and we have

$$G_0 = \frac{1}{2}\big(e^{\sqrt{\zeta}\theta} + e^{-\sqrt{\zeta}\theta}\big), \qquad H_0 = \frac{1}{2\sqrt{\zeta}}\big(e^{\sqrt{\zeta}\theta} - e^{-\sqrt{\zeta}\theta}\big), \tag{3.16}$$

$$G_1 = \frac{\frac{1}{2} - \nu}{2\sqrt{\zeta}}\big(e^{\sqrt{\zeta}\theta} - e^{-\sqrt{\zeta}\theta}\big), \qquad H_1 = 0, \tag{3.17}$$

and

$$\begin{cases} G_s = \dfrac{\nu^2 - \frac{1}{4}}{2\sqrt{\zeta}}\displaystyle\int_0^\theta \Big(\sum_{j=2}^{s}(-1)^j(j-1)\phi^{j-2}G_{s-j}\Big)\big(e^{\sqrt{\zeta}(\theta-\phi)} - e^{\sqrt{\zeta}(\phi-\theta)}\big)\,d\phi, \\ H_s = \dfrac{\nu^2 - \frac{1}{4}}{2\sqrt{\zeta}}\displaystyle\int_0^\theta \Big(\sum_{j=2}^{s}(-1)^j(j-1)\phi^{j-2}H_{s-j}\Big)\big(e^{\sqrt{\zeta}(\theta-\phi)} - e^{\sqrt{\zeta}(\phi-\theta)}\big)\,d\phi \end{cases} \tag{3.18}$$

for $s \geq 2$, where $\sqrt{\zeta} = i\sqrt{-\zeta}$ if $\zeta < 0$. Using the inequalities

$$\cosh x \sinh y \leq y \cosh(x+y) \qquad \text{and} \qquad \sinh x \sinh y \leq y \sinh(x+y),$$

it can be proved by induction that for $\zeta < 0$

$$|G_s(\theta,\zeta)| \leq (|\nu|+1)^s|\theta|^s \qquad \text{and} \qquad |H_s(\theta,\zeta)| \leq (|\nu|+1)^s|\theta|^{s+1}, \tag{3.19}$$

and that for $\zeta > 0$

$$|G_s(\theta,\zeta)| \leq (|\nu|+1)^s|\theta|^s G_0(\theta,\zeta), \qquad |H_s(\theta,\zeta)| \leq (|\nu|+1)^s|\theta|^s\,|H_0(\theta,\zeta)|. \tag{3.20}$$

Thus, the formal series in (3.11) are uniformly convergent for any bounded θ and sufficiently large N, and (3.2) follows.

Let $L_s(\theta,\zeta)$ and $K_s(\theta,\zeta)$ be given as in (3.5). By the same argument, we have

$$K_0(\theta,\zeta)=\frac{1}{2}\big(e^{\sqrt{\zeta}\theta}+e^{-\sqrt{\zeta}\theta}\big), \qquad L_0(\theta,\zeta)=\frac{1}{2\sqrt{\zeta}}\big(e^{\sqrt{\zeta}\theta}-e^{-\sqrt{\zeta}\theta}\big), \tag{3.21}$$

$$K_1(\theta,\zeta)=\frac{\nu-\frac{1}{2}}{2\sqrt{\zeta}}\big(e^{\sqrt{\zeta}\theta}-e^{-\sqrt{\zeta}\theta}\big), \qquad L_1(\theta,\zeta)=0. \tag{3.22}$$

Furthermore, for $\zeta<0$

$$|K_s(\theta,\zeta)|\le(|\nu|+1)^s|\theta|^s, \qquad |L_s(\theta,\zeta)|\le(|\nu|+1)^s|\theta|^{s+1}, \tag{3.23}$$

and for $\zeta>0$

$$|K_s(\theta,\zeta)|\le(|\nu|+1)^s|\theta|^sK_0(\theta,\zeta), \qquad |L_s(\theta,\zeta)|\le(|\nu|+1)^s|\theta|^s\,|L_0(\theta,\zeta)|. \tag{3.24}$$

This demonstrates the uniform convergence of the formal series in (3.5) for sufficiently large N, and completes the proof of the lemma. ∎

From the recursive formula (3.16) - (3.18), it can be readily shown that

$$G_s(-\theta,\zeta)=(-1)^sG_s(\theta,\zeta) \qquad \text{and} \qquad H_s(-\theta,\zeta)=(-1)^{s-1}H_s(\theta,\zeta). \tag{3.25}$$

Choosing $\theta=\pm1$ in (3.2), we have

$$\left(1\pm\frac{1}{N}\right)^{\frac{1}{2}}Z_\nu[(N\pm1)\zeta^{\frac{1}{2}}]=Z_\nu(N\zeta^{\frac{1}{2}})G\left(\zeta,\pm\frac{1}{N}\right)\pm\zeta^{\frac{1}{2}}Z_{\nu-1}(N\zeta^{\frac{1}{2}})H\left(\zeta,\pm\frac{1}{N}\right), \tag{3.26}$$

where

$$G\left(\zeta,\pm\frac{1}{N}\right):=G(N;\pm1,\zeta):=\sum_{s=0}^{\infty}(\pm1)^s\frac{G_s(\zeta)}{N^s} \tag{3.27}$$

and

$$H\left(\zeta,\pm\frac{1}{N}\right):=\pm H(N;\pm1,\zeta):=\sum_{s=0}^{\infty}(\pm1)^s\frac{H_s(\zeta)}{N^s}. \tag{3.28}$$

Similarly, it follows from (3.3) that

$$\left(1\pm\frac{1}{N}\right)^{\frac{1}{2}}Z_{\nu-1}[(N\pm1)\zeta^{\frac{1}{2}}]=\pm\zeta^{\frac{1}{2}}Z_\nu(N\zeta^{\frac{1}{2}})L\left(\zeta,\pm\frac{1}{N}\right)+Z_{\nu-1}(N\zeta^{\frac{1}{2}})K\left(\zeta,\pm\frac{1}{N}\right), \tag{3.29}$$

where

$$L\left(\zeta,\pm\frac{1}{N}\right):=\pm L(N;\pm1,\zeta):=\sum_{s=0}^{\infty}(\pm1)^s\frac{L_s(\zeta)}{N^s} \tag{3.30}$$

and

$$K\left(\zeta,\pm\frac{1}{N}\right):=K(N;\pm1,\zeta):=\sum_{s=0}^{\infty}(\pm1)^s\frac{K_s(\zeta)}{N^s}. \tag{3.31}$$

By (3.16) and (3.21), we also have

$$G_0(\zeta) = K_0(\zeta) = \cosh \zeta^{\frac{1}{2}} \tag{3.32}$$

and

$$H_0(\zeta) = L_0(\zeta) = \frac{\sinh \sqrt{\zeta}}{\sqrt{\zeta}}. \tag{3.33}$$

Later in our discussion, we also need the values $G\big(0, \dfrac{1}{N}\big), H\big(0, \dfrac{1}{N}\big), L\big(0, \dfrac{1}{N}\big)$ and $K\big(0, \dfrac{1}{N}\big)$. To this end, we note that using (3.26) with $Z_\nu = I_\nu$ and $Z_\nu = e^{i\nu\pi} K_\nu$, we have, respectively,

$$G\left(\zeta, \frac{1}{N}\right) = \left(1 + \frac{1}{N}\right)^{\frac{1}{2}} \frac{K_\nu[(N+1)\zeta^{\frac{1}{2}}]}{K_\nu(N\zeta^{\frac{1}{2}})} + \frac{\zeta^{\frac{1}{2}} K_{\nu-1}(N\zeta^{\frac{1}{2}})}{K_\nu(N\zeta^{\frac{1}{2}})} H\left(\zeta, \frac{1}{N}\right)$$

and

$$H\left(\zeta, \frac{1}{N}\right) = \left(1 + \frac{1}{N}\right)^{\frac{1}{2}} \frac{I_\nu[(N+1)\zeta^{\frac{1}{2}}]}{\zeta^{\frac{1}{2}} I_{\nu-1}(N\zeta^{\frac{1}{2}})} - \frac{I_\nu(N\zeta^{\frac{1}{2}})}{\zeta^{\frac{1}{2}} I_{\nu-1}(N\zeta^{\frac{1}{2}})} G\left(\zeta, \frac{1}{N}\right).$$

From the asymptotic relations

$$I_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)} \qquad \text{and} \qquad K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}$$

as $z \to 0$, it follows that

$$G\left(0, \frac{1}{N}\right) = \left(1 + \frac{1}{N}\right)^{-\nu+\frac{1}{2}}, \tag{3.34}$$

$$H\left(0, \frac{1}{N}\right) = \frac{N}{2\nu}\left[\left(1 + \frac{1}{N}\right)^{\nu+\frac{1}{2}} - \left(1 + \frac{1}{N}\right)^{-\nu+\frac{1}{2}}\right]. \tag{3.35}$$

In a similar manner, we obtain

$$K\left(0, \frac{1}{N}\right) = \left(1 + \frac{1}{N}\right)^{\nu-\frac{1}{2}}, \tag{3.36}$$

$$L\left(0, \frac{1}{N}\right) = \frac{N}{2(\nu-1)}\left[\left(1 + \frac{1}{N}\right)^{\nu-\frac{1}{2}} - \left(1 + \frac{1}{N}\right)^{-\nu+\frac{3}{2}}\right]. \tag{3.37}$$

4 Formal Asymptotic Solutions

Let $\zeta(x)$ be an increasing function with $\zeta(x_+) = 0$. We try a formal series solution to (1.3) in the form

$$P_n(x) = N^{\frac{1}{2}} Z_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} Z_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}; \tag{4.1}$$

cf. (2.15). For convenience, we put

$$A\Big(\zeta, \frac{1}{N}\Big) := \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s}, \qquad B\Big(\zeta, \frac{1}{N}\Big) := \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \tag{4.2}$$

and

$$\Psi\Big(x, \frac{1}{N}\Big) := A_n x + B_n = \sum_{s=0}^{\infty} \frac{\alpha_s' x + \beta_s'}{N^s}; \tag{4.3}$$

cf. (1.3) and (2.6). By Lemma 1, we have

$$\begin{aligned} P_{n\pm1}(x) =& N^{\frac{1}{2}} Z_\nu(N\zeta^{\frac{1}{2}}) \Big\{ G\Big(\zeta, \pm\frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N\pm1}\Big) \pm \zeta L\Big(\zeta, \pm\frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N\pm1}\Big) \Big\} \\ &+ N^{\frac{1}{2}} \zeta^{\frac{1}{2}} Z_{\nu-1}(N\zeta^{\frac{1}{2}}) \Big\{ K\Big(\zeta, \pm\frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N\pm1}\Big) \pm H\Big(\zeta, \pm\frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N\pm1}\Big) \Big\}. \end{aligned} \tag{4.4}$$

Substituting (4.1) and (4.4) into the recurrence relation (1.3) and matching the coefficients of Z_ν and $Z_{\nu-1}$, we obtain

$$\begin{aligned} G\Big(\zeta, \frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N+1}\Big) + G\Big(\zeta, -\frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N-1}\Big) - \Psi\Big(x, \frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N}\Big) \\ + \zeta L\Big(\zeta, \frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N+1}\Big) - \zeta L\Big(\zeta, -\frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N-1}\Big) = 0 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} H\Big(\zeta, \frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N+1}\Big) - H\Big(\zeta, -\frac{1}{N}\Big) A\Big(\zeta, \frac{1}{N-1}\Big) - \Psi\Big(x, \frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N}\Big) \\ + K\Big(\zeta, \frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N+1}\Big) + K\Big(\zeta, -\frac{1}{N}\Big) B\Big(\zeta, \frac{1}{N-1}\Big) = 0. \end{aligned} \tag{4.6}$$

By letting $N \to \infty$, the last two equations give

$$G_0(\zeta) = K_0(\zeta) = \frac{\alpha_0' x + \beta_0'}{2}. \tag{4.7}$$

Coupling (3.16) and (4.7) yields

$$\zeta^{\frac{1}{2}} = \cosh^{-1}\Big(\frac{\alpha_0' x + \beta_0'}{2}\Big). \tag{4.8}$$

Equating coefficients of like powers of $1/N$ in (4.5) and (4.6), we get

$$\sum_{2s\le p}\left[A_{p-2s}\sum_{i=0}^{2s}\binom{2s-p}{2s-i}G_i\right] + \sum_{2s+1\le p}\left[B_{p-2s-1}\sum_{i=0}^{2s+1}\binom{2s+1-p}{2s+1-i}\zeta L_i\right]-\sum_{s\le p}\frac{\alpha_s' x+\beta_s'}{2}A_{p-s}=0 \tag{4.9}$$

and

$$\sum_{2s\le p}\left[B_{p-2s}\sum_{i=0}^{2s}\binom{2s-p}{2s-i}K_i\right] + \sum_{2s+1\le p}\left[A_{p-2s-1}\sum_{i=0}^{2s+1}\binom{2s+1-p}{2s+1-i}H_i\right]-\sum_{s\le p}\frac{\alpha_s' x+\beta_s'}{2}B_{p-s}=0. \tag{4.10}$$

From (4.9) and (4.10), it follows that

$$(p-1)H_0A_{p-1}=\sum_{1\le 2s\le p}\left[B_{p-2s}\sum_{i=0}^{2s}\binom{2s-p}{2s-i}K_i\right] + \sum_{2\le 2s+1\le p}\left[A_{p-2s-1}\sum_{i=0}^{2s+1}\binom{2s+1-p}{2s+1-i}H_i\right]-\sum_{2\le s\le p}\frac{\alpha_s' x+\beta_s'}{2}B_{p-s} \tag{4.11}$$

and

$$(p-1)\zeta L_0B_{p-1}=\sum_{1\le 2s\le p}\left[A_{p-2s}\sum_{i=0}^{2s}\binom{2s-p}{2s-i}G_i\right] + \sum_{2\le 2s+1\le p}\left[B_{p-2s-1}\sum_{i=0}^{2s+1}\binom{2s+1-p}{2s+1-i}\zeta L_i\right]-\sum_{2\le s\le p}\frac{\alpha_s' x+\beta_s'}{2}A_{p-s}, \tag{4.12}$$

where we have made use of the fact that $H_1=L_1=0$. Thus, for each $p\ge 1, A_p(\zeta)$ and $B_p(\zeta)$ can be determined successively from the above two equations for any given $A_0(\zeta)$ and $B_0(\zeta)$.

LEMMA 2. *Let $\zeta(x)$ be given as in (4.8), and suppose that $|A_0(\zeta)|$ and $(1+|\zeta|^{1/2})|B_0(\zeta)|$ are bounded for $x\ge x_-+\delta$. Further, let $A_s(\zeta)$ and $B_s(\zeta)$ be successively defined as in (4.11) and (4.12). Then there exists a positive constant N_s independent of x such that*

$$|A_s(\zeta)|\le N_s, \qquad |B_s(\zeta)|\le\frac{N_s}{(1+|\zeta|^{1/2})}, \qquad s=1,\ 2,\ \cdots, \tag{4.13}$$

for all $x\ge x_-+\delta,\ \delta>0$.

Proof. Using (3.32) and (3.33), it can be shown that

$$|G_0| \le C|H_0|(1+|\zeta|^{1/2}) \tag{4.14}$$

for $x \ge x_- + \delta$, where C is a positive constant. From (3.19), (3.20), (3.23) and (3.24), it also follows that there exists a constant C_s, independent of x, such that

$$|K_s| + |G_s| \le C_s|H_0|(1+|\zeta|^{1/2}). \tag{4.15}$$

Thus, if the functions $A_p(\zeta)$ and $B_p(\zeta)$ given successively in (4.11) and (4.12) are well-defined (i.e., $B_p(\zeta)$ is bounded at $\zeta = 0$), then the estimates in (4.13) can be readily verified by induction. To show that $B_p(0)$ is bounded, we divide our discussion into three cases : (i) $2\nu \ne 0, 1, 2, \cdots$; (ii) $2\nu = 1, 2, 3, \cdots$, and (iii) $2\nu = 0$.

In case (i), we first consider the second-order linear difference equation

$$y_{N+1} + y_{N-1} - \Psi\left(x_+, \frac{1}{N}\right) y_N = 0, \tag{4.16}$$

where $\Psi(x_+,\ 1/N)$ is given in (4.3). The results in [21] infer that (4.16) has two linearly independent asymptotic solutions of the form

$$y_N^{(1)} \sim N^{-\nu+\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_s}{N^s}, \qquad\qquad y_N^{(2)} \sim N^{\nu+\frac{1}{2}} \sum_{s=0}^{\infty} \frac{d_s}{N^s}, \tag{4.17}$$

where $c_0 = d_0 = 1$,

$$c_1 = \frac{\alpha_3' x_+ + \beta_3'}{1+2\nu} \qquad \text{and} \qquad d_1 = \frac{\alpha_3' x_+ + \beta_3'}{1-2\nu} \tag{4.18}$$

on account of (2.14). We shall show that for all $p \ge 0$,

$$A_p(0) = c_p \qquad \text{and} \qquad B_p(0) = \frac{d_{p+1} - c_{p+1}}{2\nu}. \tag{4.19}$$

To this end, note that

$$\alpha_3' x_+ + \beta_3' = \alpha_3 x_+ + \beta_3 + 2\tau_0(\alpha_2 x_+ + \beta_2).$$

Since $2\nu \ne 0, 1, 2, \cdots$ in this case, we can choose τ_0 to be

$$\tau_0 = -\frac{\alpha_3 x_+ + \beta_3}{2(\alpha_2 x_+ + \beta_2)} = -\frac{\alpha_3 x_+ + \beta_3}{2(\nu^2 - \frac{1}{4})}, \tag{4.20}$$

so that $B_0(0) = 0$. Note that $\nu \ne 1/2$ in the present case, and that the last equality follows from (2.14).

Returning to (4.5) and (4.6), we set $\zeta = 0$. This yields

$$G\left(0, \frac{1}{N}\right) A\left(0, \frac{1}{N+1}\right) + G\left(0, -\frac{1}{N}\right) A\left(0, \frac{1}{N-1}\right) - \Psi\left(x_+, \frac{1}{N}\right) A\left(0, \frac{1}{N}\right) = 0 \tag{4.21}$$

and

$$H\Big(0,\frac{1}{N}\Big)A\Big(0,\frac{1}{N+1}\Big)-H\Big(0,-\frac{1}{N}\Big)A\Big(0,\frac{1}{N-1}\Big)-\Psi\Big(x_+,\frac{1}{N}\Big)B\Big(0,\frac{1}{N}\Big)$$
$$+K\Big(0,\frac{1}{N}\Big)B\Big(0,\frac{1}{N+1}\Big)+K\Big(0,-\frac{1}{N}\Big)B\Big(0,\frac{1}{N-1}\Big)=0. \tag{4.22}$$

To see that (4.21) can be written in the form of (4.16), we put

$$x_N := N^{-\nu+\frac{1}{2}}A\Big(0,\frac{1}{N}\Big). \tag{4.23}$$

From (4.21) and (3.34), it is clear that x_N satisfies

$$x_{N+1}+x_{N-1}-\Psi\Big(x_+,\frac{1}{N}\Big)x_N=0$$

and we have

$$N^{-\nu+\frac{1}{2}}A\Big(0,\frac{1}{N}\Big)=y_N^{(1)}\sim N^{-\nu+\frac{1}{2}}\sum_{s=0}^{\infty}\frac{c_s}{N^s}$$

on account of (4.17). This gives the first equation in (4.19). Analogously, we put

$$\hat{x}_N = N^{\nu-\frac{1}{2}}\Big[NA\Big(0,\frac{1}{N}\Big)+2\nu B\Big(0,\frac{1}{N}\Big)\Big]. \tag{4.24}$$

Then, by using (3.35) and (3.36), it can be shown that $\hat{x}_N$ satisfies (4.16) and

$$N^{\nu-\frac{1}{2}}\Big[NA\Big(0,\frac{1}{N}\Big)+2\nu B\Big(0,\frac{1}{N}\Big)\Big]=y_N^{(2)}\sim N^{\nu+\frac{1}{2}}\sum_{s=0}^{\infty}\frac{d_s}{N^s},$$

from which it follows

$$A_s(0)+2\nu B_{s-1}(0)=d_s \tag{4.25}$$

and we obtain the second equation in (4.19).

In case (ii), i.e., $2\nu=1,2,\cdots$, the two linearly independent asymptotic solutions of (4.16) are

$$y_N^{(1)}\sim N^{-\nu+\frac{1}{2}}\sum_{s=0}^{\infty}\frac{c_s}{N^s},\qquad y_N^{(2)}\sim N^{\nu+\frac{1}{2}}\sum_{s=0}^{\infty}\frac{d_s}{N^s}+Cy_N^{(1)}\log N, \tag{4.26}$$

where C is a constant; see Wong and Li [21]. In a similar manner, we have

$$N^{-\nu+\frac{1}{2}}A\Big(0,\frac{1}{N}\Big)=y_N^{(1)}\sim N^{-\nu+\frac{1}{2}}\sum_{s=0}^{\infty}\frac{c_s}{N^s}$$

and

$$N^{\nu-\frac{1}{2}}\Big[NA\Big(0,\frac{1}{N}\Big)+2\nu B\Big(0,\frac{1}{N}\Big)\Big]=0$$

on account of (4.26). Coupling these together gives

$$A_s(0) = c_s \qquad \text{for} \quad s = 0, 1, \cdots \tag{4.27}$$

and

$$2\nu B_s(0) = A_{s+1}(0) \qquad \text{for} \quad s = 0, 1, \cdots . \tag{4.28}$$

When $\nu \neq 1/2$, we can still choose τ_0 as in (4.20) so that $B_0(0) = 0$. When $\nu = 1/2$, there is no singularity at $\xi = 0$ in (2.13), and the functions Z_ν and $Z_{\nu-1}$ in (4.1) can be expressed in terms of hyperbolic cosine and hyperbolic sine. So $\nu = 1/2$ is a simple case, and we can choose τ_0 to be any real number.

In case (iii), $\nu = 0$ and the results in [21] infer that (4.16) has two linearly independent asymptotic solutions of the form

$$y_N^{(1)} \sim N^{\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_s}{N^s}, \qquad\qquad y_N^{(2)} \sim N^{\frac{1}{2}} \sum_{s=0}^{\infty} \frac{d_s}{N^s} + y_N^{(1)} \log N, \tag{4.29}$$

where $c_0 = d_0 = 1$ and $c_1 = -d_1 = (\alpha_3' x_+ + \beta_3')$. Similarly, we have

$$N^{\frac{1}{2}} A\left(0, \frac{1}{N}\right) = y_N^{(1)} \sim N^{\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_s}{N^s}$$

and

$$N^{-\frac{1}{2}} \left[(N \log N) A\left(0, \frac{1}{N}\right) + B\left(0, \frac{1}{N}\right) \right] = y_N^{(2)} - y_N^{(1)} \sim N^{\frac{1}{2}} \sum_{s=1}^{\infty} \frac{d_s - c_s}{N^s} + y_N^{(1)} \log N,$$

from which it follows

$$A_s(0) = c_s \qquad \text{and} \qquad B_s(0) = d_{s+1} - c_{s+1} \qquad \text{for} \quad s = 0, 1, \cdots . \tag{4.30}$$

In the case of $\nu = 0$, we can also choose τ_0 as in (4.20) so that $B_0(0) = 0$. The proof of the lemma is now complete. ∎

5 Proof of the Theorem

Since x is a fixed number in the recurrence relation (1.3), we may take

$$Z_\nu(N\zeta^{\frac{1}{2}}) = H_0^{-\frac{1}{2}}(\zeta) I_\nu(N\zeta^{\frac{1}{2}}) \qquad \text{and} \qquad Z_\nu(N\zeta^{\frac{1}{2}}) = e^{i\pi\nu} H_0^{-\frac{1}{2}}(\zeta) K_\nu(N\zeta^{\frac{1}{2}})$$

in (4.1). In view of (3.33) and (4.8), we obtain two formal solutions

$$\begin{aligned} P_n(x) = \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} \Bigg[& N^{\frac{1}{2}} I_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \\ & + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} I_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \Bigg] \end{aligned} \tag{5.1}$$

and

$$Q_n(x) = \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \left[N^{\frac{1}{2}} K_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \right. \\ \left. - N^{\frac{1}{2}} \zeta^{\frac{1}{2}} K_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}\right]. \tag{5.2}$$

Here, I_ν and K_ν are the modified Bessel functions.

For convenience, we introduce the notations

$$A_p\left(\zeta, \frac{1}{N}\right) := \sum_{s=0}^{p} \frac{A_s(\zeta)}{N^s}, \qquad B_p\left(\zeta, \frac{1}{N}\right) := \sum_{s=0}^{p} \frac{B_s(\zeta)}{N^s}, \tag{5.3}$$

$$r_n^p(x) := \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \left[N^{\frac{1}{2}} I_\nu(N\zeta^{\frac{1}{2}}) A_p\left(\zeta, \frac{1}{N}\right) \right. \\ \left. + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} I_{\nu-1}(N\zeta^{\frac{1}{2}}) B_p\left(\zeta, \frac{1}{N}\right)\right] \tag{5.4}$$

and

$$s_n^p(x) := \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \left[N^{\frac{1}{2}} K_\nu(N\zeta^{\frac{1}{2}}) A_p\left(\zeta, \frac{1}{N}\right) \right. \\ \left. - N^{\frac{1}{2}} \zeta^{\frac{1}{2}} K_{\nu-1}(N\zeta^{\frac{1}{2}}) B_p\left(\zeta, \frac{1}{N}\right)\right]. \tag{5.5}$$

By Lemma 1, we have

$$r_{n+1}^p(x) - (A_n x + B_n)\, r_n^p(x) + r_{n-1}^p(x) = \frac{R_n^p(x)}{N^{p+\frac{3}{2}}}, \tag{5.6}$$

where the nonhomogeneous term is given by

$$R_n^p(x)/N^{p+\frac{3}{2}} = \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \left[N^{\frac{1}{2}} I_\nu(N\zeta^{\frac{1}{2}}) F_{1,n}(x) \right. \\ \left. + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} I_{\nu-1}(N\zeta^{\frac{1}{2}}) F_{2,n}(x)\right] \tag{5.7}$$

with

$$F_{1,n}(x) = G\left(\zeta, \frac{1}{N}\right) A_p\left(\zeta, \frac{1}{N+1}\right) + G\left(\zeta, -\frac{1}{N}\right) A_p\left(\zeta, \frac{1}{N-1}\right) \\ - \Psi\left(x, \frac{1}{N}\right) A_p\left(\zeta, \frac{1}{N}\right) + \zeta L\left(\zeta, \frac{1}{N}\right) B_p\left(\zeta, \frac{1}{N+1}\right) \\ - \zeta L\left(\zeta, -\frac{1}{N}\right) B_p\left(\zeta, \frac{1}{N-1}\right) \tag{5.8}$$

and

$$F_{2,n}(x) = H\left(\zeta, \frac{1}{N}\right) A_p\left(\zeta, \frac{1}{N+1}\right) - H\left(\zeta, -\frac{1}{N}\right) A_p\left(\zeta, \frac{1}{N-1}\right) \\ - \Psi\left(x, \frac{1}{N}\right) B_p\left(\zeta, \frac{1}{N}\right) + K\left(\zeta, \frac{1}{N}\right) B_p\left(\zeta, \frac{1}{N+1}\right) \\ + K\left(\zeta, \frac{1}{N}\right) B_p\left(\zeta, \frac{1}{N-1}\right). \tag{5.9}$$

Recall that the series

$$A\left(\zeta, \frac{1}{N}\right) = \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \qquad \text{and} \qquad B\left(\zeta, \frac{1}{N}\right) = \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}$$

in (4.2) are formal solutions of (4.5) and (4.6). Since $A_p(\zeta, \frac{1}{N})$ and $B_p(\zeta, \frac{1}{N})$ can be written as

$$A_p\left(\zeta, \frac{1}{N}\right) = \sum_{s=0}^{\infty} \frac{A_s^*(\zeta)}{N^s} \qquad \text{and} \qquad B_p\left(\zeta, \frac{1}{N}\right) = \sum_{s=0}^{\infty} \frac{B_s^*(\zeta)}{N^s}$$

with $A_s^*(\zeta) = A_s(\zeta), B_s^*(\zeta) = B_s(\zeta)$ for $s \le p$ and $A_s^*(\zeta) = B_s^*(\zeta) = 0$ for $s \ge p+1$, terms with powers of $1/N$ less than or equal to $p+1$ in the expansions of $F_{1,n}(x)$ and $F_{2,n}(x)$ all vanish. (Note: the recurrence relations (4.11) and (4.12) were obtained when we equated coefficients of $1/N^p$ in (4.5) and (4.6) to zero.) Hence, using Lemma 2, it can be proved that there is a constant C_p such that

$$|F_{1,n}(x)| \le C_p(1+|x|)/N^{p+2} \tag{5.10}$$

and

$$(1 + |\zeta^{\frac{1}{2}}|)|F_{2,n}(x)| \le C_p(1+|x|)/N^{p+2} \tag{5.11}$$

for all $x \ge x_- + \delta$. From (5.7), it follows that

$$|R_n^p(x)| \le \widetilde{C}_p \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} (1+|x|) \left[|I_\nu(N\zeta^{\frac{1}{2}})| + \frac{|\zeta|^{\frac{1}{2}}}{1+|\zeta|^{\frac{1}{2}}} |I_{\nu-1}(N\zeta^{\frac{1}{2}})| \right] \tag{5.12}$$

for all $x \ge x_- + \delta$ and for some positive constant $\widetilde{C}_p$. Similarly, we have

$$s_{n+1}^p(x) - (A_n x + B_n) s_n^p(x) + s_{n-1}^p(x) = \frac{S_n^p(x)}{N^{p+\frac{3}{2}}}, \tag{5.13}$$

where

$$|S_n^p(x)| \le \widetilde{C}_p \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} (1+|x|) \left[|K_\nu(N\zeta^{\frac{1}{2}})| + \frac{|\zeta|^{\frac{1}{2}}}{1+|\zeta|^{\frac{1}{2}}} |K_{\nu-1}(N\zeta^{\frac{1}{2}})| \right]. \tag{5.14}$$

We now establish the existence of two solutions $P_n(x)$ and $Q_n(x)$ of (1.3) satisfying

$$P_n(x) \sim r_n^0(x) \qquad \text{and} \qquad Q_n(x) \sim s_n^0(x) \tag{5.15}$$

as $n \to \infty$ for any fixed $x > x_-$. It is easily verified that

$$r_n^0(x) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{4}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} e^{N\zeta^{\frac{1}{2}}} (A_0 + \zeta^{\frac{1}{2}} B_0), \qquad x > x_+, \tag{5.16}$$

$$\begin{aligned} r_n^0(x) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{4}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} \Big[& e^{-N\zeta^{\frac{1}{2}} + (\nu + \frac{1}{2})\pi i} (A_0 - \zeta^{\frac{1}{2}} B_0) \\ & + e^{N\zeta^{\frac{1}{2}}} (A_0 + \zeta^{\frac{1}{2}} B_0) \Big], \qquad x_- + \delta \le x < x_+, \end{aligned} \tag{5.17}$$

and

$$s_n^0(x) \sim \sqrt{\frac{\pi}{2}} \left(\frac{4}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} e^{-N\zeta^{\frac{1}{2}}} (A_0 - \zeta^{\frac{1}{2}} B_0), \qquad x \ge x_- + \delta, \tag{5.18}$$

as $n \to \infty$. If $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions of (1.3) satisfying (5.15), then it follows from (5.16), (5.17) and (5.18) that

$$P_{n+1}(x)Q_n(x) - P_n(x)Q_{n+1}(x) = P_{n+2}(x)Q_{n+1}(x) - P_{n+1}(x)Q_{n+2}(x)$$

and

$$\begin{aligned} P_{n+1}(x)Q_n(x) - P_n(x)Q_{n+1}(x) &= \lim_{m \to \infty} \left[r_{m+1}^0(x) s_m^0(x) - r_m^0(x) s_{m+1}^0(x) \right] \\ &= A_0^2 - \zeta B_0^2 \neq 0. \end{aligned}$$

Without loss of generality, we may assume $A_0^2 - \zeta B_0^2 = 1$. From this, it follows that there exists a smooth function $\Gamma(\zeta)$ such that $(1 + |\zeta|^{1/2})\Gamma(\zeta)$ is bounded for $x \ge x_- + \delta$ and

$$A_0 = \cosh\left(\sqrt{\zeta}\Gamma(\zeta)\right), \qquad B_0 = \frac{\sinh\left(\sqrt{\zeta}\Gamma(\zeta)\right)}{\sqrt{\zeta}}; \tag{5.19}$$

see the assumption in Lemma 2. With these choices, we have

$$P_{n+1}(x)Q_n(x) - P_n(x)Q_{n+1}(x) = 1. \tag{5.20}$$

This, in particular, shows that $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions.

Now define

$$\varepsilon_n^p(x) := P_n(x) - r_n^p(x) \qquad \text{and} \qquad \delta_n^p(x) := Q_n(x) - s_n^p(x). \tag{5.21}$$

We first show that the existence of $Q_n(x)$ to (1.3) satisfying (5.15) is equivalent to the existence of $\delta_n^p(x)$ to the summation formula

$$\begin{aligned} \delta_n^p(x) = & \sum_{j=n+1}^{\infty} \frac{[r_n^p(x) s_j^p(x) - s_n^p(x) r_j^p(x)] S_j^p(x)}{[r_{n+1}^p(x) s_n^p(x) - r_n^p(x) s_{n+1}^p(x)](j + \tau_0)^{p+\frac{3}{2}}} \\ & + \sum_{j=n+1}^{\infty} \frac{[r_n^p(x) S_j^p(x) - s_n^p(x) R_j^p(x)] \delta_j^p(x)}{[r_{n+1}^p(x) s_n^p(x) - r_n^p(x) s_{n+1}^p(x)](j + \tau_0)^{p+\frac{3}{2}}}. \end{aligned} \tag{5.22}$$

From (1.3) and (5.13), we obtain

$$\delta_{n+1}^p(x) - (A_n x + B_n)\delta_n^p(x) + \delta_{n-1}^p(x) = -\frac{S_n^p(x)}{N^{p+\frac{3}{2}}}. \tag{5.23}$$

Coupling (5.6) and (5.23) gives

$$\begin{aligned} r_n^p(x)\delta_{n+1}^p(x) - r_{n+1}^p(x)\delta_n^p(x) = r_{n+1}^p(x)\delta_{n+2}^p(x) - r_{n+2}^p(x)\delta_{n+1}^p(x) \\ + \frac{S_{n+1}^p(x)r_{n+1}^p(x) + R_{n+1}^p(x)\delta_{n+1}^p(x)}{(N+1)^{p+\frac{3}{2}}}. \end{aligned} \tag{5.24}$$

In exactly the same manner, we also have

$$\begin{aligned} s_n^p(x)\delta_{n+1}^p(x) - s_{n+1}^p(x)\delta_n^p(x) = s_{n+1}^p(x)\delta_{n+2}^p(x) - s_{n+2}^p(x)\delta_{n+1}^p(x) \\ + \frac{S_{n+1}^p(x)s_{n+1}^p(x) + S_{n+1}^p(x)\delta_{n+1}^p(x)}{(N+1)^{p+\frac{3}{2}}}. \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} r_{n+1}^p(x)s_n^p(x) - s_{n+1}^p(x)r_n^p(x) = r_{n+2}^p(x)s_{n+1}^p(x) - s_{n+2}^p(x)r_{n+1}^p(x) \\ + \frac{S_{n+1}^p(x)r_{n+1}^p(x) - R_{n+1}^p(x)s_{n+1}^p(x)}{(N+1)^{p+\frac{3}{2}}}. \end{aligned} \tag{5.26}$$

Repeated application of the last three equations yields, respectively,

$$\begin{aligned} r_n^p(x)\delta_{n+1}^p(x) - r_{n+1}^p(x)\delta_n^p(x) = r_{m+1}^p(x)\delta_{m+2}^p(x) - r_{m+2}^p(x)\delta_{m+1}^p(x) \\ + \sum_{j=n+1}^{m+1} \frac{R_j^p(x)\delta_j^p(x) + S_j^p(x)r_j^p(x)}{(j+\tau_0)^{p+\frac{3}{2}}}. \end{aligned} \tag{5.27}$$

$$\begin{aligned} s_n^p(x)\delta_{n+1}^p(x) - s_{n+1}^p(x)\delta_n^p(x) = s_{m+1}^p(x)\delta_{m+2}^p(x) - s_{m+2}^p(x)\delta_{m+1}^p(x) \\ + \sum_{j=n+1}^{m+1} \frac{S_j^p(x)s_j^p(x) + S_j^p(x)\delta_j^p(x)}{(j+\tau_0)^{p+\frac{3}{2}}} \end{aligned} \tag{5.28}$$

and

$$\begin{aligned} r_{n+1}^p(x)s_n^p(x) - s_{n+1}^p(x)r_n^p(x) = r_{m+2}^p(x)s_{m+1}^p(x) - s_{m+2}^p(x)r_{m+1}^p(x) \\ + \sum_{j=n+1}^{m+1} \frac{S_j^p(x)r_j^p(x) - R_j^p(x)s_j^p(x)}{(j+\tau_0)^{p+\frac{3}{2}}}. \end{aligned} \tag{5.29}$$

If $Q_n(x)$ satisfies (5.15), then $\delta_n^0(x) = o\big(s_n^0(x)\big)$ as $n \to \infty$. Since $\delta_n^p(x) = \delta_n^0(x) + s_n^0(x) - s_n^p(x)$ by (5.21), and since $A_s(\zeta)$ and $(1+|\zeta|^{\frac{1}{2}})B_s(\zeta)$ are bounded for $s = 0, \cdots, p$ by Lemma 2, we also have $\delta_n^p(x) = o\big(s_n^0(x)\big)$ as $n \to \infty$. In view of (5.16), (5.17) and (5.18), we have

$$r_{m+1}^p(x)\delta_{m+2}^p(x) - r_{m+2}^p(x)\delta_{m+1}^p(x) \to 0, \tag{5.30}$$

$$s^p_{m+1}(x)\delta^p_{m+2}(x) - s^p_{m+2}(x)\delta^p_{m+1}(x) \to 0 \tag{5.31}$$

and

$$r^p_{m+2}(x)s^p_{m+1}(x) - s^p_{m+2}(x)r^p_{m+1}(x) \to 1 \tag{5.32}$$

as $m \to \infty$. By letting $m \to \infty$ in (5.27) - (5.29), we obtain

$$r^p_n(x)\delta^p_{n+1}(x) - r^p_{n+1}(x)\delta^p_n(x) = \sum_{j=n+1}^{\infty} \frac{R^p_j(x)\delta^p_j(x) + S^p_j(x)r^p_j(x)}{(j+\tau_0)^{p+\frac{3}{2}}}, \tag{5.33}$$

$$s^p_n(x)\delta^p_{n+1}(x) - s^p_{n+1}(x)\delta^p_n(x) = \sum_{j=n+1}^{\infty} \frac{S^p_j(x)s^p_j(x) + S^p_j(x)\delta^p_j(x)}{(j+\tau_0)^{p+\frac{3}{2}}} \tag{5.34}$$

and

$$r^p_{n+1}(x)s^p_n(x) - s^p_{n+1}(x)r^p_n(x) = 1 + \sum_{j=n+1}^{\infty} \frac{S^p_j(x)r^p_j(x) - R^p_j(x)s^p_j(x)}{(j+\tau_0)^{p+\frac{3}{2}}}. \tag{5.35}$$

Upon solving (5.33) and (5.34), we obtain (5.22). The existence of a solution $\{\delta^p_n(x)\}^\infty_{n=1}$ to (5.22) is proved by using the successive approximation method. Starting with $\delta^p_{n,0}(x) = 0$, we define $\delta^p_{n,k}(x)$ by

$$\begin{aligned}\delta^p_{n,k}(x) = &\sum_{j=n+1}^{\infty} \frac{[r^p_n(x)s^p_j(x) - s^p_n(x)r^p_j(x)]S^p_j(x)}{[r^p_{n+1}(x)s^p_n(x) - r^p_n(x)s^p_{n+1}(x)](j+\tau_0)^{p+\frac{3}{2}}} \\ &+ \sum_{j=n+1}^{\infty} \frac{[r^p_n(x)S^p_j(x) - s^p_n(x)R^p_j(x)]\delta^p_{j,k-1}(x)}{[r^p_{n+1}(x)s^p_n(x) - r^p_n(x)s^p_{n+1}(x)](j+\tau_0)^{p+\frac{3}{2}}}\end{aligned} \tag{5.36}$$

for $k \geq 1$. We shall show that for fixed p and sufficiently large but also fixed n, the sequence $\{\delta^p_{n,k}(x)\}_{k\geq 0}$ is convergent as $k \to \infty$. Since $A_s(\zeta)$ and $(1+|\zeta|^{\frac{1}{2}})B_s(\zeta)$ are bounded for $x \geq x_- + \delta$, it follows from (5.3) - (5.5) that

$$\left|r^p_n(x)\right| \leq CN^{\frac{1}{2}}\left(\frac{4\zeta}{(\alpha'_0 x + \beta'_0)^2 - 4}\right)^{\frac{1}{4}}\left[\left|I_\nu(N\zeta^{\frac{1}{2}})\right| + \frac{|\zeta^{\frac{1}{2}}|}{1+|\zeta^{\frac{1}{2}}|}\left|I_{\nu-1}(N\zeta^{\frac{1}{2}})\right|\right] \tag{5.37}$$

and

$$\left|s^p_n(x)\right| \leq CN^{\frac{1}{2}}\left(\frac{4\zeta}{(\alpha'_0 x + \beta'_0)^2 - 4}\right)^{\frac{1}{4}}\left[\left|K_\nu(N\zeta^{\frac{1}{2}})\right| + \frac{|\zeta^{\frac{1}{2}}|}{1+|\zeta^{\frac{1}{2}}|}\left|K_{\nu-1}(N\zeta^{\frac{1}{2}})\right|\right] \tag{5.38}$$

for some positive constant C. Furthermore, by virtue of the behaviors of I_ν and K_ν, we have from (5.12) and (5.14)

$$\left|R^p_n(x)s^p_n(x)\right| \leq M'N^{\frac{1}{2}}\left(\frac{1+|\zeta^{\frac{1}{2}}|}{1+N|\zeta^{\frac{1}{2}}|}\right) \leq M'N^{\frac{1}{2}} \tag{5.39}$$

and

$$\left|S_n^p(x) r_n^p(x)\right| \le M' N^{\frac{1}{2}} \left(\frac{1+|\zeta^{\frac{1}{2}}|}{1+N|\zeta^{\frac{1}{2}}|} \right) \le M' N^{\frac{1}{2}}, \tag{5.40}$$

where M' is a positive constant. Thus, from (5.35) we obtain

$$\left| r_{n+1}^p(x) s_n^p(x) - s_{n+1}^p(x) r_n^p(x) - 1 \right| \le \frac{2M'}{p} \cdot \frac{1}{N^p},$$

which in turn gives

$$r_{n+1}^p(x) s_n^p(x) - s_{n+1}^p(x) r_n^p(x) > \frac{1}{2} \tag{5.41}$$

for large n, say $n > 4M' - \tau_0$. A combination of (5.36), (5.37), (5.38) and (5.41) yields

$$\begin{aligned} \left|\delta_{n,1}^p(x)\right| &\le 2 \sum_{j=n+1}^{\infty} \frac{[|r_n^p(x) s_j^p(x)| + |s_n^p(x) r_j^p(x)|] |S_j^p(x)|}{(j+\tau_0)^{p+\frac{3}{2}}} \\ &\le M'' \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} \left(\sum_{j=n+1}^{\infty} \frac{N^{\frac{1}{2}}}{(j+\tau_0)^{p+1}} \right) \left[|K_\nu(N\zeta^{\frac{1}{2}})| + |K_{\nu-1}(N\zeta^{\frac{1}{2}})| \right] \end{aligned}$$

for some constant $M'' > 0$, where we have also used (5.40) and the monotonicity properties of I_ν and K_ν. Hence

$$\left|\delta_{n,1}^p(x)\right| \le \frac{M''}{p} \cdot \frac{1}{N^{p-\frac{1}{2}}} \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} \left[|K_\nu(N\zeta^{\frac{1}{2}})| + |K_{\nu-1}(N\zeta^{\frac{1}{2}})| \right]. \tag{5.42}$$

Similarly, we can prove by induction that

$$\begin{aligned} \left|\delta_{n,k}^p(x) - \delta_{n,k-1}^p(x)\right| \le &\left(\frac{M''}{p} \cdot \frac{1}{N^p} \right)^k \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} N^{\frac{1}{2}} \\ &\left[|K_\nu(N\zeta^{\frac{1}{2}})| + |K_{\nu-1}(N\zeta^{\frac{1}{2}})| \right], \end{aligned} \tag{5.43}$$

from which it also follows that

$$\delta_{n,k}^p(x) = \sum_{m=1}^{k} \left[\delta_{n,m}^p(x) - \delta_{n,m-1}^p(x) \right] \tag{5.44}$$

converges, as $k \to \infty$, for all $n \ge 2M'' - \tau_0$. Clearly, the limit function $\delta_n^p(x)$ satisfies (5.22). Thus, $Q_n(x) = s_n^p(x) + \delta_n^p(x)$ is a solution of (1.3) satisfying (5.15). Furthermore, we have from (5.42), (5.43) and (5.44) that

$$\left|\delta_n^p(x)\right| \le M_p' \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4} \right)^{\frac{1}{4}} \left[|K_\nu(N\zeta^{\frac{1}{2}})| + |K_{\nu-1}(N\zeta^{\frac{1}{2}})| \right] / N^{p-\frac{1}{2}}.$$

Note that $A_s(\zeta)$ and $(1+|\zeta^{\frac{1}{2}}|)B_s(\zeta)$ are bounded for $x \geq x_- + \delta$. By taking an extra term in the expansion (5.5), we have

$$\begin{aligned}\left|\delta_n^p(x)\right| \leq |\delta_n^{p+1}(x)| + M_p'\left(\frac{4\zeta}{(\alpha_0'x+\beta_0')^2-4}\right)^{\frac{1}{4}}&\Big[|K_\nu(N\zeta^{\frac{1}{2}})||\frac{A_{p+1}(\zeta)}{N^{p+\frac{1}{2}}}| \\ &+ |\zeta^{\frac{1}{2}}K_{\nu-1}(N\zeta^{\frac{1}{2}})||\frac{B_{p+1}(\zeta)}{N^{p+\frac{1}{2}}}|\Big] \\ \leq M_p\left(\frac{4\zeta}{(\alpha_0'x+\beta_0')^2-4}\right)^{\frac{1}{4}}&\big[|K_\nu(N\zeta^{\frac{1}{2}})| + |K_{\nu-1}(N\zeta^{\frac{1}{2}})|\big]/N^{p+\frac{1}{2}}\end{aligned}$$

for some positive constants M_p' and M_p, and (2.20) follows; see also (5.21).

The proof of (2.19) is very similar. Like (5.34), we have

$$Q_n(x)\varepsilon_{n+1}^p(x) - Q_{n+1}(x)\varepsilon_n^p(x) = \sum_{j=n+1}^{\infty}\frac{R_j^p(x)Q_j(x)}{(j+\tau_0)^{p+\frac{3}{2}}}.$$

Since $P_n(x)\varepsilon_{n+1}^p(x) - P_{n+1}(x)\varepsilon_n^p(x) = r_n^p(x)\varepsilon_{n+1}^p(x) - r_{n+1}^p(x)\varepsilon_n^p(x)$, it follows from (5.6) that

$$\begin{aligned}P_n(x)\varepsilon_{n+1}^p(x) - P_{n+1}(x)\varepsilon_n^p(x) &= r_m^p(x)\varepsilon_{m+1}^p(x) - r_{m+1}^p(x)\varepsilon_m^p(x) \\ &\quad - \sum_{j=m+1}^{n}\frac{R_j^p(x)r_j^p(x) + R_j^p(x)\varepsilon_j^p(x)}{(j+\tau_0)^{p+\frac{3}{2}}},\end{aligned}$$

where $n > m$; cf. (5.28). We may choose $m = [n/2]$. Upon solving the last two equations, we get from (5.20)

$$\begin{aligned}\varepsilon_n^p(x) =& [r_{m+1}^p(x)\varepsilon_m^p(x) - r_m^p(x)\varepsilon_{m+1}^p(x)]Q_n(x) \\ &+ \sum_{j=n+1}^{\infty}\frac{[r_n^p(x)+\varepsilon_n^p(x)]R_j^p(x)Q_j(x)}{(j+\tau_0)^{p+\frac{3}{2}}} \\ &+ \sum_{j=m+1}^{\infty}\frac{[R_j^p(x)r_j^p(x) + R_j^p(x)\varepsilon_j^p(x)]Q_n(x)}{(j+\tau_0)^{p+\frac{3}{2}}}.\end{aligned}$$

The estimate

$$\left|\varepsilon_n^p(x)\right| \leq M_p\left(\frac{4\zeta}{(\alpha_0'x+\beta_0')^2-4}\right)^{\frac{1}{4}}\big[|I_\nu(N\zeta^{\frac{1}{2}})| + |I_{\nu-1}(N\zeta^{\frac{1}{2}})|\big]/N^{p+\frac{1}{2}}$$

is again proved by using the successive approximation method.

6 An Example

As an illustration, we consider monic polynomials $\pi_n(x)$ that are orthogonal on $[-1,1]$ with respect to a modified Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta h(x)$, where α, $\beta > -1$ and h is real analytic and strictly positive on $[-1,1]$. These polynomials satisfy a three-term recurrence relation

$$\pi_{n+1}(x) = (x - b_n)\pi_n(x) - a_n^2\pi_{n-1}(x), \tag{6.1}$$

where the coefficients a_n and b_n have asymptotic expansions of the form

$$a_n \sim \frac{1}{2} + \sum_{k=2}^{\infty} \frac{C_k}{n^k}, \tag{6.2}$$

$$b_n \sim \sum_{k=2}^{\infty} \frac{D_k}{n^k}. \tag{6.3}$$

The first few coefficients C_k and D_k are given by

$$C_2 = -\frac{4\alpha^2 - 1}{32} - \frac{4\beta^2 - 1}{32}, \tag{6.4}$$

$$C_3 = \frac{4\alpha^2 - 1}{32}(\alpha + \beta + c_0) + \frac{4\beta^2 - 1}{32}(\alpha + \beta + d_0), \tag{6.5}$$

$$D_2 = -\frac{\beta^2 - \alpha^2}{4}, \tag{6.6}$$

$$D_3 = -\frac{\beta^2 - \alpha^2}{4}(1 + \alpha + \beta) + c_0\frac{4\alpha^2 - 1}{16} - d_0\frac{4\beta^2 - 1}{16}, \tag{6.7}$$

where

$$c_0 = \frac{1}{2\pi i}\int_\gamma \frac{\log h(t)}{\sqrt{t^2 - 1}}\frac{dt}{t - 1}, \tag{6.8}$$

$$d_0 = \frac{1}{2\pi i}\int_\gamma \frac{\log h(t)}{\sqrt{t^2 - 1}}\frac{dt}{t + 1}, \tag{6.9}$$

and γ is a closed contour encircling the interval $[-1,\ 1]$ once in the positive direction; see Kuijlaars et al [12], where an asymptotic expansion for the polynomials $\pi_n(x)$ has also been given, which holds uniformly in compact subsets of $\mathbb{C} \setminus [-1,1]$. Moreover, in another paper [13], Kuijlaars and Vanlessen have presented a uniform asymptotic expansion for $\pi_n(x)$ in the interval $(-1+\delta, 1-\delta)$, which agrees with the one given by Szegö [17, p.298, Theorem 12.1.6 and footnote 59]

$$\pi_n(x) = \frac{\sqrt{2}D_\infty}{2^n w^{\frac{1}{2}}(x)(1 - x^2)^{\frac{1}{4}}}\Big[\cos(n\arccos x + \gamma(x)) + o(1)\Big], \tag{6.10}$$

where

$$D_\infty = \exp\left(\frac{1}{2\pi}\int_{-1}^{1} \frac{\log w(t)}{\sqrt{1 - t^2}}\ dt\right) \tag{6.11}$$

and

$$\gamma(x) = \frac{(1 - x^2)^{\frac{1}{2}}}{2\pi} PV \int_{-1}^{1} \frac{\log[\sqrt{1 - t^2}w(t)]}{\sqrt{1 - t^2}}\frac{dt}{t - x}\ . \tag{6.12}$$

Here the integral is taken in the Cauchy principal–value sense. Note that the weight function $w(x)$ satisfies the Szegö condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}\, dx > -\infty,$$

and that we have

$$\lim_{n\to\infty} \frac{2^n \pi_n(z)}{\varphi^n(z)} = \frac{D_\infty}{D(z)} \frac{\varphi^{\frac{1}{2}}(z)}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}, \tag{6.13}$$

holding uniformly for z in compact subsets of $\mathbb{C} \setminus [-1,1]$, where

$$\varphi(z) = z + \sqrt{z^2-1}, \qquad z \in \mathbb{C} \setminus [-1,1] \tag{6.14}$$

and

$$D(z) = \exp\biggl(-\frac{(z^2-1)^{\frac{1}{2}}}{2\pi} \int_{-1}^{1} \frac{\log w(t)}{\sqrt{1-t^2}} \frac{dt}{t-z}\biggr). \tag{6.15}$$

In (6.14) and (6.15), we take that branch of $(z^2-1)^{1/2}$ which is analytic in $\mathbb{C} \setminus [-1,1]$ and behaves like z as $z \to \infty$. The function $\varphi(z)$ in (6.14) is the familiar Joukowski or aerofoil map that maps the exterior of $[-1,1]$ conformally onto the exterior of the unit ball, and the function $D(z)$ is the so-called Szegö function associated with the weight $w(x)$; see [17, p.277].

Returning to the three-term recurrence relation (6.1), we let

$$K_n = 2^{-n} \prod_{m=0}^{\infty} (2a_{n+2m+1})^{-2}.$$

It can be readily verified that $K_{n+1}/K_{n-1} = a_n^2$ and $P_n(x) := \pi_n(x)/K_n$ satisfies the recurrence relation (1.3)

$$P_{n+1}(x) - (A_n x + B_n) P_n(x) + P_{n-1} = 0$$

with $A_n = K_n/K_{n+1}$ and $B_n = -b_n K_n/K_{n+1}$. From (6.2)–(6.7), it follows that the coefficients A_n and B_n satisfy

$$A_n \sim 2 + \sum_{s=2}^{\infty} \frac{\alpha_s}{n^s} \qquad \text{and} \qquad B_n \sim \sum_{s=2}^{\infty} \frac{\beta_s}{n^s}, \tag{6.16}$$

where

$$\alpha_2 = \frac{1}{4}(2\alpha^2 + 2\beta^2 - 1), \tag{6.17}$$

$$\alpha_3 = -\frac{4\alpha^2-1}{8}(\alpha+\beta+1+c_0) - \frac{4\beta^2-1}{8}(\alpha+\beta+1+d_0), \tag{6.18}$$

$$\beta_2 = \frac{\alpha^2-\beta^2}{2} \tag{6.19}$$

and

$$\beta_3 = -\frac{4\alpha^2-1}{8}(\alpha+\beta+1+c_0) + \frac{4\beta^2-1}{8}(\alpha+\beta+1+d_0). \tag{6.20}$$

In terms of the notations in (2.3), (2.14) and (4.20), we have

$$x_+ = 1, \qquad x_- = -1, \qquad \nu = \alpha$$

and

$$N = n + \tau_0 = n + \frac{\alpha + \beta + 1 + c_0}{2} \ .$$

By our main theorem, there are two linearly independent solutions

$$P_n(x) \sim \left(\frac{\zeta}{x^2-1}\right)^{\frac{1}{4}} \left[N^{\frac{1}{2}} I_\alpha(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} I_{\alpha-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}\right] \tag{6.21}$$

and

$$Q_n(x) \sim \left(\frac{\zeta}{x^2-1}\right)^{\frac{1}{4}} \left[N^{\frac{1}{2}} K_\alpha(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} - N^{\frac{1}{2}} \zeta^{\frac{1}{2}} K_{\alpha-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}\right] \tag{6.22}$$

for $x \geq -1 + \delta$, where

$$\zeta^{\frac{1}{2}}(x) = \log \varphi(x) = \log(x + \sqrt{x^2 - 1}) \tag{6.23}$$

for $x \geq 1$ and

$$\zeta^{\frac{1}{2}}(x) = \mathrm{e}^{\frac{\pi}{2} i} \arccos x \tag{6.24}$$

for $-1 < x < 1$; see (4.8) and (6.14). Since $\pi_n(x)/K_n$ is also a solution of (1.3), there exist two functions $C_1(x)$ and $C_2(x)$, which are independent of n, such that

$$\pi_n(x)/K_n = C_1(x) P_n(x) + C_2(x) Q_n(x). \tag{6.25}$$

To determine these two functions, we first choose the function $\Gamma(\zeta)$ in (5.19) as

$$\Gamma(\zeta) = \left(\frac{z^2-1}{\zeta}\right)^{\frac{1}{2}} \frac{1}{4\pi i} \int_\gamma \frac{\log h(t)}{\sqrt{t^2-1}} \frac{dt}{t-z} - \frac{c_0}{2} \qquad z \in \mathbb{C} \setminus (-\infty, -1], \tag{6.26}$$

where ζ is the analytic function of z given in (6.33) and γ is a closed contour encircling the interval $[-1, 1]$ in the positive direction and also the given point z. Obviously, $\Gamma(\zeta)$ is analytic in $z \in \mathbb{C} \setminus (-\infty, -1]$. From (6.12) and (6.26), it is clear that

$$\Gamma(\zeta(x)) \arccos x = \gamma(x) + \frac{\alpha\pi}{2} + \frac{\pi}{4} - \frac{1}{2}(\alpha + \beta + 1 + c_0) \arccos x \tag{6.27}$$

for $-1 < x < 1$. From (5.21) and (5.22), we have

$$P_n(x) \sim \sqrt{\frac{2}{\pi}} (1 - x^2)^{-\frac{1}{4}} \mathrm{e}^{\alpha\pi i/2} \cos\left[(N + \Gamma(\zeta)) \arccos x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] \tag{6.28}$$

and

$$\begin{aligned} Q_n(x) \sim \sqrt{\frac{\pi}{2}} (1 - x^2)^{-\frac{1}{4}} \mathrm{e}^{-(\alpha+1)\pi i/2} \Big\{ &\cos\left[(N + \Gamma(\zeta)) \arccos x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] \\ &- i \sin\left[(N + \Gamma(\zeta)) \arccos x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] \Big\} \end{aligned} \tag{6.29}$$

as $n \to \infty$, for $-1 < x < 1$. Letting $n \to \infty$ in (6.25), it follows from (6.10) and (6.27)

$$\pi_n(x)/K_n \sim \sqrt{2} D_\infty w^{-\frac{1}{2}}(x)(1-x^2)^{-\frac{1}{4}} \cos\left[(N+\Gamma(\zeta)) \arccos x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right]. \tag{6.30}$$

A combination of (6.25), (6.28), (6.29) and (6.30) gives

$$C_1(x) = \sqrt{\pi} D_\infty w^{-\frac{1}{2}}(x) \mathrm{e}^{-\frac{\alpha\pi}{2} i} \qquad \text{and} \qquad C_2(x) = 0 \tag{6.31}$$

for $-1 < x < 1$. Thus, we obtain

$$\pi_n(x)/K_n = \sqrt{\pi} D_\infty (1+x)^{-\frac{\beta}{2}} h^{-\frac{1}{2}}(x) \left[\mathrm{e}^{-\frac{\alpha\pi}{2} i}(1-x)^{-\frac{\alpha}{2}} P_n(x)\right] \tag{6.32}$$

for $-1 < x < 1$. Note that $\mathrm{e}^{-\alpha\pi i/2}(1-x)^{-\alpha/2} I_\alpha(N(-\zeta)^{1/2} i)$ can be written as $(x-1)^{-\alpha/2} I_\alpha(N\zeta^{1/2})$, where

$$\zeta^{\frac{1}{2}}(z) := \log(z + \sqrt{z^2-1}), \qquad z \in \mathbb{C} \setminus (-\infty, 1], \tag{6.33}$$

and that all functions $(z-1)^{-\alpha/2} I_\alpha(N\zeta^{1/2})$, $(z-1)^{-\alpha/2} P_n(z)$ and $(z+1)^{-\beta/2} h^{-1/2}(z)$ are analytic in $z \in \mathbb{C} \setminus (-\infty, -1]$. By analytic continuation, we have from (6.32)

$$\pi_n(z)/K_n = \sqrt{\pi} D_\infty (z+1)^{-\frac{\beta}{2}} h^{-\frac{1}{2}}(z) \left[(z-1)^{-\frac{\alpha}{2}} P_n(z)\right] \tag{6.34}$$

for $z \in \mathbb{C} \setminus (-\infty, -1]$. In particular, when $z = x = \cos\theta$, $\theta \in (0, \pi - \delta)$, we obtain the uniform asymptotic expansion

$$\begin{aligned} w^{\frac{1}{2}}(\cos\theta)\left(\frac{\sin\theta}{\theta}\right)^{\frac{1}{2}} \pi_n(\cos\theta) \sim K_n \sqrt{\pi} D_\infty \Bigg[& N^{\frac{1}{2}} J_\alpha(N\theta) \sum_{s=0}^{\infty} \frac{A_s(-\theta^2)}{N^s} \\ & - N^{\frac{1}{2}} \theta J_{\alpha-1}(N\theta) \sum_{s=0}^{\infty} \frac{B_s(-\theta^2)}{N^s} \Bigg], \end{aligned} \tag{6.35}$$

where J_α is the Bessel function of first kind.

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