CONTINUITY OF A DEFORMATION IN H^1 AS A FUNCTION OF ITS CAUCHY-GREEN TENSOR IN L^1

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Abstract. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. The Cauchy-Green, or metric, tensor field associated with a deformation of the set Ω, i.e., a smooth enough orientation-preserving mapping $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^n$, is the $n \times n$ symmetric matrix field defined by $\boldsymbol{\nabla\Theta}^T(x)\boldsymbol{\nabla\Theta}(x)$ at each point $x \in \Omega$. We show that, under appropriate assumptions, the deformations depend continuously on their Cauchy-Green tensors, the topologies being those of the spaces $\boldsymbol{H}^1(\Omega)$ for the deformations and $\boldsymbol{L}^1(\Omega)$ for the Cauchy-Green tensors. When $n = 3$ and Ω is viewed as a reference configuration of an elastic body, this result has potential applications to nonlinear three-dimensional elasticity, since the stored energy function of an hyperelastic material depends on the deformation gradient field $\boldsymbol{\nabla\Theta}$ through the Cauchy-Green tensor.

1. Introduction

Let Ω be a bounded and connected subset of $\mathbb{R}^3$, and let $\mathcal{B}$ be an elastic body with Ω as its *reference configuration*. Thanks mostly to the landmark existence theory of Ball [3], it is now customary in nonlinear three-dimensional elasticity to view any mapping $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^3)$ that is almost-everywhere injective and satisfies $\det \boldsymbol{\nabla\Theta} > 0$ a.e. in Ω as a possible *deformation* of $\mathcal{B}$ when $\mathcal{B}$ is subjected to *ad hoc* applied forces and boundary conditions. The almost-everywhere injectivity of $\boldsymbol{\Theta}$ (understood in the sense of Ciarlet & Nečas [12]) and the restriction on the sign of $\det \boldsymbol{\nabla\Theta}$ mathematically express (in an arguably weak way) the non-interpenetrability and orientation-preserving conditions that any physically realistic deformation should satisfy.

Let $\mathbb{S}^n$ designate the set of all symmetric matrices of order n. The *Cauchy-Green tensor field* $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} \in L^1(\Omega; \mathbb{S}^3)$ associated with a deformation $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^3)$ plays a major rôle in the modeling of three-dimensional nonlinear elasticity, since the response function of a frame-indifferent elastic material, or the stored energy function of a frame-indifferent hyperelastic material, necessarily depends on the deformation gradient $\boldsymbol{\nabla\Theta}$ through the Cauchy-Green tensor (for details, see, e.g., Ciarlet [4, Chapters 3 and 4]).

Conceivably, an alternative approach to the existence theory in three-dimensional elasticity could thus regard *the Cauchy-Green tensor as the primary unknown*, instead of the deformation itself as is usually the case.

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This observation, already made by Antman [2], is one of the reasons underlying the present study, the other being differential geometry *per se*. As such, it is a continuation of the works initiated in Ciarlet & Laurent [7] and Ciarlet & Mardare [9]. Note that a similar study, this time motivated by *nonlinear shell theory* and accordingly carried out for *surfaces in* $\mathbb{R}^3$ has been also undertaken in Ciarlet [5] and then extended in Ciarlet & Mardare [10].

Clearly, *the Cauchy-Green tensors depend continuously on the deformations*, since Cauchy-Schwarz inequality immediately shows that the mapping

$$\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^3) \to \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta} \in L^1(\Omega; \mathbb{S}^3)$$

is continuous (irrespectively of whether the mappings $\boldsymbol{\Theta}$ are almost-everywhere injective and orientation-preserving).

The purpose of this paper is to show that, *under appropriate smoothness and orientation-preserving assumptions, the converse holds, i.e., the deformations depend continuously on their Cauchy-Green tensors, the topologies being those of the same spaces* $H^1(\Omega; \mathbb{R}^3)$ *and* $L^1(\Omega; \mathbb{S}^3)$ (by contrast with the orientation-preserving condition, the issue of non-interpenetrability turns out to be irrelevant to our subsequent developments). In fact, we shall directly establish this continuity result in an arbitrary dimension n, at no extra cost in its proof. For convenience, we shall continue to call *Cauchy-Green tensor field* the matrix field $\boldsymbol{\nabla\Phi}^T \boldsymbol{\nabla\Phi} \in L^1(\Omega; \mathbb{S}^n)$ associated with any mapping $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^n)$ and we shall likewise say that a mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ is *orientation-preserving* if it satisfies $\det \boldsymbol{\nabla\Phi} > 0$ a.e. in Ω. Note that the Cauchy-Green tensor is simply the extension to a Sobolev space setting of the familiar *metric tensor* of a manifold in classical differential geometry.

This continuity result is itself a simple consequence of a *key inequality*, which constitutes the main result of this paper (see Theorem 1): Let Ω be a bounded and connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ be a mapping satisfying $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$. Then there exists a constant $C(\boldsymbol{\Theta})$ with the following property: For each orientation-preserving mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$, there exist a $n \times n$ rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ (i.e., an orthogonal matrix with a determinant equal to one) and a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ in $\mathbb{R}^n$ such that

$$\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R\Theta})\|_{H^1(\Omega;\mathbb{R}^n)} \leq C(\boldsymbol{\Theta}) \|\boldsymbol{\nabla\Phi}^T \boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta}\|_{L^1(\Omega;\mathbb{S}^n)}^{1/2}.$$

That a rotation $\boldsymbol{R}$ and a vector $\boldsymbol{b}$ should appear in the left-hand side of such an inequality is the least one could expect, in light of the classical *rigidity theorem*. This well-known result (for a proof, see, e.g., Ciarlet & Larsonneur [6, Theorem 3]) asserts that, if two mappings $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ and $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0$ and $\det \boldsymbol{\nabla\Theta} > 0$ in an open connected subset Ω of $\mathbb{R}^n$ have the same Cauchy-Green tensor field, then there exist a vector $\boldsymbol{b}$ in $\mathbb{R}^n$ and a $n \times n$ rotation $\boldsymbol{R}$ such that $\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \boldsymbol{R\Theta}(x)$ for all $x \in \Omega$ (the converse clearly holds). In other words, $\widetilde{\boldsymbol{\Theta}} = \boldsymbol{J} \circ \boldsymbol{\Theta}$, where the mapping $\boldsymbol{J}$ is an isometry of $\mathbb{R}^n$; for this reason, the mappings $\widetilde{\boldsymbol{\Theta}}$ and $\boldsymbol{\Theta}$ are said to be isometrically equivalent.

More generally, we shall say that two orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega;\mathbb{R}^n)$ and $\boldsymbol{\Theta} \in H^1(\Omega;\mathbb{R}^n)$ are *isometrically equivalent* if there exist a vector $\boldsymbol{b}$ in $\mathbb{R}^n$ and a $n \times n$ rotation such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

One application of the key inequality of Theorem 1 is the following *sequential continuity property* (in the same spirit, the same inequality can be also recast as one involving distances; see Theorem 2): Let $\boldsymbol{\Theta}^k \in H^1(\Omega;\mathbb{R}^n)$, $k \geq 1$, and $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ be orientation-preserving mappings. Then there exist a constant $C(\boldsymbol{\Theta})$ and orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}}^k \in H^1(\Omega;\mathbb{R}^n)$, $k \geq 1$, that are isometrically equivalent to $\boldsymbol{\Theta}^k$ such that

$$\|\widetilde{\boldsymbol{\Theta}}^k - \boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)} \leq C(\boldsymbol{\Theta}) \|(\boldsymbol{\nabla}\boldsymbol{\Theta}^k)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^k - \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Hence *the sequence* $(\widetilde{\boldsymbol{\Theta}}^k)_{k=1}^\infty$ *converges to* $\boldsymbol{\Theta}$ *in* $H^1(\Omega;\mathbb{R}^n)$ *as* $k \to \infty$ *if the sequence* $((\boldsymbol{\nabla}\boldsymbol{\Theta}^k)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^k)_{k=1}^\infty$ *converges to* $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ *in* $L^1(\Omega;\mathbb{S}^n)$ *as* $k \to \infty$.

Should the Cauchy-Green strain tensor be viewed as the primary unknown (as suggested above), such a sequential continuity could thus prove to be useful when considering *infimizing sequences* of the total energy, in particular for handling the part of the energy that takes into account the applied forces and the boundary conditions, which are both naturally expressed in terms of the deformation itself.

The key inequality is first established in the special case where $\boldsymbol{\Theta}$ is the identity mapping of the set $\overline{\Omega}$ (see Lemmas 1 and 2), by making use in particular of a fundamental *"geometric rigidity lemma"* recently proved by Friesecke, James & Müller [13]. It is then extended to an arbitrary mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega},\mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$ (see Lemmas 3 to 7), thanks in particular to a methodology already used by Ciarlet & Laurent [7] for establishing the continuity of (equivalence classes of isometrically equivalent) mappings in the space $\mathcal{C}^3(\Omega;\mathbb{R}^n)$ as functions of their Cauchy-Green tensor in the space $\mathcal{C}^2(\Omega;\mathbb{S}^n)$, both spaces being equipped with their standard Fréchet topologies. Note that, in the same spirit but by means of a different approach, the local Lipschitz-continuity of (equivalence classes of isometrically equivalent) mappings in the Banach space $\mathcal{C}^3(\overline{\Omega};\mathbb{R}^n)$ as functions of their Cauchy-Green tensor in the Banach space $\mathcal{C}^2(\overline{\Omega};\mathbb{S}^n)$ has been recently established by Ciarlet & Mardare [9].

Such results are to be compared with the earlier, pioneering estimates of John [15, 16] and Kohn [17], which implied *continuity at rigid body deformations*, i.e., at a mapping $\boldsymbol{\Theta}$ that is isometrically equivalent to the identity mapping of $\overline{\Omega}$. The recent and noteworthy result of Reshetnyak [19] for *quasi-isometric mappings* is in a sense complementary to the one obtained here (it also deals with Sobolev type norms) and is thus particularly relevant to the present study.

The results of this paper were announced in Ciarlet & Mardare [11].

2. Notations and other preliminaries

All spaces, matrices, etc., are real. The symbols $\mathbb{M}^n$, $\mathbb{S}^n$, $\mathbb{S}^n_>$, and $\mathbb{O}^n_+$ respectively designate the sets of all square matrices of order n, of all symmetric matrices of order n, of all positive-definite symmetric matrices of order n, and of all orthogonal matrices $\boldsymbol{Q}$ of order n with $\det \boldsymbol{Q} = 1$. A matrix $\boldsymbol{Q} \in \mathbb{O}^n_+$ will be called a *rotation*.

The Euclidean norm of a vector $\boldsymbol{b} \in \mathbb{R}^n$ is denoted $|\boldsymbol{b}|$ and $|\boldsymbol{A}| := \sup_{|\boldsymbol{b}|=1} |\boldsymbol{A}\boldsymbol{b}|$ denotes the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^n$. The Euclidean and spectral norms are invariant under rotations, in the sense that $|\boldsymbol{b}| = |\boldsymbol{Q}\boldsymbol{b}|$ and $|\boldsymbol{A}| = |\boldsymbol{Q}\boldsymbol{A}| = |\boldsymbol{A}\boldsymbol{Q}|$ for all rotations $\boldsymbol{Q} \in \mathbb{O}^n_+$.

The restriction of a mapping f to a set U is denoted $f_{|U}$. The identity mapping of a set X is denoted $\boldsymbol{id}_X$.

Let Ω be an open subset of $\mathbb{R}^n$. Given any matrix-valued mapping $\boldsymbol{F} \in L^2(\Omega; \mathbb{M}^n)$, we let

$$\|\boldsymbol{F}\|_{L^2(\Omega;\mathbb{M}^n)} := \left\{ \int_\Omega |\boldsymbol{F}(x)|^2 dx \right\}^{1/2},$$

and, given any vector-valued mapping $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^n)$, we let

$$\|\boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)} := \left\{ \int_\Omega \left(|\boldsymbol{\Theta}(x)|^2 + |\boldsymbol{\nabla}\boldsymbol{\Theta}(x)|^2 \right) dx \right\}^{1/2},$$

where $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) \in \mathbb{M}^n$ denotes the gradient matrix of the mapping $\boldsymbol{\Theta}$ at x. *These norms are thus also invariant under rotations in* $\mathbb{R}^n$, in the sense that $\|\boldsymbol{F}\|_{L^2(\Omega;\mathbb{M}^n)} = \|\boldsymbol{Q}\boldsymbol{F}\|_{L^2(\Omega;\mathbb{M}^n)} = \|\boldsymbol{F}\boldsymbol{Q}\|_{L^2(\Omega;\mathbb{M}^n)}$ and $\|\boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)} = \|\boldsymbol{Q}\boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)}$ for all rotations $\boldsymbol{Q} \in \mathbb{O}^n_+$.

In this paper, the space $\mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ is defined as that consisting of all vector-valued functions $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ that, together with their partial derivatives of the first order, possess continuous extentions to the closure $\overline{\Omega}$ of Ω, and the definition of a *bounded open set with a Lipschitz-continuous boundary* is the usual one, as found for instance in Nečas [18], Adams [1], or Grisvard [14].

3. A key inequality

The following theorem is the main result of this paper.

Theorem 1. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $C(\boldsymbol{\Theta})$ with the following property: Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla}\boldsymbol{\Phi} > 0$ *a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta})\|_{H^1(\Omega;\mathbb{R}^n)} \leq C(\boldsymbol{\Theta}) \|\boldsymbol{\nabla}\boldsymbol{\Phi}^T\boldsymbol{\nabla}\boldsymbol{\Phi} - \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}. \qquad \square$$

The proof of Theorem 1, which for clarity is broken into those of seven lemmas, essentially aims at demonstrating the existence of a constant $c(\boldsymbol{\Theta})$ and a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(\Omega;\mathbb{M}^n)} \leq c(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)},$$

the proof of such an *"intermediary inequality"* occupying Lemmas 1 to 6.

In what follows, many constants, vectors, matrices, etc., will explicitly display dependences on sets such as Ω, on mappings such as $\boldsymbol{\Phi}$, $\boldsymbol{\Theta}$, etc., but in order to avoid lengthy formulas, only the dependences that are crucial to the argument will be displayed, however. Thus for instance, the dependence on the set Ω does not appear in the constant $C(\boldsymbol{\Theta})$ of Theorem 1, but the same dependence is explicit in the constant $\Lambda(\Omega)$ of Lemma 2 because it plays a key rôle in the proof of Lemma 3, etc.

To begin with, we establish a simple, yet crucial, result about matrices.

Lemma 1. *Let a matrix* $\boldsymbol{F} \in \mathbb{M}^n$ *be such that* $\det \boldsymbol{F} > 0$. *Then*

$$\operatorname{dist}(\boldsymbol{F}, \mathbb{O}^n_+) := \inf_{\boldsymbol{Q}\in\mathbb{O}^n_+} |\boldsymbol{F} - \boldsymbol{Q}| \leq |\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}|^{1/2}.$$

Proof. It is known that

$$\operatorname{dist}(\boldsymbol{F}, \mathbb{O}^n_+) = |(\boldsymbol{F}^T\boldsymbol{F})^{1/2} - \boldsymbol{I}|.$$

Let $0 < v_1 \leq v_2 \leq ... \leq v_n$ denote the singular values of the matrix $\boldsymbol{F}$. Then

$$\begin{aligned}|(\boldsymbol{F}^T\boldsymbol{F})^{1/2} - \boldsymbol{I}| &= \max\{|v_1 - 1|, |v_n - 1|\}\\ &\leq \max\{|v_1^2 - 1|^{1/2}, |v_n^2 - 1|^{1/2}\} = |\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}|^{1/2}.\end{aligned}$$

$\square$

Thanks to Lemma 1 and to a fundamental *"geometric rigidity lemma"*, due to G. Friesecke, R.D. James and S. Müller, it is an easy matter to show that the "intermediary inequality" already holds in the special case where $\boldsymbol{\Theta} = \boldsymbol{id}_{\overline{\Omega}}$.

Lemma 2. *Let* Ω *be a bounded connected open subset of* $\mathbb{R}^n$ *with a Lipschitz-continuous boundary. Then there exists a constant* $\Lambda(\Omega)$ *with the following property: Given any mapping* $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ *satisfying* $\det \boldsymbol{\nabla\Phi} > 0$ *a.e. in* Ω*, there exists a rotation* $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}) \in \mathbb{O}^n_+$ *such that*

$$\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\|_{L^2(\Omega;\mathbb{M}^n)} \leq \Lambda(\Omega)\|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{I}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. By the *"geometric rigidity lemma"* of Friesecke, James & Müller [13, Theorem 3.1], there exists a constant $\Lambda(\Omega)$ depending only on the set Ω with the following property: For each $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\|_{L^2(\Omega;\mathbb{M}^n)} \leq \Lambda(\Omega)\|\operatorname{dist}(\boldsymbol{\nabla\Phi}, \mathbb{O}^n_+)\|_{L^2(\Omega)}.$$

If in addition the mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$ satisfies $\det \boldsymbol{\nabla\Phi} > 0$ a.e. in Ω, then Lemma 1 implies that

$$\mathrm{dist}(\boldsymbol{\nabla\Phi}(x), \mathbb{O}^n_+) \leq |\boldsymbol{\nabla\Phi}(x)^T\boldsymbol{\nabla\Phi}(x) - \boldsymbol{I}|^{1/2}$$

for almost all $x \in \Omega$. Hence

$$\|\mathrm{dist}(\boldsymbol{\nabla\Phi}, \mathbb{O}^n_+)\|_{L^2(\Omega)} \leq \|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{I}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

□

To proceed from the identity mapping to a "general" mapping $\boldsymbol{\Theta}$, we now follow a pattern inspired by that proposed in Ciarlet & Laurent [7, Lemma 2.2], although the technical details are for the most part different. To begin with, we consider the case where the mapping $\boldsymbol{\Theta}$ is *injective in* $\overline{\Omega}$.

Lemma 3. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any injective mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $\widehat{C}(\boldsymbol{\Theta})$ with the following property: Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla\Phi} > 0$ *a.e. in Ω, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi},\boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(\Omega;\mathbb{M}^n)} \leq \widehat{C}(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. Since the boundary of Ω is Lipschitz-continuous, any mapping $\boldsymbol{\Theta}$ in the space $\mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ as defined in Sect. 2 can be extended to a mapping $\boldsymbol{\Theta}^\flat$ in the space $\mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$ (for a proof, see, e.g., Ciarlet & Mardare [9, Theorem 4.2], where this property is derived from the extension theorem of Whitney [20] combined with *ad hoc* Taylor formulas along paths). Moreover, since $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$ and Ω is bounded, there exists a connected open subset $\Omega^\sharp$ containing $\overline{\Omega}$ such that the restriction $\boldsymbol{\Theta}^\sharp \in \mathcal{C}^1(\Omega^\sharp;\mathbb{R}^n)$ to $\Omega^\sharp$ of such an extension $\boldsymbol{\Theta}^\flat$ satisfies $\det \boldsymbol{\nabla\Theta}^\sharp > 0$ in $\Omega^\sharp$.

Consequently, the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is also a bounded connected open subset of $\mathbb{R}^n$ whose boundary $\boldsymbol{\Theta}(\partial\Omega) = \boldsymbol{\Theta}^\sharp(\partial\Omega)$ is Lipschitz-continuous. Besides, the inverse mapping $\widehat{\boldsymbol{\Theta}} : \{\widehat{\Omega}\}^- \to \overline{\Omega}$ of $\boldsymbol{\Theta}$ belongs to the space $\mathcal{C}^1(\{\widehat{\Omega}\}^-;\mathbb{R}^n)$ (the notation $\{\widehat{\Omega}\}^-$ designates the closure of the set $\widehat{\Omega}$), since each point of the boundary of $\widehat{\Omega}$ possesses a neighborhood $\widehat{N}$ over which $\boldsymbol{\Theta}^\sharp|_{\widehat{N}}$ is invertible and $\widehat{\boldsymbol{\Theta}}|_{\widehat{N}\cap\{\widehat{\Omega}\}^-} = (\boldsymbol{\Theta}^\sharp|_{\widehat{N}})^{-1}|_{\widehat{N}\cap\{\widehat{\Omega}\}^-}$.

Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$, the composite mapping $\widehat{\boldsymbol{\Phi}} := \boldsymbol{\Phi} \circ \widehat{\boldsymbol{\Theta}}$ belongs to the space $H^1(\widehat{\Omega};\mathbb{R}^n)$ since the bijection $\boldsymbol{\Theta} : \overline{\Omega} \to \{\widehat{\Omega}\}^-$ is bi-Lipschitzian. Moreover,

$$\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\hat{x}) = \boldsymbol{\nabla\Phi}(x)\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Theta}}(\hat{x}) = \boldsymbol{\nabla\Phi}(x)\boldsymbol{\nabla\Theta}(x)^{-1}$$

for almost all $\hat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$, the notation $\widehat{\boldsymbol{\nabla}}$ indicating that differentiation is performed with respect to the variable $\hat{x}$. Hence $\det \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}} > 0$ a.e. in $\widehat{\Omega}$ if in addition $\det \boldsymbol{\nabla\Phi} > 0$ a.e. in Ω.

By Lemma 2, there exists a constant $c_0(\boldsymbol{\Theta}) := \Lambda(\widehat{\Omega})$ with the following property: Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Phi} > 0$ a.e.

in Ω, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that the mapping $\widehat{\boldsymbol{\Phi}} = \boldsymbol{\Phi} \circ \widehat{\boldsymbol{\Theta}}$ satisfies

$$\|\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}} - \boldsymbol{R}\|_{L^2(\widehat{\Omega};\mathbb{M}^n)} \leq c_0(\boldsymbol{\Theta}) \|\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}^T \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I}\|^{1/2}_{L^1(\widehat{\Omega};\mathbb{S}^n)}.$$

The injectivity of the mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ and the relation $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$ together imply that

$$\begin{aligned}
\|\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}} - \boldsymbol{R}\|^2_{L^2(\widehat{\Omega};\mathbb{M}^n)} &= \int_{\widehat{\Omega}} |\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\hat{x}) - \boldsymbol{R}|^2 d\hat{x} \\
&= \int_{\Omega} |\boldsymbol{\nabla\Phi}(x)\boldsymbol{\nabla\Theta}(x)^{-1} - \boldsymbol{R}|^2 \det \boldsymbol{\nabla\Theta}(x) dx \\
&\geq \int_{\Omega} |\boldsymbol{\nabla\Phi}(x) - \boldsymbol{R}\boldsymbol{\nabla\Theta}(x)|^2 |\boldsymbol{\nabla\Theta}(x)|^{-2} \det \boldsymbol{\nabla\Theta}(x) dx \\
&\geq c_1(\boldsymbol{\Theta}) \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|^2_{L^2(\Omega;\mathbb{M}^n)},
\end{aligned}$$

where $c_1(\boldsymbol{\Theta}) := \inf_{x\in\overline{\Omega}} \{|\boldsymbol{\nabla\Theta}(x)|^{-2} \det \boldsymbol{\nabla\Theta}(x)\} > 0$. Likewise,

$$\begin{aligned}
&\|\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}^T \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I}\|_{L^1(\widehat{\Omega};\mathbb{S}^n)} = \int_{\widehat{\Omega}} |\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\hat{x})^T \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\hat{x}) - \boldsymbol{I}| d\hat{x} \\
&\quad = \int_{\Omega} |\boldsymbol{\nabla\Theta}(x)^{-T} \left(\boldsymbol{\nabla\Phi}(x)^T \boldsymbol{\nabla\Phi}(x) - \boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x)\right) \boldsymbol{\nabla\Theta}(x)^{-1}| \det \boldsymbol{\nabla\Theta}(x) dx \\
&\quad \leq c_2(\boldsymbol{\Theta}) \|\boldsymbol{\nabla\Phi}^T \boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta}\|_{L^1(\Omega;\mathbb{S}^n)},
\end{aligned}$$

where $c_2(\boldsymbol{\Theta}) := \sup_{x\in\overline{\Omega}} \{|\boldsymbol{\nabla\Theta}(x)^{-T}||\boldsymbol{\nabla\Theta}(x)^{-1}| \det \boldsymbol{\nabla\Theta}(x)\} < \infty$. The announced inequality thus holds with $\widehat{C}(\boldsymbol{\Theta}) := c_0(\boldsymbol{\Theta}) c_1(\boldsymbol{\Theta})^{-1/2} c_2(\boldsymbol{\Theta})^{1/2}$. $\square$

In view of establishing the "intermediate inequality" in the general case, we first prove two technical results, which make up Lemmas 4 and 5.

Lemma 4. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$, there exist finitely many connected open subsets $V_j = V_j(\boldsymbol{\Theta})$, $1 \leq j \leq N$, of Ω, each with a Lipschitz-continuous boundary, such that $\Omega = \cup_{j=1}^N V_j$ and such that, for each $1 \leq k \leq N$, the set $\cup_{j=1}^k V_j$ is connected and the restriction of $\boldsymbol{\Theta}$ to $\overline{V}_k$ is injective.*

Proof. Given any point $x \in \Omega$, there exists by the inverse mapping theorem an open ball $V(x)$ centered at x such that $V(x) \subset \Omega$ and the restriction of $\boldsymbol{\Theta}$ to $\overline{V(x)}$ is injective. Each such open set $V(x)$ is thus connected and has a Lipschitz-continuous boundary.

It was established in the proof of Lemma 3 that there exists a connected open set $\Omega^\sharp$ containing $\overline{\Omega}$ and an extension $\boldsymbol{\Theta}^\sharp \in \mathcal{C}^1(\Omega^\sharp; \mathbb{R}^n)$ of $\boldsymbol{\Theta}$ from $\overline{\Omega}$ to $\Omega^\sharp$ that satisfies $\det \boldsymbol{\nabla\Theta}^\sharp > 0$ in $\Omega^\sharp$. Hence, given any point $x \in \partial\Omega$, there exists, again by the inverse mapping theorem, but applied this time to the extension $\boldsymbol{\Theta}^\sharp$, an open ball $B(x) \subset \Omega^\sharp$ centered at x such that the restriction of $\boldsymbol{\Theta}^\sharp$ to $\overline{B(x)}$ is injective. The assumption that the boundary of Ω is Lipschitz-continuous furthermore implies that there exist a local Cartesian frame centered at x, with basis vectors $\boldsymbol{b}_i$, $1 \le i \le n$, and local coordinates $\boldsymbol{y}' = (y_1, ..., y_{n-1})$ and y_n, constants $\alpha > 0$ and $\beta > 0$, and a Lipschitz-continuous mapping $\varphi : \{\boldsymbol{y}' \in \mathbb{R}^{n-1}; |\boldsymbol{y}'| < \alpha\} \to \mathbb{R}$, such that

$$W(x) := \{\sum_{i=1}^{n} y_i \boldsymbol{b}_i; |\boldsymbol{y}'| < \alpha, |y_n| < \beta\} \subset B(x),$$
$$\Omega \cap W(x) = \{(\boldsymbol{y}', y_n) \in \mathbb{R}^n; |\boldsymbol{y}'| < \alpha, -\beta < y_n < \varphi(\boldsymbol{y}')\}.$$

In addition, it is easily seen that each open set $V(x) := W(x) \cap \Omega$ is connected and has a Lipschitz-continuous boundary.

Since the set $\overline{\Omega}$ is compact, there exist finitely many points $x_j \in \Omega$, $1 \le j \le M$, and $x_j \in \partial\Omega$, $M+1 \le j \le N$, such that

$$\Omega \subset \overline{\Omega} \subset \left\{\cup_{j=1}^{M} V(x_j)\right\} \cup \left\{\cup_{j=M+1}^{N} W(x_j)\right\}.$$

Consequently,

$$\Omega = \left\{\cup_{j=1}^{M} (V(x_j) \cap \Omega)\right\} \cup \left\{\cup_{j=M+1}^{N} (W(x_j) \cap \Omega)\right\} = \cup_{j=1}^{N} V(x_j),$$

where each open subset $V(x_j)$ of Ω is connected, has a Lipschitz-continuous boundary, and is such that the restriction of $\boldsymbol{\Theta}$ to $\overline{V(x_j)}$ is injective.

A simple recursion argument then shows that there exists a bijection σ from the set $\{1, 2, ..., n\}$ onto itself such that, for each $1 \le k \le N$, the set $\cup_{j=1}^{k} V(x_{\sigma(j)})$ is connected. The assertion then follows by letting $V_j := V(x_{\sigma(j)})$, $1 \le j \le N$. $\square$

Lemma 5. *Let U be a bounded open subset of $\mathbb{R}^n$. Then, given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{U}; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{U}$ and given any open subset ω of U, there exists a constant $\widetilde{C}(\omega, \boldsymbol{\Theta})$ with the following property: Given any mapping $\boldsymbol{\Phi} \in H^1(U; \mathbb{R}^n)$, there exists a rotation $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\Phi}_{|\omega}, \boldsymbol{\Theta}_{|\omega}) \in \mathbb{O}^n_+$ such that*

$$\|\boldsymbol{\nabla\Phi} - \boldsymbol{Q}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \le \widetilde{C}(\omega, \boldsymbol{\Theta}) \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \text{ for all } \boldsymbol{R} \in \mathbb{O}^n_+.$$

Proof. Given any mapping $\boldsymbol{\Phi} \in H^1(U; \mathbb{R}^n)$, define the matrix $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{\Phi}_{|\omega}, \boldsymbol{\Theta}_{|\omega}) \in \mathbb{M}^n$ by

$$\boldsymbol{M} := \frac{1}{\int_\omega \det \boldsymbol{\nabla\Theta} dx} \int_\omega \boldsymbol{\nabla\Phi} (\boldsymbol{\nabla\Theta})^{-1} \det \boldsymbol{\nabla\Theta} dx,$$

and let $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\Phi}_{|\omega}, \boldsymbol{\Theta}_{|\omega}) \in \mathbb{O}^n_+$ by any rotation that satisfies

$$|\boldsymbol{M} - \boldsymbol{Q}| = \operatorname{dist}(\boldsymbol{M}, \mathbb{O}^n_+).$$

Then, for any rotation $\boldsymbol{R} \in \mathbb{O}^n_+$,

$$
\begin{aligned}
|\boldsymbol{M} - \boldsymbol{R}| &\leq \frac{1}{\int_\omega \det \boldsymbol{\nabla\Theta} dx} \int_\omega |\boldsymbol{\nabla\Phi}(\boldsymbol{\nabla\Theta})^{-1} - \boldsymbol{R}| \det \boldsymbol{\nabla\Theta} dx \\
&\leq \frac{1}{\int_\omega \det \boldsymbol{\nabla\Theta} dx} \int_\omega |\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}||(\boldsymbol{\nabla\Theta})^{-1}| \det \boldsymbol{\nabla\Theta} dx \\
&\leq \frac{\|(\boldsymbol{\nabla\Theta})^{-1} \det \boldsymbol{\nabla\Theta}\|_{L^2(\omega;\mathbb{M}^n)}}{\| \det \boldsymbol{\nabla\Theta}\|_{L^1(\omega)}} \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(\omega;\mathbb{M}^n)},
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
&\|\boldsymbol{\nabla\Phi} - \boldsymbol{Q}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \leq \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} + |\boldsymbol{R} - \boldsymbol{Q}|\|\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \\
&\quad \leq \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} + (|\boldsymbol{M} - \boldsymbol{R}| + |\boldsymbol{M} - \boldsymbol{Q}|)\|\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \\
&\quad \leq \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} + 2|\boldsymbol{M} - \boldsymbol{R}|\|\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \\
&\quad \leq \widetilde{C}(\omega, \boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)},
\end{aligned}
$$

with

$$
\widetilde{C}(\omega, \boldsymbol{\Theta}) := 1 + 2\frac{\|(\boldsymbol{\nabla\Theta})^{-1} \det \boldsymbol{\nabla\Theta}\|_{L^2(\omega;\mathbb{M}^n)}\|\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)}}{\| \det \boldsymbol{\nabla\Theta}\|_{L^1(\omega)}}.
$$

□

Lemma 5 thus shows that, even though the rotation $\boldsymbol{Q}$ is determined by the knowledge of the mappings $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ solely on the subset ω of U, it is "almost as good as" any rotation $\boldsymbol{R}$ that would minimise the norm $\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)}$, in the sense that $\boldsymbol{Q}$ satisfies

$$
\begin{aligned}
\widetilde{C}(\omega, \boldsymbol{\Theta})^{-1}\|\boldsymbol{\nabla\Phi} - \boldsymbol{Q}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} &\leq \inf_{\boldsymbol{R}\in\mathbb{O}^n_+} \|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)} \\
&\leq \|\boldsymbol{\nabla\Phi} - \boldsymbol{Q}\boldsymbol{\nabla\Theta}\|_{L^2(U;\mathbb{M}^n)}.
\end{aligned}
$$

We are now in a position to establish the "intermediary inequality" in the general case, i.e., without assuming that the mapping $\boldsymbol{\Theta}$ is injective on $\overline{\Omega}$.

Lemma 6. *Let Ω be a bounded open connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $c(\boldsymbol{\Theta})$ with the following property: Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla\Phi} > 0$ *a.e. in Ω, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$
\|\boldsymbol{\nabla\Phi} - \boldsymbol{R}\boldsymbol{\nabla\Theta}\|_{L^2(\Omega;\mathbb{M}^n)} \leq c(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.
$$

Proof. The connected open subsets $V_j = V_j(\boldsymbol{\Theta})$, $1 \leq j \leq N$ of Ω being those constructed in Lemma 4, let

$$
U_k := \cup_{j=1}^k V_j,\ \ \boldsymbol{\Theta}_k := \boldsymbol{\Theta}_{|\overline{U}_k},\ \ \boldsymbol{\Phi}_k := \boldsymbol{\Phi}_{|U_k},\ \ 1 \leq k \leq N.
$$

By Lemma 3 applied to the set U_1, there exist a constant $\widehat{C}_1(\boldsymbol{\Theta})$ and a rotation $\boldsymbol{R}_1 = \boldsymbol{R}_1(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\Phi}_1 - \boldsymbol{R}_1\boldsymbol{\nabla\Theta}_1\|_{L^2(U_1;\mathbb{M}^n)} \leq \widehat{C}_1(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}_1^T\boldsymbol{\nabla\Phi}_1 - \boldsymbol{\nabla\Theta}_1^T\boldsymbol{\nabla\Theta}_1\|^{1/2}_{L^1(U_1;\mathbb{S}^n)}.$$

Assume more generally that, for some $1 \leq k < N$, there exist a constant $\widehat{C}_k(\boldsymbol{\Theta})$ and a rotation $\boldsymbol{R}_k = \boldsymbol{R}_k(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\Phi}_k - \boldsymbol{R}_k\boldsymbol{\nabla\Theta}_k\|_{L^2(U_k;\mathbb{M}^n)} \leq \widehat{C}_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}_k^T\boldsymbol{\nabla\Phi}_k - \boldsymbol{\nabla\Theta}_k^T\boldsymbol{\nabla\Theta}_k\|^{1/2}_{L^1(U_k;\mathbb{S}^n)}.$$

By Lemma 5 applied to the subset

$$\omega_k := U_k \cap V_{k+1}$$

of the set U_k, there exists a constant $\widetilde{C}_k(\boldsymbol{\Theta})$ and a rotation $\boldsymbol{Q}_k = \boldsymbol{Q}_k(\boldsymbol{\Phi}|_{\omega_k}, \boldsymbol{\Theta}|_{\omega_k}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\Phi}_k - \boldsymbol{Q}_k\boldsymbol{\nabla\Theta}_k\|_{L^2(U_k;\mathbb{M}^n)} \leq \widetilde{C}_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}_k - \boldsymbol{R}_k\boldsymbol{\nabla\Theta}_k\|_{L^2(U_k;\mathbb{M}^n)}.$$

Therefore,

$$\begin{aligned}\|\boldsymbol{\nabla\Phi}_k - \boldsymbol{Q}_k\boldsymbol{\nabla\Theta}_k\|_{L^2(U_k;\mathbb{M}^n)} \\ \leq \widehat{C}_k(\boldsymbol{\Theta})\widetilde{C}_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}_k^T\boldsymbol{\nabla\Phi}_k - \boldsymbol{\nabla\Theta}_k^T\boldsymbol{\nabla\Theta}_k\|^{1/2}_{L^1(U_k;\mathbb{S}^n)}.\end{aligned}$$

Let

$$\widetilde{\boldsymbol{\Theta}}_{k+1} := \boldsymbol{\Theta}|_{\overline{V}_{k+1}} \text{ and } \widetilde{\boldsymbol{\Phi}}_{k+1} := \boldsymbol{\Phi}|_{V_{k+1}}.$$

By Lemma 3 applied to the set V_{k+1}, there exists a constant $\widehat{C}'_k(\boldsymbol{\Theta})$ and a rotation $\widetilde{\boldsymbol{R}}_{k+1} = \widetilde{\boldsymbol{R}}_{k+1}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\begin{aligned}\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \widetilde{\boldsymbol{R}}_{k+1}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|_{L^2(V_{k+1};\mathbb{M}^n)} \\ \leq \widehat{C}'_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}^T_{k+1}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T_{k+1}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|^{1/2}_{L^1(V_{k+1};\mathbb{S}^n)}.\end{aligned}$$

By Lemma 5 applied to the set ω_k, now viewed as a subset of the set V_{k+1}, there exists a constant $\widetilde{C}'_k(\boldsymbol{\Theta})$ such that

$$\begin{aligned}\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{Q}_k\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|_{L^2(V_{k+1};\mathbb{M}^n)} \\ \leq \widetilde{C}'_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \widetilde{\boldsymbol{R}}_{k+1}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|_{L^2(V_{k+1};\mathbb{M}^n)},\end{aligned}$$

where the rotation $\boldsymbol{Q}_k$ is the same as before, since it only depends on the restrictions $\boldsymbol{\Phi}|_{\omega_k}$ and $\boldsymbol{\Theta}|_{\omega_k}$. Consequently,

$$\begin{aligned}\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{Q}_k\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|_{L^2(V_{k+1};\mathbb{M}^n)} \\ \leq \widehat{C}'_k(\boldsymbol{\Theta})\widetilde{C}'_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}^T_{k+1}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T_{k+1}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|^{1/2}_{L^1(V_{k+1};\mathbb{S}^n)}.\end{aligned}$$

Combining the above inequalities, we thus obtain

$$\begin{aligned}
&\|\boldsymbol{\nabla\Phi}_{k+1} - \boldsymbol{Q}_k \boldsymbol{\nabla\Theta}_{k+1}\|_{L^2(U_{k+1};\mathbb{M}^n)} \\
&\quad \le \|\boldsymbol{\nabla\Phi}_k - \boldsymbol{Q}_k \boldsymbol{\nabla\Theta}_k\|_{L^2(U_k;\mathbb{M}^n)} + \|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{Q}_k \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|_{L^2(V_{k+1};\mathbb{M}^n)} \\
&\quad \le \widehat{C}_k(\boldsymbol{\Theta})\widetilde{C}_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}_k^T \boldsymbol{\nabla\Phi}_k - \boldsymbol{\nabla\Theta}_k^T \boldsymbol{\nabla\Theta}_k\|^{1/2}_{L^1(U_k;\mathbb{S}^n)} \\
&\quad + \widehat{C}'_k(\boldsymbol{\Theta})\widetilde{C}'_k(\boldsymbol{\Theta})\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{k+1}\|^{1/2}_{L^1(V_{k+1};\mathbb{S}^n)}.
\end{aligned}$$

Hence the constant $\widehat{C}_{k+1}(\boldsymbol{\Theta}) := (\widehat{C}_k(\boldsymbol{\Theta})\widetilde{C}_k(\boldsymbol{\Theta}) + \widehat{C}'_k(\boldsymbol{\Theta})\widetilde{C}'_k(\boldsymbol{\Theta}))$ and the rotation $\boldsymbol{R}_{k+1} = \boldsymbol{R}_{k+1}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) := \boldsymbol{Q}_k(\boldsymbol{\Phi}|_{\omega_k}, \boldsymbol{\Theta}|_{\omega_k}) \in \mathbb{O}^n_+$ satisfy

$$\begin{aligned}
&\|\boldsymbol{\nabla\Phi}_{k+1} - \boldsymbol{R}_{k+1} \boldsymbol{\nabla\Theta}_{k+1}\|_{L^2(U_{k+1};\mathbb{M}^n)} \\
&\qquad\qquad \le \widehat{C}_{k+1}(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}_{k+1}^T \boldsymbol{\nabla\Phi}_{k+1} - \boldsymbol{\nabla\Theta}_{k+1}^T \boldsymbol{\nabla\Theta}_{k+1}\|^{1/2}_{L^1(U_{k+1};\mathbb{S}^n)}.
\end{aligned}$$

A recursion argument thus shows that the announced property holds with

$$c(\boldsymbol{\Theta}) := \widehat{C}_N(\boldsymbol{\Theta}) \text{ and } \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) := \boldsymbol{R}_N(\boldsymbol{\Phi}, \boldsymbol{\Theta}).$$

□

The next lemma concludes the proof of Theorem 1.

Lemma 7. *Let the assumptions on the set Ω and the mapping $\boldsymbol{\Theta}$ be as in Lemma 6. Then there exists a constant $C(\boldsymbol{\Theta})$ with the following property: Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Phi} > 0$ a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R\Theta})\|_{H^1(\Omega;\mathbb{R}^n)} \le C(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T \boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. Let there be given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Phi} > 0$ a.e. in Ω. By Lemma 6, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\Phi} - \boldsymbol{R\nabla\Theta}\|_{L^2(\Omega;\mathbb{M}^n)} \le c(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T \boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Let the vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ be defined by

$$\boldsymbol{b} := \left(\int_\Omega dx\right)^{-1} \int_\Omega (\boldsymbol{\Phi} - \boldsymbol{R\Theta}) dx.$$

By the generalized Poincaré inequality, there exists a constant d such that, for all $\boldsymbol{\Psi} \in H^1(\Omega; \mathbb{R}^n)$,

$$\|\boldsymbol{\Psi}\|_{H^1(\Omega;\mathbb{R}^n)} \le d \left(\|\boldsymbol{\nabla\Psi}\|_{L^2(\Omega;\mathbb{M}^n)} + \left| \int_\Omega \boldsymbol{\Psi} dx \right| \right).$$

Applying this inequality to the mapping $\boldsymbol{\Psi} := \boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R\Theta})$ yields the desired conclusion, with $C(\boldsymbol{\Theta}) := dc(\boldsymbol{\Theta})$. □

4. The key inequality revisited

Define the set

$$H^1_+(\Omega;\mathbb{R}^n) := \{\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n); \det \boldsymbol{\nabla\Phi} > 0 \text{ a.e. in } \Omega\}$$

and the quotient set

$$\dot{H}^1_+(\Omega;\mathbb{R}^n) := H^1_+(\Omega;\mathbb{R}^n)/\mathcal{R},$$

where $(\boldsymbol{\Phi},\boldsymbol{\Theta}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{b} \in \mathbb{R}^n$ and a rotation $\boldsymbol{R} \in \mathbb{O}^n_+$ such that

$$\boldsymbol{\Phi}(x) = \boldsymbol{b} + \boldsymbol{R\Theta}(x) \text{ for almost all } x \in \Omega.$$

The equivalence class of $\boldsymbol{\Theta} \in H^1_+(\Omega;\mathbb{R}^n)$ modulo $\mathcal{R}$ will be denoted $\dot{\boldsymbol{\Theta}}$.

Two mappings $\boldsymbol{\Phi} \in H^1_+(\Omega;\mathbb{R}^n)$ and $\boldsymbol{\Theta} \in H^1_+(\Omega;\mathbb{R}^n)$ satisfying $(\boldsymbol{\Phi},\boldsymbol{\Theta}) \in \mathcal{R}$ are thus *isometrically equivalent*, according to the definition given in the Introduction. Note that, while isometrically equivalent mappings clearly share the same Cauchy-Green tensor, the converse does *not* hold for mappings in the space $H^1_+(\Omega;\mathbb{R}^n)$. This was pointed out to us by Hervé Le Dret and one of the referees, who provided similar counter-exemples (consider, e.g., $\Omega = \{x = (x_1,x_2) \in \mathbb{R}^2; |x| < 1\}$, $\boldsymbol{\Phi}(x) = (x_1x_2^2, x_2)$, and $\boldsymbol{\Theta}(x) = \boldsymbol{\Phi}(x)$ if $x_2 \geq 0$ and $\boldsymbol{\Theta}(x) = -\boldsymbol{\Phi}(x)$ if $x_2 < 0$). The converse *does* hold, however, if one of the mappings is in $\mathcal{C}^1(\Omega;\mathbb{R}^n)$; see Ciarlet & Mardare [8, Theorem 2.1].

The question thus remains to find minimal assumptions on mappings from Ω into $\mathbb{R}^n$ guaranteeing that they are isometrically equivalent if they share the same Cauchy-Green tensor. In this respect, the set of orientation-preserving mappings in $\mathcal{C}^1(\Omega;\mathbb{R}^n)$ is "too small", while the set $H^1_+(\Omega;\mathbb{R}^n)$ is "too big".

Thanks to the invariance of the norm $\|\cdot\|_{H^1(\Omega;\mathbb{R}^n)}$ under rotations, it is easily verified that the mapping $d_{1,\Omega} : \dot{H}^1_+(\Omega;\mathbb{R}^n) \times \dot{H}^1_+(\Omega;\mathbb{R}^n) \to \mathbb{R}^n_+$ defined by

$$d_{1,\Omega}(\dot{\boldsymbol{\Phi}},\dot{\boldsymbol{\Theta}}) := \inf_{\begin{cases} \boldsymbol{b} \in \mathbb{R}^n \\ \boldsymbol{R} \in \mathbb{O}^n_+ \end{cases}} \|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R\Theta})\|_{H^1_+(\Omega;\mathbb{R}^n)}$$

is a distance on the quotient set $\dot{H}^1_+(\Omega;\mathbb{R}^n)$. The key inequality of Theorem 1 can then be recast as one involving distances in metric spaces (one of them a Banach space).

Theorem 2. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$, there exists a constant $C(\boldsymbol{\Theta})$ with the following property: For any equivalence class $\dot{\boldsymbol{\Phi}} \in \dot{H}^1(\Omega;\mathbb{R}^n)$,*

$$d_{1,\Omega}(\dot{\boldsymbol{\Phi}},\dot{\boldsymbol{\Theta}}) \leq C(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

□

Naturally, the sequential continuity property mentionned in the Introduction can also be recoved from the inequality of Theorem 2.

5. Concluding remarks

As stated in the introduction, the continuity result established in this paper has potential applications to differential geometry and to three-dimensional nonlinear elasticity. From the viewpoint of differential geometry, this result is a mathematical expression of a natural idea: If the metrics of two manifolds in $\mathbb{R}^n$ are close, then the two manifolds are also close (up to isometries, of course). While the previous results in this direction involved "simpler" topologies, viz., those of spaces of continuously differentiable mappings (see [7] and [9]), the present one can be considered as a genuine improvement over these, inasmuch as the norm for the Cauchy-Green tensor fields is in a sense much "weaker". By contrast with the former ones, however, the latter requires the seemingly unavoidable assumptions that both mappings be orientation-preserving and that one of them be sufficiently regular.

From the viewpoint of three-dimensional nonlinear elasticity, the present result represents a first step toward considering the Cauchy-Green tensor field as the primary unknown, even though much further work is clearly needed until a full-fledged theory can be developed in this spirit. Nevertheless, the inequalities of Theorems 1 and 2 can be considered as genuine, and new to the best of our knowledge, nonlinear Korn inequalities. Similar inequlities have been indeed already established, but in the special case where one of the mapping is the identity; see John [15,16] and Kohn [17]. By contrast with the linear case, however, proving that such an inequality holds "at the identity" evidently does not imply that it holds elsewhere (to establish such an implication was precisely the purpose of Lemmas 3 to 7).

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