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Philippe G. CIARLET and Liliana GRATIE

Abstract

Using techniques from asymptotic analysis, the second author has recently identified equations that generalize the classical Marguerre-von Kármán equations for a nonlinearly elastic shallow shell by allowing more realistic boundary conditions, which may change their type along the lateral face of the shell. We first reduce these more general equations to a single "cubic" operator equation, whose sole unknown is the vertical displacement of the shell. This equation generalizes a cubic operator equation introduced by M. S. Berger and P. Fife for analyzing the von Kármán equations for a nonlinearly elastic plate. We then establish the existence of a solution to this operator equation by means of a compactness method due to J. L. Lions.

Key words: nonlinear shallow shell theory, nonlinear partial differential equations

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1 Introduction

Let ω be a domain in the "horizontal" plane $\boldsymbol{R}^2$, with a smooth boundary γ, and let γ_1 be a portion of γ that satisfies $0 < \textit{length}\ \gamma_1 < \textit{length}\ \gamma$. We consider a nonlinearly elastic shallow shell with middle surface $\{(y, \theta^\varepsilon(y)) \in \boldsymbol{R}^3; y \in \overline{\omega}\}$ and thickness 2ε, where $\theta^\varepsilon : \overline{\omega} \to \boldsymbol{R}$ is a smooth function that satisfies $\theta^\varepsilon = \partial_\nu \theta^\varepsilon = 0$ on γ_1. The portion of the lateral face of the shell with γ_1 as its middle line is subjected to boundary conditions "of von Kármán type" of the form proposed by Ciarlet [1980], the remaining portion of the lateral face being free.

Under the basic assumption that $\theta^\varepsilon = O(\varepsilon)$ (this constitutes the "shallowness" assumption as originally proposed and justified by Ciarlet & Paumier [1986]), the second author (cf. Gratie [2002]) has shown, by means of the formal asymptotic expansion method, that the following *generalized Marguerre-von Kármán equations* constitute a two-dimensional model for such a shell:

$$-\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2 \xi) = [\Phi, \xi + \widetilde{\theta}] + f \text{ in } \omega,$$
$$\Delta^2 \Phi = -[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega,$$
$$\xi = \partial_\nu \xi = 0 \text{ on } \gamma_1,$$

$$m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2,$$
$$\partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \text{ on } \gamma_2,$$
$$\Phi = \Phi_0 \text{ and } \partial_\nu\Phi = \Phi_1 \text{ on } \gamma,$$

where

$$m_{\alpha\beta}(\nabla^2\xi) = -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\xi\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\xi\right\},$$
$$[\Phi,\xi] = \partial_{11}\Phi\partial_{22}\xi + \partial_{22}\Phi\partial_{11}\xi - 2\partial_{12}\Phi\partial_{12}\xi.$$

The function f is, up to a constant factor, the resultant of the vertical forces acting on the shell. The functions Φ_0 and Φ_1 are known functions of the appropriate "scaled" density $(h_\alpha) : \gamma_1 \to \boldsymbol{R}^2$ of the resultant of the horizontal forces acting on the portion of the lateral face of the shell with γ_1 as its middle line. The *unknowns* $\xi : \overline{\omega} \to \boldsymbol{R}$ and $\Phi : \overline{\omega} \to \boldsymbol{R}$ are, up to constant factors, the *vertical component of the displacement field of the middle surface of the shell* and the *Airy function*.

The derivation from three-dimensional elasticity of the classical *Marguerre-von Kármán equations*, which correspond to the special case where $\gamma_1 = \gamma$ have been justified by Ciarlet & Paumier [1986] by means of a formal asymptotic analysis.

These classical equations have been studied from the mathematical viewpoint by many authors; see, e.g., Rupprecht [1981], Kesavan & Srikanth [1983], Rao [1995a, 1995b], Paumier & Rao [1989], Kavian & Rao [1993], Ciarlet [1997, Section 5.12], and Vorovich [1999]. Note that these equations owe their name to Marguerre [1938] and von Kármán & Tsien [1939], who proposed the two partial differential equations in ω (no mention was made of boundary conditions in their works).

The main novelty is thus that *boundary conditions that change their type along the lateral face* can be handled by the *generalized Marguerre-von Kármán equations*.

In Section 2, we briefly review (see Propositions 1, 2, and 3) the main steps of the derivation of the above two-dimensional equations from the three-dimensional equations of nonlinear elasticity that model the actual three-dimensional shell, thus shedding a useful light on the mechanical interpretation of these equations.

In Section 3, we prove in Theorem 1 the main result of this paper, which asserts that solving the generalized Marguerre-von Kármán equations amounts to solving a single "cubic" operator equation. This equation generalizes an operator equation introduced by Berger[1967] and Berger & Fife [1968] in their study of the von Kármán plates, as well as the simpler one used by Ciarlet, Gratie & Sabu [2001] in their mathematical analysis of the *generalized von Kármán equations* (corresponding to the special case $\theta^\varepsilon = 0$) proposed and justified by Ciarlet & Gratie [2001].

Applying as in Ciarlet, Gratie & Sabu [2001], a *compactness method of J. L. Lions* [1969, p.54] to this operator equation, we then establish in Section 4 the existence of solutions to the generalized Marguerre-von Kármán equations.

More specifically, we show in Theorem 2 that, if the domain ω is simply-connected, the functions $h_\alpha : \gamma_1 \to \boldsymbol{R}$ satisfy natural compatibility conditions, and their norms $||h_\alpha||_{L^2(\gamma_1)}$ are small enough, then the generalized Marguerre-von Kármán equations have at least one solution $(\xi, \Phi) \in H^2(\omega) \times H^2(\omega)$ in the sense of distributions.

2 The Generalized Marguerre-Von Kármán Equations

Greek indices, corresponding to the coordinates in the "horizontal" plane vary in $\{1, 2\}$, and Latin indices in $\{1, 2, 3\}$, except if they are used for indexing sequences. The summation convention with respect to repeated indices is systematically used. All the notions needed below from three-dimensional nonlinear elasticity are detailed in, e.g., Ciarlet [1988].

Let ω be a *domain* in the "horizontal" plane $\boldsymbol{R}^2$, i.e., a bounded and connected subset ω of $\boldsymbol{R}^2$ with a sufficiently smooth boundary γ, the set ω being locally on a single side of γ. Let γ_1 and γ_2 be two disjoint relatively open subsets of γ such that *length* $\gamma_1 > 0$, *length* $\gamma_2 > 0$, and *length* $(\gamma - \{\gamma_1 \cup \gamma_2\}) = 0$. Let $y = (y_\alpha)$ denote a generic point in $\overline{\omega}$, and let $\partial_\alpha = \frac{\partial}{\partial y_\alpha}$ and $\partial_{\alpha\beta} = \frac{\partial^2}{\partial y_\alpha \partial y_\beta}$. Let (ν_α) denote the unit outer normal vector along γ, let (τ_α) denote the unit tangent vector along γ defined by $\tau_1 = -\nu_2, \tau_2 = \nu_1$, and finally, let $\partial_\nu = \nu_\alpha \partial_\alpha$ and $\partial_\tau = \tau_\alpha \partial_\alpha$ the outer normal and tangential derivative operators along γ. Consider a *nonlinearly elastic shell* occupying in its reference configuration the set $\{\widehat{\Omega}^\varepsilon\}^-$, where $\widehat{\Omega}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\Omega^\varepsilon), \Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, and the mapping $\boldsymbol{\Theta}^\varepsilon : \{\Omega^\varepsilon\}^- \to \boldsymbol{R}^3$ is defined by

$$\boldsymbol{\Theta}^\varepsilon(y, x_3^\varepsilon) = (y, \theta^\varepsilon(y)) + x_3^\varepsilon \boldsymbol{a}_3^\varepsilon(y),$$

for all $(y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon$, where $\boldsymbol{a}_3^\varepsilon$ is a unit normal vector to the surface $\boldsymbol{\Theta}^\varepsilon(\overline{\omega})$ and $\theta^\varepsilon : \overline{\omega} \to \boldsymbol{R}$ is a function of class C^3 that satisfies $\theta^\varepsilon = \partial_v \theta^\varepsilon = 0$ along γ_1. The surface $\boldsymbol{\Theta}^\varepsilon(\overline{\omega})$ is the *middle surface* of the shell and 2ε is its *thickness*. The relation $\theta^\varepsilon = \partial_v \theta^\varepsilon = 0$ on γ_1 imply that, along each connected portion of γ_1, the middle line of the lateral face of the shell is in the horizontal plane and the lateral face is vertical.

The shell is called *shallow* if the function θ^ε that describes the shape of the middle surface of the shell in its reference configuration is of the order of the thickness of the shell, i.e., of the order ε. This definition was first proposed by Ciarlet & Paumier [1986], when they justified the "classical" Marguerre-von Kármán equations. They also showed that, for $\varepsilon > 0$ small enough, the mapping $\boldsymbol{\Theta}^\varepsilon : \overline{\Omega}^\varepsilon \to \boldsymbol{\Theta}^\varepsilon(\overline{\Omega}^\varepsilon)$ is a C^1-diffeomorphism.

We assume that the nonlinearly elastic material constituting the shell is a *St Venant-Kirchhoff material* with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. In particular then, the material constituting the plate is *homogeneous* and *isotropic*, and the reference configuration $\boldsymbol{\Theta}^\varepsilon(\overline{\Omega}^\varepsilon)$ of the shell is a *natural state*.

The shell is subjected to body forces of density $(\hat{f}_i^\varepsilon) = (0, 0, \hat{f}_3^\varepsilon) : \hat{\Omega}^\varepsilon \to \boldsymbol{R}^3$ in its interior $\boldsymbol{\Theta}^\varepsilon(\overline{\Omega}^\varepsilon)$; to *surface forces* of density $(\hat{g}_i^\varepsilon) = (0, 0, \hat{g}_3^\varepsilon) : \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon \to \boldsymbol{R}^3$ on its upper and lower faces $\hat{\Gamma}_-^\varepsilon := \boldsymbol{\Theta}^\varepsilon(\Gamma_-^\varepsilon), \hat{\Gamma}_+^\varepsilon := \boldsymbol{\Theta}^\varepsilon(\Gamma_+^\varepsilon)$, where $\Gamma_+^\varepsilon := \omega \times \{+\varepsilon\}, \Gamma_-^\varepsilon := \omega \times \{-\varepsilon\}$; and to *applied surface forces of "von Kármán's type"* on the portion $\boldsymbol{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the remaining portion $\boldsymbol{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being *free*.

The surface forces of von Kármán type are "horizontal" and only their resultant $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0) : \hat{\gamma}_1^\varepsilon \to \boldsymbol{R}^3$ after integration across the thickness is given along $\hat{\gamma}_1^\varepsilon := \boldsymbol{\Theta}^\varepsilon(\gamma_1)$.

Let $\hat{x}^\varepsilon = (\hat{x}_i^\varepsilon)$ denote a generic point in the set $\{\hat{\Omega}^\varepsilon\}^-$ and let $\hat{\partial}_i^\varepsilon = \partial/\partial \hat{x}_i^\varepsilon$.

The unknown in the three-dimensional formulation is the displacement field $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}_i^\varepsilon) : \{\hat{\Omega}^\varepsilon\}^- \to \boldsymbol{R}^3$, where the functions $\hat{u}_i^\varepsilon : \{\hat{\Omega}^\varepsilon\}^- \to \boldsymbol{R}$ are thus its Cartesian components. The unknown displacement field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \boldsymbol{R}^3$ then satisfies the following *three-dimensional boundary value problem*:

$$
\begin{aligned}
& -\hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \quad \text{in } \hat{\Omega}^\varepsilon, \\
& \hat{u}_\alpha^\varepsilon \quad \text{independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \boldsymbol{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon]), \\
& \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k \hat{u}_\alpha^\varepsilon) \circ \boldsymbol{\Theta}^\varepsilon\} v_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon \quad \text{on } \gamma_1, \\
& (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon = 0 \quad \text{on } \gamma_2 \times [-\varepsilon, \varepsilon], \\
& (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon \quad \text{on } \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\sigma}_{ij}^\varepsilon &:= \lambda^\varepsilon \hat{E}_{pp}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon), \\
\hat{E}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon) &:= \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{u}_m^\varepsilon \cdot \hat{\partial}_j^\varepsilon \hat{u}_m^\varepsilon),
\end{aligned}
$$

$(\hat{n}_i^\varepsilon)$ is the unit outer normal vector along the boundary of the set $\hat{\Omega}^\varepsilon$, and (ν_α) is the unit outer normal vector along the boundary of the set ω.

The *stresses* $\hat{\sigma}_{ij}^\varepsilon : \{\hat{\Omega}^\varepsilon\}^- \to \boldsymbol{R}$ are the components of the *second Piola-Kirchhoff stress tensor and the strains* $\hat{E}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)$ are the components of the *Green-St Venant strain tensor*. The relations between the stresses $\hat{\sigma}_{ij}^\varepsilon$ and the strains $\hat{E}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)$ form the *constitutive equation* of the St Venant-Kirchhoff material constituting the shallow shell.

The boundary conditions along the portion of the lateral face with $\hat{\gamma}_1^\varepsilon$ as its middle line, viz.,

$$\hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \boldsymbol{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$$

mean that only horizontal displacements of equal direction and magnitude are allowed along each vertical segment of the subset $\boldsymbol{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of the lateral face of the shell.

The boundary conditions on $\boldsymbol{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ for the displacement field and those on γ_1 for the stress tensor field are of the form proposed by Ciarlet [1980] for justifying the well-known *von Kármán equations*, which correspond to the special case where $\gamma_1 = \gamma$ and $\theta^\varepsilon = 0$. In the form considered here, they have been later put to use by Ciarlet & Paumier [1986] for justifying the "classical" Marguerre- von Kármán equations, which correspond to the special case where $\gamma_1 = \gamma$. Another special case where $\gamma_2 = \gamma$, which corresponds to a nonlinearly elastic shallow shell *clamped along its entire lateral face*, has also been treated in *ibid.*

Following a by now well-established procedure (see, e.g., Ciarlet [1997, Chaps.4 and 5]), Gratie [2002] has then applied the method of formal asymptotic expansions to this problem, according to the following steps:

First, the above boundary value problem is put in a *variational*, or *weak*, form. *Second*, the resulting variational problem is "*scaled* " over a domain that is independent of ε. More specifically, we let $\Omega = \omega \times]-1, 1[,\ \Gamma_+ = \omega \times \{1\}, \Gamma_- = \omega \times \{-1\}$, and with each point $x \in \Omega$, we associate the point $x^\varepsilon \in \overline{\Omega}^\varepsilon$ through the bijection

$$\pi^\varepsilon : x = (y, x_3) \in \overline{\Omega} \to x^\varepsilon = (x_i^\varepsilon) = (y, \varepsilon x_3) \in \overline{\Omega}^\varepsilon.$$

With the displacement field $\hat{\boldsymbol{u}}^\varepsilon : \{\hat{\Omega}^\varepsilon\}^- \to \boldsymbol{R}^3$, we next associate the *scaled displacement field* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \boldsymbol{R}^3$ defined by means of the *scalings*:

$$\hat{u}_\alpha^\varepsilon(\hat{x}^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \ \ \hat{u}_3^\varepsilon(\hat{x}^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \ \ \text{for all } \ \hat{x}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\pi^\varepsilon x) \in \{\hat{\Omega}^\varepsilon\}^-.$$

Finally, we assume that the following *assumptions on the data*, i.e., the Lamé constants and the applied force densities, are satisfied:

$$\begin{aligned}
\lambda^\varepsilon = \lambda \quad &\text{and} \quad \mu^\varepsilon = \mu, \\
\hat{f}_3^\varepsilon(\hat{x}^\varepsilon) = \varepsilon^3 f_3(x), &\ \ \text{for all } \ \hat{x}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\pi^\varepsilon x) \in \hat{\Omega}^\varepsilon, \\
\hat{g}_3^\varepsilon(\hat{x}^\varepsilon) = \varepsilon^4 g_3(x), &\ \ \text{for all } \ \hat{x}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\pi^\varepsilon x) \in \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon, \\
\hat{h}_\alpha^\varepsilon(\hat{y}^\varepsilon) = \varepsilon^2 h_\alpha(y), &\ \ \text{for all } \ \hat{y}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\pi^\varepsilon y) \in \hat{\gamma}_1^\varepsilon, \\
\theta^\varepsilon(y) = \varepsilon \theta(y), &\quad\ \ \text{for all } \ y \in \overline{\omega},
\end{aligned}$$

where the functions $f_3 \in L^2(\Omega), g_3 \in L^2(\Gamma_+ \cup \Gamma_-), h_\alpha \in L^2(\gamma_1)$, and $\theta \in C^3(\overline{\omega})$ are all *independent of* ε and $\theta = \partial_\nu \theta = 0$ along γ_1. Note that the last relation above precisely defines the *shallowness* of the shell.

Taking all the above relations into account thus yields a variational problem $P(\varepsilon; \Omega)$ posed over he *fixed* domain Ω and satisfied by the scaled displacement field $\boldsymbol{u}(\varepsilon)$.

Third, it is assumed that $\boldsymbol{u}(\varepsilon)$ admits a *formal asymptotic expansion*, of the form

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 + \varepsilon \boldsymbol{u}^1 + \varepsilon^2 \boldsymbol{u}^2 + \varepsilon^3 \boldsymbol{u}^3 + \varepsilon^4 \boldsymbol{u}^4 + \dots.$$

The method of formal asymptotic expansions then consists in equating to zero the factors of the successive powers of ε, arranged by increasing order, found in problem $P(\varepsilon; \Omega)$, until the *leading term* $\boldsymbol{u}^0$ of the expansion can be fully identified as the solution of an *ad hoc* variational problem (to this end, the successive terms $\boldsymbol{u}^0, \boldsymbol{u}^1, \dots$ are assumed to possess whatever smoothness is needed). Carried out in the present situation, this method leads to the following outcome (cf. Gratie [2002, Theorem 3]):

Proposition 1. (a) *Define the space*:

$$\boldsymbol{V}(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_3 = \partial_\nu \eta_3 = 0 \ \text{ on } \ \gamma_1\}.$$

Then there exists $\boldsymbol{\zeta} = (\zeta_i) \in \boldsymbol{V}(\omega)$ *such that the components of the leading term* $\boldsymbol{u}^0 = (u_i^0)$ *are of the form*

$$u_\alpha^0 = \zeta_\alpha - x_3 \partial_\alpha \zeta_3 \quad \textit{and} \quad u_3^0 = \zeta_3.$$

(b) *Define the functions*:

$$
\begin{aligned}
m_{\alpha\beta} &:= -\frac{1}{3}\left\{\frac{4\gamma\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\right\} \in L^2(\omega),\\
\overline{E}^0_{\alpha\beta} &:= \frac{1}{2}\left\{\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\zeta_3\partial_\beta\zeta_3 + \partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3\right\} \in L^2(\omega),\\
\overline{N}_{\alpha\beta} &:= \frac{4\lambda\mu}{\lambda+2\mu}\overline{E}^0_{\sigma\sigma}(\boldsymbol{\zeta})\delta_{\alpha\beta} + 4\mu\overline{E}^0_{\alpha\beta}(\boldsymbol{\zeta}) \in L^2(\omega),\\
p_3 &:= \int_{-1}^{1} f_3 dx_3 + g_3(.,+1) + g_3(.,-1) \in L^2(\omega).
\end{aligned}
$$

Then, the field $\boldsymbol{\zeta} = (\zeta_i) \in \boldsymbol{V}(\omega)$ *satisfies the following variational equations*:

$$
\begin{aligned}
-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \overline{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega + \int_\omega \overline{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \\
= \int_\omega p_3\eta_3 d\omega + \int_{\gamma_1} h_\alpha\eta_\alpha d\gamma,
\end{aligned}
$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$.

Remarks. (1) It is most likely, although this claim remains to be substantiated by a proof, that the equations obtained in Proposition 1 by means of a *formal* asymptotic procedure can be rigorously justified from the equations of three-dimensional nonlinear elasticity by means of a *convergence theorem as* $\varepsilon \to 0$ based on *gamma-convergence theory*, in the spirit of the landmark contributions of Le Dret & Raoult [1995] and Friesecke, Miller & James [2002a, 2002b].

(2) Similar equations can be also established in *curvilinear coordinates*; see Andreoiu-Banica [1998]. ∎

The variational problem found in Proposition 1 is, at least *formally*, equivalent to a boundary value problem:

Proposition 2. *Assume that the boundary* γ *is smooth enough. Then any smooth enough solution* $\boldsymbol{\zeta} = (\zeta_i)$ *of the variational problem found in* Proposition 1 *also satisfies the following boundary value problem*:

$$
\begin{aligned}
-\partial_{\alpha\beta}m_{\alpha\beta} - \overline{N}_{\alpha\beta}(\zeta_3+\theta) &= p_3 \ \text{ in } \omega,\\
\partial_\beta\overline{N}_{\alpha\beta} &= 0 \ \text{ in } \omega,\\
\zeta_3 = \partial_\nu\zeta_3 &= 0 \ \text{ on } \gamma_1,\\
\overline{N}_{\alpha\beta}\nu_\beta &= h_\alpha \ \text{ on } \gamma_1,\\
m_{\alpha\beta}\nu_\alpha\nu_\beta &= 0 \ \text{ on } \gamma_2,\\
\partial_\alpha m_{\alpha\beta} + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) &= 0 \ \text{ on } \gamma_2,\\
\overline{N}_{\alpha\beta}\nu_\beta &= 0 \ \text{ on } \gamma_2.
\end{aligned}
$$

Note that

$$
-\partial_{\alpha\beta}m_{\alpha\beta} = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3,
$$

so that the first equation may be also written as

$$\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 - \overline{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) = p_3 \ \ in \ \omega.$$

Under the crucial assumption that the set ω is simply-connected, it is then shown that the boundary value problem found in Proposition 2 is equivalent, within the class of smooth solutions, to another boundary value problem, involving this time only *two* unknown functions (cf. Gratie [2002, Theorem 6]).

It is henceforth assumed that the origin of $\boldsymbol{R}^2$ belongs to the boundary γ. Given any point $y \in \gamma$, the arc oriented in the usual manner, joining 0 to y along the boundary γ is denoted $\gamma(y)$.

Proposition 3. *Assume that the set ω is simply connected and that its boundary γ is smooth enough. Assume that there exists a solution $\boldsymbol{\zeta} = (\zeta_i)$ of the boundary value problem found in* Proposition 2, *that possesses the regularity*

$$(\zeta_\alpha) \in H^3(\omega) \ \ and \ \ \zeta_3 \in H^4(\omega).$$

Then the functions $\widetilde{h}_\alpha \in L^2(\gamma)$ defined by $\widetilde{h}_\alpha = h_\alpha$ on γ_1 and $\widetilde{h}_\alpha = 0$ on γ_2 *necessarily satisfy the compatibility relations*

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma = 0.$$

*Moreover, there exists a **scaled Airy function** $\Phi \in H^4(\omega)$, uniquely determined by the conditions $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$,* such that

$$\overline{N}_{11} = \partial_{22}\Phi, \overline{N}_{12} = \overline{N}_{21} = -\partial_{12}\Phi, \overline{N}_{22} = \partial_{11}\Phi \ in \ \omega.$$

*Finally, the pair $(\zeta_3, \Phi) \in H^4(\omega)\times H^4(\omega)$ satisfies the following **scaled generalized Marguerre-von Kármán equations**:*

$$\begin{aligned}
\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 &= [\Phi, \zeta_3+\theta] + p_3 \text{ in } \omega,\\
\Delta^2\Phi &= -\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3, \zeta_3+2\theta] \text{ in } \omega,\\
\zeta_3 = \partial_\nu\zeta_3 &= 0 \ \text{ on } \gamma_1,\\
m_{\alpha\beta}\nu_\alpha\nu_\beta &= 0 \ \text{ on } \gamma_2,\\
\partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) &= 0 \ \text{ on } \gamma_2,\\
\Phi = \Phi_0 \text{ and } \partial_\nu\Phi &= \Phi_1 \ \text{ on } \gamma,
\end{aligned}$$

where

$$\Phi_0(y) := -y_1\int_{\gamma(y)} \widetilde{h}_2 d\gamma + y_2\int_{\gamma(y)} \widetilde{h}_1 d\gamma + \int_{\gamma(y)} (x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma, \ y \in \gamma$$

$$\Phi_1(y) := -\nu_1\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)} \widetilde{h}_1 d\gamma, \ \ y \in \gamma,$$

and

$$[\Phi, \xi] = \partial_{11}\Phi\partial_{22}\xi + \partial_{22}\Phi\partial_{11}\xi - 2\partial_{12}\Phi\partial_{12}\xi.$$

Remark. Not only do the compatibility conditions satisfied by the functions $\widetilde{h}_\alpha$, or equivalently by the functions h_α, have a *mathematical* justification, viz., insuring in particular that the functions Φ_0 and Φ_1 are unambiguously defined, but they also have a *mechanical* interpretation, simply expressing that the horizontal forces acting on the shell are in static equilibrium.

∎

In order that this "scaled" boundary value problem be expressed in terms of "physical" quantities, it remains to "de-scale" the unknowns. To this end we are naturally led, in view of the scalings made on the unknowns of the three-dimensional problem, to define the "*limit*" *displacement field* $\boldsymbol{\zeta} = (\zeta_i^\varepsilon) : \overline{\omega} \to \boldsymbol{R}^3$ of the middle surface of the shallow shell through the *de-scalings*:

$$\zeta_\alpha^\varepsilon = \varepsilon^2\zeta_\alpha \text{ and } \zeta_3^\varepsilon = \varepsilon\zeta_3 \text{ in } \overline{\omega}.$$

Together with the *assumptions on the data* made in Section 2, these de-scalings immediately lead to the following corollary to Proposition 3:

Proposition 4. *Let the assumptions be as in* Proposition 3 *and let*

$$\begin{aligned} m_{\alpha\beta}^\varepsilon =& -\frac{\varepsilon^3}{3}\left\{\frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon}\Delta\zeta_3^\varepsilon\delta_{\alpha\beta} + 4\mu^\varepsilon\partial_{\alpha\beta}\zeta_3^\varepsilon\right\}, \\ \overline{N}_{\alpha\beta}^\varepsilon =& \varepsilon\left\{\frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon}\overline{E}_{\sigma\sigma}^0(\boldsymbol{\zeta}^\varepsilon)\delta_{\alpha\beta} + 4\mu^\varepsilon\overline{E}_{\alpha\beta}^0(\boldsymbol{\zeta}^\varepsilon)\right\}, \\ \overline{E}_{\alpha\beta}^0(\boldsymbol{\zeta}^\varepsilon) =& \frac{1}{2}\{\partial_\alpha\zeta_\beta^\varepsilon + \partial_\beta\zeta_\alpha^\varepsilon + \partial_\alpha\zeta_3^\varepsilon\partial_\beta\zeta_3^\varepsilon + \partial_\alpha\theta^\varepsilon\partial_\beta\zeta_3^\varepsilon + \partial_\beta\theta^\varepsilon\partial_\alpha\zeta_3^\varepsilon\}. \end{aligned}$$

Then there exists an **Airy function** $\Phi^\varepsilon \in H^4(\omega)$, *uniquely determined by the requirements that* $\Phi^\varepsilon(0) = \partial_\alpha\Phi^\varepsilon(0) = 0$, *such that*

$$\overline{N}_{11}^\varepsilon = \varepsilon\partial_{22}\Phi^\varepsilon, \overline{N}_{12}^\varepsilon = \overline{N}_{21}^\varepsilon = -\varepsilon\partial_{12}\Phi^\varepsilon, \overline{N}_{22}^\varepsilon = \varepsilon\partial_{11}\Phi^\varepsilon \text{ in } \omega,$$

and the pair $(\zeta_3^\varepsilon, \Phi^\varepsilon) \in H^4(\omega) \times H^4(\omega)$ *satisfies the following* **generalized Marguerre-von Kármán equations**:

$$\begin{aligned} \frac{8\mu^\varepsilon(\lambda^\varepsilon + \mu^\varepsilon)}{3(\lambda^\varepsilon + 2\mu^\varepsilon)}\varepsilon^3\Delta^2\zeta_3^\varepsilon &= \varepsilon[\Phi^\varepsilon, \zeta_3^\varepsilon + \theta^\varepsilon] + p_3 \text{ in } \omega, \\ \Delta^2\Phi^\varepsilon &= -\frac{\mu^\varepsilon(3\lambda^\varepsilon + 2\mu^\varepsilon)}{\lambda^\varepsilon + \mu^\varepsilon}[\zeta_3^\varepsilon, \zeta_3^\varepsilon + 2\theta^\varepsilon] \text{ in } \omega, \\ \zeta_3^\varepsilon = \partial_\nu\zeta_3^\varepsilon &= 0 \text{ on } \gamma_1, \\ m_{\alpha\beta}^\varepsilon\nu_\alpha\nu_\beta &= 0 \text{ on } \gamma_2, \\ (\partial_\alpha m_{\alpha\beta}^\varepsilon)\nu_\beta + \partial_\tau(m_{\alpha\beta}^\varepsilon\nu_\alpha\tau_\beta) &= 0 \text{ on } \gamma_2, \\ \Phi^\varepsilon = \Phi_0^\varepsilon \text{ and } \partial_\nu\Phi^\varepsilon &= \Phi_1^\varepsilon \text{ on } \gamma, \end{aligned}$$

where

$$\Phi_0^\varepsilon(y) := -y_1 \int_{\gamma(y)} \widetilde{h}_2^\varepsilon d\gamma + y_2 \int_{\gamma(y)} \widetilde{h}_1 d\gamma + \int_{\gamma(y)} (x_1 \widetilde{h}_2^\varepsilon - x_2 \widetilde{h}_1^\varepsilon) d\gamma, \ \ y \in \gamma,$$

$$\Phi_1^\varepsilon(y) := -\nu_1(y) \int_{\gamma(y)} \widetilde{h}_2^\varepsilon d\gamma + \nu_2(y) \int_{\gamma(y)} \widetilde{h}_1^\varepsilon d\gamma, \ \ y \in \gamma.$$

The functions $m_{\alpha\beta}^\varepsilon, \overline{N}_{\alpha\beta}^\varepsilon$ and $\overline{E}_{\alpha\beta}^0(\boldsymbol{\zeta}^\varepsilon)$ are the *bending moments*, the *stress resultants*, and the *two-dimensional strains* inside the shallow shell. The coefficient $\frac{8\mu^\varepsilon(\lambda^\varepsilon+\mu^\varepsilon)}{3(\lambda^\varepsilon+2\mu^\varepsilon)}\varepsilon^3$ of $\Delta^2\zeta_3^\varepsilon$ in the first partial differential equation is the *flexural rigidity* of the shallow shell.

3 An Equivalent "Cubic" Operator Equation

Once the generalized Marguerre-von Kármán equations are derived under the assumption that their solutions are smooth, they can be studied for their own sake, in particular regarding the existence of less smooth solutions. The purpose of this paper is precisely to establish that they possess such solutions. As it is evidently enough to consider the *scaled* generalized Marguerre-von Kármán equations, we shall take these as our point of departure, thus benefiting from their simpler notations.

Our first task consists in showing that finding a solution (ξ_3, Φ) in the sense of distributions to these equations amounts to solving a single "*cubic*" *operator equation*, whose single unknown is (proportional to) the function $(\xi_3 + \theta)$, the scaled Airy function Φ being then obtained by solving a linear boundary value problem.

We recall that the given function $\theta \in C^3(\overline{\omega})$, which is used for defining the "geometry" of the shallow shell, satisfies $\theta = \partial_\nu\theta = 0$ on γ_1 by assumption.

Theorem 1. *Assume that the set ω is simply-connected and that the functions $\widetilde{h}_\alpha \in L^2(\gamma)$ defined by $\widetilde{h}_\alpha = h_\alpha$ on γ_1 and $\widetilde{h}_\alpha = 0$ on γ_2 satisfy the compatibility relations*

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1) d\gamma = 0.$$

Let $E := \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \xi := \sqrt{E}\,\zeta_3, \widetilde{\theta} = \sqrt{E}\,\theta$, and $f := \sqrt{E}p_3$, and define the space

$$V(\omega) := \{\eta \in H^2(\omega); \eta = \partial_\nu\eta = 0 \text{ on } \gamma_1\}.$$

Then there exist a "cubic" mapping $\widetilde{C} : V(\omega) \to V(\omega)$ ("cubic" in the sense that $\widetilde{C}(\alpha\eta) = \alpha^3\widetilde{C}(\eta)$ for all $\alpha \in \boldsymbol{R}$ and $\eta \in V(\omega)$), a linear mapping $\widetilde{L} : V(\omega) \to V(\omega)$, and an element $\widetilde{F} \in V(\omega)$ (their definitions are given at the end of the proof below) such that a pair $(\zeta_3, \Phi) \in V(\omega) \times H^2(\omega)$ satisfies the scaled generalized Marguerre-von Kármán equations (cf. Proposition 3) *in the sense of distributions if and only if the function*

$$\widetilde{\xi} := (\xi + \widetilde{\theta}) \in V(\omega)$$

satisfies the "cubic" operator equation

$$\widetilde{C}(\widetilde{\xi}) + (I - \widetilde{L})\widetilde{\xi} - \widetilde{F} = 0.$$

The scaled Airy function $\Phi \in H^2(\omega)$ is then given as the unique solution in the sense of distributions of

$$\begin{aligned} \Delta^2\Phi =& - [\widetilde{\xi} - \widetilde{\theta}, \widetilde{\xi} + \widetilde{\theta}] \text{ in } \omega, \\ \Phi =& \Phi_0 \text{ and } \partial_\nu \Phi = \Phi_1 \quad \text{on } \gamma. \end{aligned}$$

Proof. (i) For any $\eta \in H^2(\omega)$, let

$$m_{\alpha\beta}(\nabla^2\eta) := -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda + 2\mu}\Delta\eta\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\eta\right\} \in L^2(\omega),$$

and let the functions ξ and $\widetilde{\theta}$ be defined as in the statement of the theorem. Expressed in terms of these functions, the scaled generalized Marguerre-von Kármán equations found in Proposition 3 read:

$$\begin{aligned} -\partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\xi) &= [\Phi, \xi + \widetilde{\theta}] + f \text{ in } \omega, \\ \Delta^2\Phi &= -[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega, \\ \xi = \partial_\nu \xi &= 0 \quad \text{on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta &= 0 \quad \text{on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) &= 0 \quad \text{on } \gamma_2, \\ \Phi = \Phi_0 \text{ and } \partial_v\Phi &= \Phi_1 \quad \text{on } \gamma. \end{aligned}$$

Note that $f \in L^2(\omega)$ and that the functions Φ_0 and Φ_1, which are defined by

$$\begin{aligned} \Phi_0(y); &= -y_1 \int_{\gamma(y)} \widetilde{h}_2 d\gamma + y_2 \int_{\gamma(y)} \widetilde{h}_1 d\gamma + \int_{\gamma(y)} (x_1\widetilde{h}_2 - x_2\widetilde{h}_1) d\gamma, \ y \in \gamma \\ \Phi_1(y); &= -\nu_1 \int_{\gamma(y)} \widetilde{h}_2 d\gamma + \nu_2 \int_{\gamma(y)} \widetilde{h}_1 d\gamma, \quad y \in \gamma, \end{aligned}$$

respectively belong to the spaces $H^{\frac{3}{2}}(\gamma)$ and $H^{\frac{1}{2}}(\gamma)$ (the compatibility relations satisfied by the functions h_α and the simple-connectedness of ω play a crucial rôle here; cf. Ciarlet [1997, Theorem 5.6-1] for details).

(ii) Let $\widetilde{\chi} \in H^2(\omega)$ denote the unique solution in the sense of distributions of

$$\begin{aligned} \Delta^2\widetilde{\chi} &= [\widetilde{\theta}, \widetilde{\theta}] \text{ in } \omega, \\ \widetilde{\chi} = \Phi_0 &\text{ and } \partial_\nu\widetilde{\chi} = \Phi_1 \text{ on } \gamma. \end{aligned}$$

Let $F \in V(\omega)$ denote the unique solution in the sense of distributions of

$$\begin{aligned} -\partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2 F) &= f \text{ in } \omega, \\ F = \partial_\nu F &= 0 \quad \text{on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2 F)\nu_\alpha\nu_\beta &= 0 \quad \text{on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2 F)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2 F)\nu_\alpha\tau_\beta) &= 0 \quad \text{on } \gamma_2. \end{aligned}$$

Let the bilinear mapping $B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega)$ be defined as follows: For each pair $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B(\xi, \eta) \in H_0^2(\omega)$ is the unique solution in the sense of distributions of

$$\Delta^2 B(\xi,\eta) = [\xi,\eta] \text{ in } \omega,$$
$$B(\xi,\eta) = \partial_\nu B(\xi,\eta) = 0 \text{ on } \gamma.$$

Finally, let the bilinear mapping $\widetilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega)$ be defined as follows: For each pair $(\Phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\varsigma := \widetilde{B}(\xi, \eta) \in V(\omega)$ is defined as the unique solution in the sense of distributions of

$$\begin{aligned} -\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2 \varsigma) &= [\Phi, \xi] \text{ in } \omega, \\ \varsigma = \partial_\nu \varsigma &= 0 \quad \text{on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2\varsigma)\nu_\alpha\nu_\beta &= 0 \quad \text{on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\varsigma)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\varsigma)\nu_\alpha\tau_\beta) &= 0 \quad \text{on } \gamma_2. \end{aligned}$$

On the one hand, the definition of the function $\widetilde{\chi}$ and that of the mapping B imply that the equations

$$\Delta^2\Phi = -[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega,$$
$$\Phi = \Phi_0 \text{ and } \partial_\nu\Phi = \Phi_1 \quad \text{on } \gamma,$$

which are part of the scaled generalized Marguerre-von Kármán equations, are equivalent to the operator equation (viewed as an equality in the space $H^2(\omega)$):

$$\Phi = -B(\widetilde{\xi}, \widetilde{\xi}) + \widetilde{\chi}.$$

On the other hand, the definition of the function F and that of the mapping $\widetilde{B}$ imply that the remaining equations in the scaled generalized Marguerre-von Kármán equations, viz.,

$$\begin{aligned} -\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2 \xi) &= [\Phi, \widetilde{\xi}] + f \text{ in } \omega, \\ \xi = \partial_\nu \xi &= 0 \quad \text{on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta &= 0 \qquad \text{on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) &= 0 \quad \text{on } \gamma_2, \end{aligned}$$

are equivalent to the operator equation (viewed as an equality in the space $V(\omega)$):

$$\widetilde{\xi} - (\widetilde{\theta} + F) = \widetilde{B}(\Phi, \widetilde{\xi}).$$

Eliminating Φ between these two operator equations yields the single operator equation (viewed as an equality in the space $V(\omega)$):

$$\widetilde{B}(B(\widetilde{\xi}, \widetilde{\xi}), \widetilde{\xi}) + \widetilde{\xi} - \widetilde{B}(\widetilde{\chi}, \widetilde{\xi}) - (\widetilde{\theta} + F) = 0.$$

This equation thus takes the announced form, with the cubic mapping $\widetilde{C} : V(\omega) \to V(\omega)$, the linear mapping $\widetilde{L} : V(\omega) \to V(\omega)$, and the element $\widetilde{F} \in V(\omega)$ being defined by:

$$\begin{aligned}
&\widetilde{C}(\eta) := \widetilde{B}(B(\eta,\eta),\eta), \quad \text{for all } \eta \in V(\omega),\\
&\widetilde{L}\eta := \widetilde{B}(\widetilde{\chi},\eta), \quad \text{for all } \eta \in V(\omega),\\
&\widetilde{F} := \widetilde{\theta} + F.
\end{aligned}$$

Since the cubic operator $\widetilde{C} : V(\omega) \to V(\omega)$ does not depend on the function $\widetilde{\theta}$, it coincides with the operator found when $\widetilde{\theta} = 0$, i.e., the situation where the shell is a plate. As a result, a noticeable outcome of Theorem 1 is that the vertical displacement of the shell when *measured from the horizontal plane*, viz., $\xi_3^\varepsilon + \theta^\varepsilon = \varepsilon E^{-\frac{1}{2}}\Big(\xi + \widetilde{\theta}\Big)$ according to Section 2, satisfies an operator equation that has the same *structure* as that found for a generalized von Kármán plate.

4 Existence Theory

Our second task consists in establishing the existence of solutions to the generalized Marguerre-von Kármán equations, by making use of the operator equation found in Theorem1.

Theorem 2. *Let the assumptions be as in* Theorem 1. *If the norms* $\| h_\alpha \|_{L^2(\gamma_1)}$ *are small enough, the generalized Marguerre-von Kármán equations have at least one solution* $(\xi, \Phi) \in V(\omega) \times H^2(\omega)$ *in the sense of distributions.*

Proof. By Theorem 1, it suffices to establish that the cubic operator found in *ibid.* has at least one solution $\widetilde{\xi} \in V(\omega)$. Accordingly, we first assemble in parts (i) to (iv) relevant properties of the mappings $B, \widetilde{B}, \widetilde{L}$, and $\widetilde{C}$ introduced in the proof of Theorem 1, before addressing the existence theory proper in part (v).

(i) The bilinear mapping $B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega)$ is sequentially compact (and hence *a fortiori* continuous), in the sense that (weak convergence is denoted $\rightharpoonup$):

$$\Big(\xi^n, \eta^n\Big) \rightharpoonup \big(\xi, \eta\big) \text{ in } H^2(\omega) \times H^2(\omega) \Longrightarrow B\Big(\xi^n, \eta^n\Big) \to B\big(\xi, \eta\big) \text{ in } H_0^2(\omega).$$

For a proof, see e.g. Ciarlet & Rabier [1980, Section 2.2] or Ciarlet [1997, Theorem 5.8-2].

(ii) The definition of the bilinear mapping $\widetilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega)$ shows that it is continuous and it satisfies

$$((\widetilde{B}(\Phi, \xi), \eta)) = \int_\omega [\Phi, \xi]\eta d\omega \text{ for all } (\Phi, \xi, \eta) \in H^2(\omega) \times H^2(\omega) \times V(\omega),$$

where for any $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, we let

$$((\xi, \eta)) := -\int_\omega m_{\alpha\beta}(\bigtriangledown^2 \xi)\partial_{\alpha\beta}\eta d\omega = \frac{1}{3}\int_\omega \left\{ \frac{4\lambda\mu}{\lambda + 2\mu}\Delta\xi\Delta\eta + 4\mu\partial_{\alpha\beta}\xi\partial_{\alpha\beta}\eta \right\} d\omega.$$

Note that the space $V(\omega)$ will be henceforth considered as equipped with this inner product, which makes it a Hilbert space since its associated norm, denoted $\| \, . \, \|$, is equivalent to the norm $\| \, . \, \|_{H^2(\omega)}$ over $V(\omega)$.

(iii) The definition of the mapping $\widetilde{B}$ and that of the function $\widetilde{\chi}$ together imply the continuity of the linear mapping $\widetilde{L} : V(\omega) \to V(\omega)$ and the existence of a constant $c_0 > 0$ such that

$$\| \widetilde{L} \|_{\mathcal{L}(V(\omega))} \leqslant c_0 \sum_{\alpha} \| h_\alpha \|_{L^2(\gamma_1)} .$$

(iv) The cubic mapping $\widetilde{C} : V(\omega) \to V(\omega)$ satisfies:

$$((\widetilde{C}(\eta), \eta)) \geq 0 \text{ for all } \eta \in V(\omega).$$

To see this, we first note that the definitions of the mappings $\widetilde{B}$ and $\widetilde{C}$ imply that, for all $\eta \in V(\omega)$,

$$((\widetilde{C}(\eta), \eta)) = ((\widetilde{B}(B(\eta, \eta), \eta), \eta)) = \int_\omega [B(\eta, \eta), \eta]\eta d\omega.$$

Noting that $B(\eta, \eta) \in H_0^2(\omega)$ (by definition of the mapping B), we also have (see Ciarlet [1997, Theorem 5.8-2]),

$$\int_\omega [B(\eta, \eta), \eta]\eta d\omega = \int_\omega B(\eta, \eta)[\eta, \eta] d\omega.$$

Hence (again by definition of B),

$$((\widetilde{C}(\eta), \eta)) = \| \Delta B(\eta, \eta) \|^2_{L^2(\omega)} \geq 0 \text{ for all } \eta \in V(\omega).$$

(v) Proceeding as in Ciarlet, Gratie & Sabu [2001], we next adapt an elegant *compactness method*, due to J. L. Lions [1969, Chapter 1, Section 4.3]. Let w_i, $i \geq 1$, denote an orthonormal basis in the Hilbert space $V(\omega)$ and let V^m denote for each integer $m \geq 1$, the subspace of $V(\omega)$ spanned by the functions w_i, $1 \leq i \leq m$.

For each $m \geq 1$, we define the mapping $J^m : \boldsymbol{R}^m \to V^m$ by letting

$$J^m(\boldsymbol{X}) := \sum_{j=1}^m X_j w_j \in V^m \text{ for all } \boldsymbol{X} = (X_j)_{j=1}^m \in \boldsymbol{R}^m.$$

Let $\boldsymbol{X} \cdot \boldsymbol{Y}$ and $|\boldsymbol{X}|$ denote the Euclidean inner product of $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{R}^m$ and norm of $\boldsymbol{X} \in \boldsymbol{R}^m$. Finally, we define for each $m \geq 1$, a mapping

$$\boldsymbol{\Phi}^m = \left(\Phi_i^m\right)_{i=1}^m : \boldsymbol{R}^m \to \boldsymbol{R}^m$$

by letting, for all $\boldsymbol{X} \in \boldsymbol{R}^m$,

$$\Phi_i^m(\boldsymbol{X}) := ((\widetilde{C}(J^m(\boldsymbol{X})) + (I - \widetilde{L})(J^m(\boldsymbol{X})) - \widetilde{F}, w_i)), 1 \leq i \leq m.$$

Then the properties established in (iii) and (iv) imply that, for all $\boldsymbol{X} \in \boldsymbol{R}^m$ and all $m \geq 1$,

$$\begin{aligned}\boldsymbol{\Phi}^m(\boldsymbol{X}) \cdot \boldsymbol{X} =& ((\widetilde{C}(J^m(\boldsymbol{X})) + (I - \widetilde{L})(J^m(\boldsymbol{X})) - \widetilde{F}, J^m(\boldsymbol{X})))\\ &\geq (1 - \| \widetilde{L} \|_{L(V(\omega))})|\boldsymbol{X}|^2 - \| \widetilde{F} \| \, |\boldsymbol{X}|\\ &\geq (1 - c_0 \sum_\alpha \| h_\alpha \|_{L^2(\gamma_1)})|\boldsymbol{X}|^2 - \| \widetilde{F} \| \, |\boldsymbol{X}|.\end{aligned}$$

Assume henceforth that $\sum_\alpha \| h_\alpha \|_{L^2(\gamma_1)} < c_0^{-1}$ and let

$$M := (1 - c_0 \sum_\alpha \| h_\alpha \|_{L^2(\gamma_1)})^{-1} \| \widetilde{F} \|,$$

so that

$$\boldsymbol{\Phi}^m(\boldsymbol{X}) \cdot \boldsymbol{X} \geq 0 \text{ for all } \boldsymbol{X} \in \boldsymbol{R}^m \text{ that satisfy } |\boldsymbol{X}| = M.$$

Then a corollary of the *Brouwer fixed point theorem* (see, e.g., Lions [1969, Chapter 1, Lemma 4.3]) applied to each continuous mapping $\boldsymbol{\Phi}^m : \boldsymbol{R}^m \to \boldsymbol{R}^m$ shows that, for each $m \geq 1$, there exists $\boldsymbol{X}^m \in \boldsymbol{R}^m$ such that

$$|\boldsymbol{X}^m| \leq M \text{ and } \boldsymbol{\Phi}^m(\boldsymbol{X})^m = 0.$$

Equivalently, there exists for each $m \geq 1$ a function $\xi^m := J^m(\boldsymbol{X}^m) \in V^m$ such that

$$\| \xi^m \| \leq M \text{ and } ((\widetilde{C}(\xi^m) + (I - \widetilde{L})\xi^m - \widetilde{F}, \eta)) = 0, \text{ for all } \eta \in V^m(\omega).$$

Therefore there exist a subsequence $\left(\xi^n\right)_{n=1}^\infty$ of the sequence $\left(\xi^m\right)_{m=1}^\infty$ and an element $\widetilde{\xi} \in V(\omega)$ such that

$$\xi^n \rightharpoonup \widetilde{\xi} \text{ in } V(\omega).$$

Given any $\eta \in V(\omega)$, there exist functions $\eta^n \in V^n$ such that

$$\eta^n \to \eta \in V(\omega),$$

so that

$$((\widetilde{C}(\xi^n) + (I - \widetilde{L})\xi^n - \widetilde{F}, \eta^n)) = 0, \text{ for all } n \geq 1.$$

By definition of the cubic mapping $\widetilde{C}$ and by (ii), we know that

$$((\widetilde{C}(\xi^n), \eta^n)) = \int_\omega \Big[B(\xi^n, \xi^n), \xi^n\Big]\eta^n d\omega,$$

and by (i), we know that $B\Big(\xi^n, \xi^n\Big) \to B(\xi, \xi)$ in $H_0^2(\omega)$. Hence we infer that

$$((\widetilde{C}(\xi^n), \eta^n)) \to ((\widetilde{C}(\widetilde{\xi}), \eta)),$$

since $H^2(\omega)$ is continuously imbedded in $C^0(\bar{\omega})$. The linear mapping $\widetilde{L} : V(\omega) \to V(\omega)$ being also continuous with respect to the weak topology of $H^2(\omega)$, we also infer that

$$((\widetilde{L}\xi^n, \eta^n)) \to ((\widetilde{L}\widetilde{\xi}, \eta)).$$

Passing to the limit as $n \to \infty$, we have thus shown that

$$((\widetilde{C}(\widetilde{\xi}) + (I - \widetilde{L})\widetilde{\xi} - \widetilde{F}, \eta)) = 0, \text{ for all } \eta \in V(\omega).$$

Hence $\widetilde{\xi}$ is a solution to the cubic operator equation and the proof is complete.

∎

A characteristic of the operator equation is the *"loss of strict positivity"* incurred by its cubic part, since the relation

$$((\widetilde{C}(\eta), \eta)) = \int_\omega B(\eta, \eta)[\eta, \eta]d\omega = \| \Delta B(\eta, \eta) \|^2_{L^2(\omega)} \geq 0 \text{ for all } \eta \in V(\omega),$$

established in part (iv) of the above proof, shows that there exist (easily constructed) nonzero functions $\eta \in V(\omega)$ that satisfy $[\eta, \eta] = 0$ in ω when $length \gamma_2 > 0$. By contrast, $((\widetilde{C}(\eta), \eta)) > 0$ for all $\eta \in V(\omega) = H_0^2(\omega)$ when $\gamma_2 = \phi$(see Ciarlet [1997, Theorem5.8-2]). This observation thus precludes the recourse to the topological degree as in Goeleven, Nguyen & Théra [1993] or to pseudo-monotone operators as in Gratie [2000], for solving the operator equation.

Another feature of this equation is that, in general, the linear mapping $\widetilde{L} \in L(V(\omega))$ is *not symmetric* with respect to the inner product ((.,.)), since the number

$$((\widetilde{L}\xi, \eta)) = ((\widetilde{B}(\widetilde{\chi}, \xi), \eta)) = \int_\omega [\widetilde{\chi}, \xi]\eta d\omega$$

is *not* necessarily equal to $\int_\omega [\widetilde{\chi}, \eta]\xi d\omega$ for arbitrary functions $\xi, \eta \in V(\omega)$ (such an equality holds if at least one of the three functions $\widetilde{\chi}, \xi, \eta$ is in the space $H_0^2(\omega)$, a condition not satisfied here; see Ciarlet [1997, Theorem5.8-2]). This second observation prevents the usage of an associated functional as a means to obtain a solution to the operator equation as that of a *minimization problem* (as in the case $\gamma_2 = \phi$; see Ciarlet & Rabier [1980] or Ciarlet [1997, Theorem 5.8-3]). Interestingly, the cubic term poses no problem in this respect, since it is easily verified that, for arbitrary functions $\xi, \eta \in V(\omega)$, the Gâteaux derivative $j'(\xi)\eta$ of the functional $j : V(\omega) \to \boldsymbol{R}$ defined by $j(\eta) := \frac{1}{4}((\widetilde{C}(\eta), \eta))$ is indeed equal to $((\widetilde{C}(\xi), \eta))$.

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