Mathematical Models and Methods in Applied Sciences

LINEARIZED ELASTICITY AND KORN'S INEQUALITY REVISITED

PHILIPPE G. CIARLET*
Department of Mathematics, City University of Hong Kong,
83 Tat Chee Avenue, Kowloon, Hong Kong
mapgc@cityu.edu.hk

PATRICK CIARLET, JR.
Ecole Nationale Supérieure de Techniques Avancées
32 Boulevard Victor, 75015 Paris, France
ciarlet@ensta.fr

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This article is dedicated to the memory of Michèle Miollany

We describe and analyze an approach to the pure traction problem of three-dimensional linearized elasticity, whose novelty consists in considering the linearized strain tensor as the "primary" unknown, instead of the displacement itself as is customary. This approach leads to a well-posed minimization problem, constrained by a weak form of the St Venant compatibility conditions. Interestingly, it also provides a new proof of Korn's inequality.

Keywords: Three-dimensional linearized elasticity; St Venant compatibility conditions; Korn's inequality.

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1. Introduction

The notations used in this introduction are defined in the next section.

Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Consider a homogeneous, isotropic, linearly elastic body with Lamé constants $\lambda > 0$ and $\mu > 0$, with $\overline{\Omega}$ as its reference configuration, and subjected to applied body forces of density $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ in its interior.

Then the weak formulation of the associated *pure traction problem of linearized elasticity* classically consists in finding a displacement vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ that

*Corresponding author

satisfies the variational equations

$$\int_\Omega \{\lambda\, tr\, \boldsymbol{e}(\boldsymbol{u})\, tr\, \boldsymbol{e}(\boldsymbol{v}) + 2\mu \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{v})\} dx = L(\boldsymbol{v})$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, where $L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx$ and

$$\boldsymbol{e}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v}) \in \boldsymbol{L}^2_{sym}(\Omega)$$

denotes the *linearized strain tensor* field associated with any vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

Clearly, the above variational equations can have solutions only if the applied body forces satisfy the *compatibility condition* $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{r}\, dx = 0$ for all $\boldsymbol{r} \in \boldsymbol{H}^1(\Omega)$ satisfying $\boldsymbol{e}(\boldsymbol{r}) = \boldsymbol{0}$ in Ω, such vector fields $\boldsymbol{r}$ being called *infinitesimal rigid displacements of the set* Ω.

It is well-known that this compatibility condition is also sufficient for the existence of solutions to the above variational equations, as a consequence of *Korn's inequality*. Besides, such solutions are unique up to the addition of any vector field $\boldsymbol{r} \in \boldsymbol{H}^1(\Omega)$ satisfying $\boldsymbol{e}(\boldsymbol{r}) = \boldsymbol{0}$ in Ω. In other words, if we let

$$\boldsymbol{R}(\Omega) = \{\boldsymbol{r} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{e}(\boldsymbol{r}) = \boldsymbol{0} \text{ in } \Omega\} = \{\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}_\Omega;\ \boldsymbol{a} \in \mathbb{R}^3, \boldsymbol{b} \in \mathbb{R}^3\}$$

denote the space of infinitesimal rigid displacements of the set Ω, there exists a unique equivalence class $\dot{\boldsymbol{u}}$ in the quotient space $\dot{\boldsymbol{H}}^1(\Omega) = \boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega)$ that satisfies the above variational equations, or equivalently such that

$$J(\dot{\boldsymbol{u}}) = \inf_{\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)} J(\dot{\boldsymbol{v}}),$$

where

$$J(\dot{\boldsymbol{v}}) = \frac{1}{2} \int_\Omega \{\lambda\, tr\, \boldsymbol{e}(\dot{\boldsymbol{v}})\, tr\, \boldsymbol{e}(\dot{\boldsymbol{v}}) + 2\mu \boldsymbol{e}(\dot{\boldsymbol{v}}) : \boldsymbol{e}(\dot{\boldsymbol{v}})\} dx - L(\dot{\boldsymbol{v}}).$$

For the sake of later comparison with our approach (see the discussion in Section 6), we begin by briefly reviewing this classical existence theory in Section 2.

The objective of this paper is to describe and analyze another (and new to the best of our knowledge) approach to the above pure traction problem that consists in *considering the linearized strain tensor as the "primary" unknown instead of the displacement itself.*

To this end, we first characterize in Section 3 those symmetric 3×3 matrix fields $\boldsymbol{e} \in \boldsymbol{L}^2_{sym}(\Omega)$ that can be written as $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v})$ for some vector fields $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, again uniquely defined up to infinitesimal rigid displacements. As shown in Theorem 3.2, this is possible if (and only if) the components e_{ij} of the field $\boldsymbol{e}$ satisfy the following *weak form of the classical St Venant compatibility conditions*:

$$\partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega) \text{ for all } i, j, k, l \in \{1, 2, 3\}.$$

The proof crucially hinges on an H^{-2}*-version of a classical theorem of Poincaré* (Theorem 3.1). We prove this result here under the simplifying assumption that the set Ω is simply-connected. Note, however, that this assumption can in fact be disposed of with some extra care; see Ref.[9]. Note that another, *albeit* of a completely different nature, characterization of such matrix fields $\boldsymbol{e} = (e_{ij})$ has also been given by Ting[16] (see Remark 3.2).

Let $\boldsymbol{E}(\Omega)$ denote the closed subspace of $\boldsymbol{L}^2_{sym}(\Omega)$ formed by the 3×3 matrix fields that satisfy the above weak St Venant compatibility conditions. We then show that *the mapping* $\boldsymbol{\mathcal{F}} : \boldsymbol{e} \in \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$*, where* $\dot{\boldsymbol{v}}$ *is such that* $\boldsymbol{e}(\dot{\boldsymbol{v}}) = \boldsymbol{e}$*, is an isomorphism between the Hilbert spaces* $\boldsymbol{E}(\Omega)$ *and* $\dot{\boldsymbol{H}}^1(\Omega)$, a property that in turn yields a *new proof of Korn's inequality*; see Theorem 4.1 and its Corollary.

We conclude this analysis by showing (Theorem 5.1) that, thanks to the isomorphism $\boldsymbol{\mathcal{F}}$, the pure traction problem of linearized elasticity may be equivalently posed in terms of the new unknown $\boldsymbol{e} \in \boldsymbol{L}^2_{sym}(\Omega)$ as a *constrained minimization problem.* More specifically, we now seek a 3×3 *matrix field* $\boldsymbol{\varepsilon} \in \boldsymbol{E}(\Omega)$ *that satisfies*

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \boldsymbol{E}(\Omega)} j(\boldsymbol{e}),$$

where

$$j(\boldsymbol{e}) = \frac{1}{2} \int_\Omega \{\lambda \operatorname{tr} \boldsymbol{e} \operatorname{tr} \boldsymbol{e} + 2\mu \boldsymbol{e} : \boldsymbol{e}\} dx - \Lambda(\boldsymbol{e}),$$

and the continuous linear form $\Lambda : \boldsymbol{E}(\Omega) \to \mathbb{R}$ is defined by $\Lambda = L \circ \boldsymbol{\mathcal{F}}$.

The results of this paper have been announced in Ref [4].

2. The classical approach to existence theory in linearized elasticity

To begin with, we list some notation and conventions that will be used throughout the article. Except when they are used for indexing sequences, Latin indices range over the set $\{1, 2, 3\}$ and the summation convention with respect to repeated indices is used in conjunction with this rule. The Euclidean and exterior products of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ are denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$. The matrix inner product of two 3×3 matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{e}$ is denoted $\boldsymbol{\varepsilon} : \boldsymbol{e} = \operatorname{tr} \boldsymbol{\varepsilon}^T \boldsymbol{e}$. The identity mapping of a set X is denoted $\boldsymbol{id}_X$. The restriction of a mapping f to a set X is denoted $f|_X$.

Given an open subset Ω of $\mathbb{R}^3$, spaces of vector-valued or matrix-valued functions or distributions defined on Ω are denoted by boldface letters. The norm in the space $L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$ is denoted $||\cdot||_{0,\Omega}$ and that in the space $H^1(\Omega)$ or $\boldsymbol{H}^1(\Omega)$ is denoted $||\cdot||_{1,\Omega}$. If V is a vector space and R a subspace of V, the quotient space of V modulo R is denoted V/R and the equivalence class of $v \in V$ modulo R is denoted $\dot{v}$.

Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$, let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$. Given a vector field $\boldsymbol{v} = (v_i)$, the 3×3 matrix with $\partial_j v_i$ as its element at the i-th row and j-th column is denoted $\boldsymbol{\nabla v}$.

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^3$ whose boundary Γ is Lipschitz-continuous in the sense of Nečas[13] or Adams[1]. Assume that the set $\overline{\Omega}$ is the *reference configuration* occupied by a *linearly elastic body* in the absence of applied forces. The elastic material constituting the body, which may be nonhomogeneous and isotropic, is thus characterized by its *elasticity tensor* $\boldsymbol{A} = (A_{ijkl}) \in \boldsymbol{L}^\infty(\Omega)$, whose elements possess the symmetries $A_{ijkl} = A_{jikl} = A_{klij}$, and which is uniformly positive-definite a.e. in Ω, in the sense that there exists a constant $\alpha > 0$ such that

$$\boldsymbol{A}(x)\boldsymbol{t} : \boldsymbol{t} \geq \alpha \boldsymbol{t} : \boldsymbol{t}$$

for almost all $x \in \Omega$ and all 3×3 symmetric matrices $\boldsymbol{t} = (t_{ij})$, where $(\boldsymbol{A}(x)\boldsymbol{t})_{ij} := A_{ijkl}(x)t_{kl}$.

Remark 2.1. If the body is homogeneous and isotropic (as that considered in the Introduction), the elements of the elasticity tensor are given by

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are the Lamé constants of the constituting material. □

The body is assumed to be subjected to *applied body forces* in its interior with density $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ and to *applied surface forces* on its boundary with density $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma)$. The assumed regularity on the vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ thus ensure that the linear form $L : \boldsymbol{H}^1(\Omega) \to \mathbb{R}$ defined by

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}\, d\Gamma \text{ for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$$

is continuous.

Then it is well known that the *unknown displacement field* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfies the following variational equations, which constitute the weak formulation of the *pure traction problem of linearized elasticity*:

$$\int_\Omega \boldsymbol{A}\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{v})dx = L(\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),$$

where

$$\boldsymbol{e}(\boldsymbol{v}) := \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{v}^T + \boldsymbol{\nabla}\boldsymbol{v}) = (\frac{1}{2}(\partial_i v_j + \partial_j v_i)) \in \boldsymbol{L}^2_{sym}(\Omega)$$

denotes the *linearized strain tensor field* associated with an arbitrary vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, and

$$\boldsymbol{L}^2_{sym}(\Omega) := \{\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2(\Omega);\ e_{ij} = e_{ji} \quad \text{in } \Omega\}.$$

Let

$$\boldsymbol{R}(\Omega) := \{\boldsymbol{r} \in \boldsymbol{H}^1(\Omega); \boldsymbol{e}(\boldsymbol{r}) = 0 \text{ in } \Omega\} = \{\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}_\Omega;\ \boldsymbol{a} \in \mathbb{R}^3, \boldsymbol{b} \in \mathbb{R}^3\}$$

denote the *space of infinitesimal rigid displacements of the set* Ω. The applied forces are also assumed to be such that the associated linear form L satisfies the (clearly necessary) relation $L(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$. Hence the above variational problem amounts to finding $\dot{\boldsymbol{u}} \in \dot{\boldsymbol{H}}^1(\Omega) := \boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega)$ such that

$$\int_\Omega \boldsymbol{A}\boldsymbol{e}(\dot{\boldsymbol{u}}) : \boldsymbol{e}(\dot{\boldsymbol{v}})dx = L(\dot{\boldsymbol{v}}) \quad \text{for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega),$$

where $\boldsymbol{e}(\dot{\boldsymbol{v}}) := \boldsymbol{e}(\boldsymbol{v})$ and $L(\dot{\boldsymbol{v}}) := L(\boldsymbol{v})$ for all $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$. In order to apply the Lax-Milgram lemma, it thus remains to show that the mapping $\dot{\boldsymbol{v}} \to ||\boldsymbol{e}(\dot{\boldsymbol{v}})||_{0,\Omega}$ is a norm over the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ equivalent to the quotient norm, which is defined by

$$||\dot{\boldsymbol{v}}||_{1,\Omega} := \inf_{\boldsymbol{r} \in \boldsymbol{R}(\Omega)} ||\boldsymbol{v} + \boldsymbol{r}||_{1,\Omega} \text{ for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega).$$

The proof comprises two stages, whose proofs are well known. We nevertheless record these here (see Theorems 2.1 and 2.2) so that they can be fruitfully compared with those found in the present approach.

The first stage consists in establishing the classical *Korn inequality in the space* $\boldsymbol{H}^1(\Omega)$:

Theorem 2.1. *There exists a constant C such that*

$$||\boldsymbol{v}||_{1,\Omega} \leq C\{||\boldsymbol{v}||^2_{0,\Omega} + ||\boldsymbol{e}(\boldsymbol{v})||^2_{0,\Omega}\}^{1/2} \text{ for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

Proof. As shown in Theorem 3.2, Chapter 3 of Duvaut & Lions[10], the essence of this remarkable inequality is that the *two Hilbert spaces $\boldsymbol{H}^1(\Omega)$ and*

$$\boldsymbol{K}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \boldsymbol{e}(\boldsymbol{v}) \in \boldsymbol{L}^2_{sym}(\Omega)\}$$

coincide. The Korn inequality in $\boldsymbol{H}^1(\Omega)$ then becomes an immediate consequence of the *closed graph theorem* applied to the identity mapping from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{K}(\Omega)$, which is thus surjective and otherwise clearly continuous. To show that $\boldsymbol{K}(\Omega) \subset \boldsymbol{H}^1(\Omega)$ (the other inclusion evidently holds), let $\boldsymbol{v} = (v_i) \in \boldsymbol{K}(\Omega)$. Then

$$\partial_k v_i \in H^{-1}(\Omega) \text{ and } \partial_{jk} v_i = \partial_j e_{ik}(\boldsymbol{v}) + \partial_k e_{ij}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v}) \in H^{-1}(\Omega).$$

Hence $\partial_k v_i \in L^2(\Omega)$ since a fundamental *lemma of J. L. Lions* asserts that, if a distribution $v \in H^{-1}(\Omega)$ is such that $\partial_j v \in H^{-1}(\Omega)$, then $v \in L^2(\Omega)$ (this implication was first proved - though remained unpublished until its appearance in Theorem 3.2, Chapter 3 of Duvaut & Lions[10] $-$ by J. L. Lions ca. 1958 for domains with smooth boundaries; it was later extended and generalized to Lipschitz-continuous boundaries by various authors, the "last word" in this respect being seemingly due to Amrouche & Girault[2]). □

The second stage consists in establishing the (equally classical) *Korn inequality in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$:

Theorem 2.2. *There exists a constant $\dot{C}$ such that*

$$||\dot{\boldsymbol{v}}||_{1,\Omega} \leq \dot{C}||\boldsymbol{e}(\dot{\boldsymbol{v}})||_{0,\Omega} \text{ for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega).$$

Proof. By the Hahn-Banach theorem, there exist six continuous linear forms l_α on $\boldsymbol{H}^1(\Omega), 1 \leq \alpha \leq 6$, with the following property: An element $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$ is equal to $\boldsymbol{0}$ if and only if $l_\alpha(\boldsymbol{r}) = 0, 1 \leq \alpha \leq 6$. We then claim that there exists a constant D such that

$$||\boldsymbol{v}||_{1,\Omega} \leq D(||\boldsymbol{e}(\boldsymbol{v})||_{0,\Omega} + \sum_{\alpha=1}^{6} |l_\alpha(\boldsymbol{v})|) \text{ for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

This inequality in turn implies Korn's inequality in $\dot{\boldsymbol{H}}^1(\Omega)$: Given any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, let $\boldsymbol{r}(\boldsymbol{v}) \in \boldsymbol{R}(\Omega)$ be such that $l_\alpha(\boldsymbol{v} + \boldsymbol{r}(\boldsymbol{v})) = 0, 1 \leq \alpha \leq 6$; then

$$||\dot{\boldsymbol{v}}||_{1,\Omega} = \inf_{\boldsymbol{r} \in \boldsymbol{R}(\Omega)} ||\boldsymbol{v} + \boldsymbol{r}||_{1,\Omega} \leq ||\boldsymbol{v} + \boldsymbol{r}(\boldsymbol{v})||_{1,\Omega} \leq D||\boldsymbol{e}(\boldsymbol{v})||_{0,\Omega} = D||\boldsymbol{e}(\dot{\boldsymbol{v}})||_{0,\Omega}.$$

To establish the existence of such a constant D, assume the contrary. Then there exist $\boldsymbol{v}^k \in \boldsymbol{H}^1(\Omega), k \geq 1$, such that

$$||\boldsymbol{v}^k||_{1,\Omega} = 1 \text{ for all } k \geq 1 \text{ and } (||\boldsymbol{e}(\boldsymbol{v}^k)||_{0,\Omega} + \sum_{\alpha=1}^{6} |l_\alpha(\boldsymbol{v}^k)|) \underset{k\to\infty}{\longrightarrow} 0.$$

By Rellich theorem, there exists a subsequence $(\boldsymbol{v}^l)_{l=1}^\infty$ that converges in $\boldsymbol{L}^2(\Omega)$. Since the sequence $(\boldsymbol{e}(\boldsymbol{v}^l))_{l=1}^\infty$ also converges in $\boldsymbol{L}^2(\Omega)$, the subsequence $(\boldsymbol{v}^l)_{l=1}^\infty$ is a Cauchy sequence with respect to the norm $\boldsymbol{v} \to \{||\boldsymbol{v}||_{0,\Omega}^2 + ||\boldsymbol{e}(\boldsymbol{v})||_{0,\Omega}^2\}^{1/2}$, hence also with respect to the norm $||\cdot||_{1,\Omega}$ by *Korn's inequality in* $\boldsymbol{H}^1(\Omega)$ (Theorem 2.1). Consequently, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $||\boldsymbol{v}^l - \boldsymbol{v}||_{1,\Omega} \underset{l\to\infty}{\longrightarrow} 0$. But then $\boldsymbol{v} = \boldsymbol{0}$ since $\boldsymbol{e}(\boldsymbol{v}) = 0$ and $l_\alpha(\boldsymbol{v}) = 0, 1 \leq \alpha \leq 6$, in contradiction with the relations $||\boldsymbol{v}^l||_{1,\Omega} = 1$ for all $l \geq 1$. □

Remark 2.2 Another proof of Theorem 2.2 is found in Theorem 3.4, Chapter 3 of Duvant & Lions[10]. □

Interestingly, our subsequent analysis will provide "as a by-product" an essentially different proof of Korn inequalities in both spaces $\boldsymbol{H}^1(\Omega)$ and $\dot{\boldsymbol{H}}^1(\Omega)$ (see the corollary to Theorem 4.1).

3. Weak versions of a classical theorem of Poincaré and of St Venant compatibility conditions

A classical *theorem of Poincaré* (see, e.g., page 235 in Schwartz[14]) asserts that, if functions $h_k \in \mathcal{C}^1(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in a simply-connected open subset Ω of $\mathbb{R}^3$ (or $\mathbb{R}^n$ for that matter), then there exists a function $p \in \mathcal{C}^2(\Omega)$ such that $h_k = \partial_k p$ in Ω. This theorem was extended by Girault & Raviart[12] (see Theorem

2.9 in Chapter 1), who showed that, if functions $h_k \in L^2(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-1}(\Omega)$ on a bounded, connected and simply-connected open subset Ω of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, then there exists $p \in H^1(\Omega)$ such that $h_k = \partial_k p$ in $L^2(\Omega)$. We now carry out this extension one step further.

Theorem 3.1. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Let $h_k \in H^{-1}(\Omega)$ be distributions that satisfy*

$$\partial_l h_k = \partial_k h_l \text{ in } H^{-2}(\Omega).$$

Then there exists a function $p \in L^2(\Omega)$, unique up to an additive constant, such that

$$h_k = \partial_k p \text{ in } H^{-1}(\Omega).$$

Proof. In this proof, we use the following notations: For any $p \in \mathcal{D}'(\Omega)$, we let $\textbf{grad}\ p := (\partial_i p) \in \boldsymbol{\mathcal{D}}'(\Omega)$, for any $\boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega)$, we let $\textbf{curl}\ \boldsymbol{v} := (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1) \in \boldsymbol{\mathcal{D}}'(\Omega)$, and we let ${}_{X'} < \cdot, \cdot >_X$ denote the duality pairing between a topological vector space X and its dual X'. We have to show that, if $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$ satisfies $\textbf{curl}\ \boldsymbol{h} = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, then there exists $p \in L^2(\Omega)$ such that $\boldsymbol{h} = \textbf{grad}\ p$. To this end, we proceed in three stages:

(i) Given *any* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, Theorem 5.1, Chapter 1 of Girault & Raviart[12] shows that there exist $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in L^2(\Omega)$ such that (the assumptions that Ω is bounded and has a Lipschitz-continuous boundary are used here):

$$-\Delta \boldsymbol{u} + \textbf{grad}\ p = \boldsymbol{h} \text{ in } \boldsymbol{H}^{-1}(\Omega) \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega.$$

Our proof will then consist in showing that, if *in addition* $\textbf{curl}\ \boldsymbol{h} = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, then $\boldsymbol{u} = \boldsymbol{0}$. In this direction, we first infer from the relation $\textbf{curl}\ \textbf{curl}\ \boldsymbol{u} = -\Delta \boldsymbol{u} + \textbf{grad}\ \operatorname{div} \boldsymbol{u}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$ that $\boldsymbol{h} = \textbf{curl}\ \textbf{curl}\ \boldsymbol{u} + \textbf{grad}\ p$, hence that

$$\textbf{curl}\ \textbf{curl}\ \textbf{curl}\ \boldsymbol{u} = \textbf{curl}\ \boldsymbol{h} - \textbf{curl}\ \textbf{grad}\ p = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\Omega)$$

if $\textbf{curl}\ \boldsymbol{h} = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, since $\textbf{curl}\ \textbf{grad}\ p = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$.

(ii) We next show that, *if a vector field* $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ *satisfies* $\operatorname{div} \boldsymbol{u} = 0$ in Ω, *then there exists a vector field* $\boldsymbol{v} \in \boldsymbol{H}_0^2(\Omega)$ *such that*

$$\boldsymbol{u} = \textbf{curl}\ \boldsymbol{v} \in \Omega.$$

To this end, let $\widetilde{\boldsymbol{u}}$ denote the extension of $\boldsymbol{u}$ by $\boldsymbol{0}$ in $\mathbb{R}^3 - \Omega$, which thus satisfies $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}^1(\mathbb{R}^3)$ and $\operatorname{div} \widetilde{\boldsymbol{u}} = 0$ in $\mathbb{R}^3$. By Theorem 3.2 in Girault[11], there thus exists an open ball $\widetilde{B}$ containing $\overline{\Omega}$ and a vector field $\widetilde{\boldsymbol{w}} \in \boldsymbol{H}^2(\widetilde{B})$ such that $\widetilde{\boldsymbol{u}} = \textbf{curl}\ \widetilde{\boldsymbol{w}}$ in $\widetilde{B}$.

The open set $\Omega' := \widetilde{B} - \overline{\Omega}$ is bounded, has a Lipschitz-continuous boundary, and is simply-connected since Ω is simply-connected by assumption (this is the only place where this assumption is needed). Furthermore, the vector field $\boldsymbol{w}' := \widetilde{\boldsymbol{w}}|_{\Omega'} \in$

$\boldsymbol{H}^2(\Omega')$ satisfies $\mathbf{curl}\ \boldsymbol{w}' = \mathbf{0}$ in Ω'. Hence Theorem 2.9, Chapter 1, of Girault & Raviart[12] shows that there exists a function $\phi' \in H^1(\Omega')$ such that $\boldsymbol{w}' = \mathbf{grad}\ \phi'$ in Ω', so that in fact $\phi' \in H^3(\Omega')$.

Since the function $\phi' \in H^3(\Omega')$ can be extended to a function in $H^3(\mathbb{R}^3)$, there exists a function $\widetilde{\phi} \in H^3(\widetilde{B})$ such that $\phi' = \widetilde{\phi}|_{\Omega'}$. The vector field $\widetilde{\boldsymbol{v}} := \widetilde{\boldsymbol{w}} -$ $\mathbf{grad}$ $\widetilde{\phi} \in \boldsymbol{H}^2(\widetilde{B})$ thus satisfies

$$\widetilde{\boldsymbol{v}}|_{\Omega'} = \boldsymbol{w}' - \mathbf{grad}\ \phi' = \mathbf{0} \text{ in } \Omega'.$$

Hence the vector field $\boldsymbol{v} := \widetilde{\boldsymbol{v}}|_\Omega$ is in the space $\boldsymbol{H}_0^2(\Omega)$. Besides,

$$\mathbf{curl}\ \widetilde{\boldsymbol{v}} = \mathbf{curl}\ \widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{u}} \text{ in } \widetilde{B},$$

since $\mathbf{curl\ grad}\ \widetilde{\phi} = \mathbf{0}$ in $\mathcal{D}'(\widetilde{B})$. Consequently, $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{v}$ in Ω as desired.

(iii) Since $\mathbf{curl\ curl\ curl}\ \boldsymbol{u} = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\Omega)$ by (i) and $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{v}$ in Ω with $\boldsymbol{v} \in \boldsymbol{H}_0^2(\Omega)$ by (ii), we conclude that

$$\begin{aligned} 0 &= {}_{\boldsymbol{H}^{-2}(\Omega)} < \mathbf{curl\ curl\ curl}\ \boldsymbol{u}, \boldsymbol{v} >_{\boldsymbol{H}_0^2(\Omega)} = {}_{\boldsymbol{H}^{-1}(\Omega)} < \mathbf{curl\ curl}\ \boldsymbol{u},\ \mathbf{curl}\ \boldsymbol{v} >_{\boldsymbol{H}_0^1(\Omega)} \\ &= {}_{\boldsymbol{H}^{-1}(\Omega)} < \mathbf{curl\ curl}\ \boldsymbol{u}, \boldsymbol{u} >_{\boldsymbol{H}_0^1(\Omega)} = {}_{\boldsymbol{L}^2(\Omega)} < \mathbf{curl}\ \boldsymbol{u}, \mathbf{curl}\ \boldsymbol{u} >_{\boldsymbol{L}^2(\Omega)} = ||\mathbf{curl}\ \boldsymbol{u}||_{0,\Omega}^2. \end{aligned}$$

Noting that

$$||\text{div}\ \boldsymbol{v}||_{0,\Omega}^2 + ||\mathbf{curl}\ \boldsymbol{v}||_{0,\Omega}^2 = \int_\Omega \sum_{i,j} |\partial_j v_i|^2 dx \text{ for any } \boldsymbol{v} = (v_i) \in \boldsymbol{H}_0^1(\Omega),$$

we thus conclude that $\boldsymbol{u} = \mathbf{0}$ in Ω since $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, div $\boldsymbol{u} = 0$ in Ω by (i), and $\mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$ in Ω as shown above. □

In 1864, A. J. C. B. de Saint Venant showed that, if functions $e_{ij} = e_{ji} \in \mathcal{C}^3(\Omega)$ satisfy in Ω ad hoc *compatibility relations* that since then bear his name, then there exists a vector field $(v_i) \in \mathcal{C}^4(\Omega)$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ in Ω. Thanks to Theorem 3.1, we are now in a position to establish that these *St Venant compatibility relations* are also sufficient conditions *in the sense of distributions*, according to the following result:

Theorem 3.2. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Let $\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}_{sym}^2(\Omega)$ be a symmetric matrix field that satisfies the following compatibility relations:*

$$\mathcal{R}_{ijkl}(\boldsymbol{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega).$$

Then there exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ such that

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j) \text{ in } L^2(\Omega),$$

and all other solutions $\widetilde{\boldsymbol{v}} = (\widetilde{v}_i) \in \boldsymbol{H}^1(\Omega)$ of the equations $e_{ij} = \frac{1}{2}(\partial_j \widetilde{v}_i + \partial_i \widetilde{v}_j)$ are of the form $\widetilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}$, with $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$.

Proof. The compatibility relations $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$ may be equivalently rewritten as

$$\partial_l h_{ijk} = \partial_k h_{ijl} \text{ in } H^{-2}(\Omega) \text{ with } h_{ijk} := \partial_j e_{ik} - \partial_i e_{jk} \in H^{-1}(\Omega).$$

Hence Theorem 3.1 shows that there exist functions $p_{ij} \in L^2(\Omega)$, unique up to additive constants, such that

$$\partial_k p_{ij} = h_{ijk} = \partial_j e_{ik} - \partial_i e_{jk} \text{ in } H^{-1}(\Omega).$$

Besides, since $\partial_k p_{ij} = -\partial_k p_{ji}$ in $H^{-1}(\Omega)$, we have the freedom of choosing the functions p_{ij} in such a way that $p_{ij} + p_{ji} = 0$ in $L^2(\Omega)$.

Noting that the functions $q_{ij} := (e_{ij} + p_{ij}) \in L^2(\Omega)$ satisfy

$$\begin{aligned}\partial_k q_{ij} &= \partial_k e_{ij} + \partial_k p_{ij} = \partial_k e_{ij} + \partial_j e_{ik} - \partial_i e_{jk}\\ &= \partial_j e_{ik} + \partial_j p_{ik} = \partial_j q_{ik} \text{ in } H^{-1}(\Omega),\end{aligned}$$

we again resort to Theorem 3.1 to assert the existence of functions $v_i \in H^1(\Omega)$, unique up to additive constants, such that

$$\partial_j v_i = q_{ij} = e_{ij} + p_{ij} \text{ in } L^2(\Omega).$$

Consequently,

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} + \frac{1}{2}(p_{ij} + p_{ji}) = e_{ij} \text{ in } L^2(\Omega),$$

as required. That all other solutions are of the indicated form follow from the well-known relation (for a proof, see, e.g., Theorem 6.3-4 in Ref.[3]):
$\{\boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega); \partial_j v_i + \partial_i v_j = 0 \text{ in } \mathcal{D}'(\Omega)\} = \{\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}_\Omega; \boldsymbol{a} \in \mathbb{R}^3, \boldsymbol{b} \in \mathbb{R}^3\}$. □

Remark 3.1 A different necessary and sufficient condition for a tensor $\boldsymbol{e} \in \boldsymbol{L}^2_{sym}(\Omega)$ to be of the form $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v})$ for some $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ has been given by Ting[16] in the following form: The tensor $\boldsymbol{e}$ should lie in the orthogonal complement in $\boldsymbol{L}^2_{sym}(\Omega)$ of the closure of the space spanned by all symmetric tensors $\boldsymbol{\sigma}$ that have all their components in $\mathcal{D}(\Omega)$ and that satisfy $\mathbf{div}\ \boldsymbol{\sigma} = \mathbf{0}$ in Ω. □

4. A basic isomorphism and a new proof of Korn's inequality

Let a symmetric matrix field $\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2_{sym}(\Omega)$ satisfy $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$, i.e., the weak form of St Venant's compatibility conditions considered in Theorem 3.2. We recall (see *ibid.*) that there then exists a unique equivalence class $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) = \boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{e}(\dot{\boldsymbol{v}})$ in $\boldsymbol{L}^2_{sym}(\Omega)$. We now show the mapping $\boldsymbol{\mathcal{F}} : \boldsymbol{e} \to \dot{\boldsymbol{v}}$ defined in this fashion has a remarkable property.

Theorem 4.1. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Define the space*

$$\boldsymbol{E}(\Omega) := \{\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2_{sym}(\Omega); \mathcal{R}_{ijkl}(\boldsymbol{e}) = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\Omega)\},$$

and let $\boldsymbol{\mathcal{F}} : \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ *be the linear mapping defined for each* $\boldsymbol{e} \in \boldsymbol{E}(\Omega)$ *by* $\boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \dot{\boldsymbol{v}}$, *where* $\dot{\boldsymbol{v}}$ *is the unique element in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$ *that satisfies* $\boldsymbol{e}(\dot{\boldsymbol{v}}) = \boldsymbol{e}$. *Then* $\boldsymbol{\mathcal{F}}$ *is an isomorphism between the Hilbert spaces* $\boldsymbol{E}(\Omega)$ *and* $\dot{\boldsymbol{H}}^1(\Omega)$.

Proof. Clearly, $\boldsymbol{E}(\Omega)$ is a Hilbert space as a closed subspace of $\boldsymbol{L}^2_{sym}(\Omega)$. The mapping $\boldsymbol{\mathcal{F}}$ is injective since $\boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \dot{\boldsymbol{0}}$ means that $\boldsymbol{e} = \boldsymbol{e}(\dot{\boldsymbol{0}}) = \boldsymbol{0}$ and surjective since, given any $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$, the matrix field $\boldsymbol{e}(\dot{\boldsymbol{v}}) \in \boldsymbol{L}^2_{sym}(\Omega)$ necessarily satisfies $\mathcal{R}_{ijkl}(\boldsymbol{e}(\dot{\boldsymbol{v}})) = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\Omega)$. Finally, the inverse mapping $\boldsymbol{\mathcal{F}}^{-1} : \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) \to \boldsymbol{e}(\dot{\boldsymbol{v}}) \in \boldsymbol{E}(\Omega)$ is continuous, since there evidently exists a constant c such that

$$||\boldsymbol{e}(\dot{\boldsymbol{v}})||_{0,\Omega} = ||\boldsymbol{e}(\boldsymbol{v}+\boldsymbol{r})||_{0,\Omega} \leq c||\boldsymbol{v}+\boldsymbol{r}||_{1,\Omega}$$

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and any $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$, so that

$$||\boldsymbol{e}(\dot{\boldsymbol{v}})||_{0,\Omega} \leq c \inf_{\boldsymbol{r}\in\boldsymbol{R}(\Omega)} ||\boldsymbol{v}+\boldsymbol{r}||_{1,\Omega} = c||\dot{\boldsymbol{v}}||_{1,\Omega}.$$

The conclusion thus follows from the *closed graph theorem*. □

Remarkably, the classical Korn's inequalitites of Section 2 can now be very simply recovered:

Corollary to Theorem 4.1 *That the mapping* $\boldsymbol{\mathcal{F}} : \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ *is an isomorphism implies Korn's inequalities in both spaces* $\boldsymbol{H}^1(\Omega)$ *and* $\dot{\boldsymbol{H}}^1(\Omega)$ (see Theorems 2.1 and 2.2).

Proof. (i) Since $\boldsymbol{\mathcal{F}}$ is an isomorphism, there exists a constant $\dot{C}$ such that

$$||\boldsymbol{\mathcal{F}}(\boldsymbol{e})||_{1,\Omega} \leq \dot{C}||\boldsymbol{e}||_{0,\Omega} \text{ for all } \boldsymbol{e} \in \boldsymbol{E}(\Omega),$$

or equivalently such that

$$||\dot{\boldsymbol{v}}||_{1,\Omega} \leq \dot{C}||\boldsymbol{e}(\dot{\boldsymbol{v}})||_{0,\Omega} \text{ for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega).$$

But this is exactly *Korn's inequality in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$, obtained by different means in Theorem 2.2.

(ii) We now show that *Korn's inequality in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$ *implies Korn's inequality in the space* $\boldsymbol{H}^1(\Omega)$ (Theorem 2.1).

Assume the contrary. Then there exist $\boldsymbol{v}^k \in \boldsymbol{H}^1(\Omega), k \geq 1$, such that

$$||\boldsymbol{v}^k||_{1,\Omega} = 1 \text{ for all } k \geq 1 \text{ and } (||\boldsymbol{v}^k||_{0,\Omega} + ||\boldsymbol{e}(\boldsymbol{v}^k)||_{0,\Omega}) \underset{k\to\infty}{\longrightarrow} 0.$$

Let $\boldsymbol{r}^k \in \boldsymbol{R}(\Omega)$ denote for each $k \geq 1$ the projection of $\boldsymbol{v}^k$ on $\boldsymbol{R}(\Omega)$ with respect to the inner-product of $\boldsymbol{H}^1(\Omega)$, which thus satisfies:

$$||\boldsymbol{v}^k - \boldsymbol{r}^k||_{1,\Omega} = \inf_{\boldsymbol{r}\in\boldsymbol{R}(\Omega)} ||\boldsymbol{v}^k - \boldsymbol{r}||_{1,\Omega} \text{ and } ||\boldsymbol{v}^k||^2_{1,\Omega} = ||\boldsymbol{v}^k - \boldsymbol{r}^k||^2_{1,\Omega} + ||\boldsymbol{r}^k||^2_{1,\Omega}.$$

The space $\boldsymbol{R}(\Omega)$ being finite-dimensional, the inequalities $||\boldsymbol{r}^k||_{1,\Omega} \leq 1$ for all $k \geq 1$ imply the existence of a subsequence $(\boldsymbol{r}^l)_{l=1}^{\infty}$ that converges in $\dot{\boldsymbol{H}}^1(\Omega)$ to an element $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$. Besides, *Korn's inequality in* $\dot{\boldsymbol{H}}^1(\Omega)$ implies that $||\boldsymbol{v}^l - \boldsymbol{r}^l||_{1,\Omega} \underset{l\to\infty}{\longrightarrow} 0$, so that $||\boldsymbol{v}^l - \boldsymbol{r}||_{1,\Omega} \underset{l\to\infty}{\longrightarrow} 0$. Hence $||\boldsymbol{v}^l - \boldsymbol{r}||_{0,\Omega} \underset{l\to\infty}{\longrightarrow} 0$, which forces $\boldsymbol{r}$ to be $\mathbf{0}$, since $||\boldsymbol{v}^l||_{0,\Omega} \to 0$ on the other hand. We thus reach the conclusion that $||\boldsymbol{v}^l||_{1,\Omega} \to 0$, a contradiction. □

Remark 4.1 By contrast with the implication of Theorem 2.2, the implication established in part (ii) of the above proof does not seem to have been previously recorded. □

5. Another approach to existence theory in linearized elasticity

Since the bilinear form $(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}) \in \dot{\boldsymbol{H}}^1(\Omega) \times \dot{\boldsymbol{H}}^1(\Omega) \to \int_\Omega \boldsymbol{A}\boldsymbol{e}(\dot{\boldsymbol{u}}) : \boldsymbol{e}(\dot{\boldsymbol{v}}) dx$ is symmetric, solving the "classical" variational formulation of the *pure traction problem of linearized elasticity* (see Section 2.2) is equivalent to solving the following *minimization problem*: Find $\dot{\boldsymbol{u}} \in \dot{\boldsymbol{H}}^1(\Omega)$ such that

$$J(\dot{\boldsymbol{u}}) = \inf_{\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)} J(\dot{\boldsymbol{v}}), \text{ where } J(\dot{\boldsymbol{v}}) := \frac{1}{2}\int_\Omega \boldsymbol{A}\boldsymbol{e}(\dot{\boldsymbol{v}}) : \boldsymbol{e}(\dot{\boldsymbol{v}}) dx - L(\dot{\boldsymbol{v}}).$$

Thanks to the isomorphism $\boldsymbol{\mathcal{F}} : \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ introduced in Theorem 4.1, this problem can be recast as *another minimization problem*, this time in terms of an unknown that lies in the space $\boldsymbol{E}(\Omega)$:

Theorem 5.1. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. The minimization problem : Find $\boldsymbol{\varepsilon} \in \boldsymbol{E}(\Omega)$ such that*

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \boldsymbol{E}(\Omega)} j(\boldsymbol{e}), \text{ where } j(\boldsymbol{e}) := \frac{1}{2}\int_\Omega \boldsymbol{A}\boldsymbol{e} : \boldsymbol{e}\, dx - \Lambda(\boldsymbol{e})$$

and the linear form $\Lambda : \boldsymbol{E}(\Omega) \to \mathbb{R}$ is defined by $\Lambda := L \circ \boldsymbol{\mathcal{F}}$, has one and only one solution $\boldsymbol{\varepsilon}$. Besides, $\boldsymbol{\varepsilon} = \boldsymbol{e}(\dot{\boldsymbol{u}})$ where $\dot{\boldsymbol{u}}$ is the unique solution to the "classical" variational formulation of the pure traction problem of linearized elasticity.

Proof. By assumption (Section 2), there exists $\alpha > 0$ such that $\int_\Omega \boldsymbol{A}\boldsymbol{e} : \boldsymbol{e}\, dx \geq \alpha||\boldsymbol{e}||^2_{0,\Omega}$ for all $\boldsymbol{e} \in \boldsymbol{L}^2_{sym}(\Omega)$. The linear form Λ is continuous since L and $\boldsymbol{\mathcal{F}}$ are continuous. Finally, $\boldsymbol{E}(\Omega)$ is a closed subspace of $\boldsymbol{L}^2_{sym}(\Omega)$. Consequently, there exists one, and only one, minimizer of the functional j over $\boldsymbol{E}(\Omega)$.

That $\dot{\boldsymbol{u}}$ minimizes the functional J over $\dot{\boldsymbol{H}}^1(\Omega)$ implies that $\boldsymbol{e}(\dot{\boldsymbol{u}})$ minimizes j over $\boldsymbol{E}(\Omega)$. Hence $\boldsymbol{\varepsilon} = \boldsymbol{e}(\dot{\boldsymbol{u}})$ since the minimizer is unique. □

6. Miscellaneous remarks

(a) While the minimization problem over the space $\dot{\boldsymbol{H}}^1(\Omega)$ is an *unconstrained one* with three unknowns, that found in Theorem 5.1 over the space $\boldsymbol{E}(\Omega)$ is in effect a *constrained minimization problem* over the space $\boldsymbol{L}^2_{sym}(\Omega)$ with six unknowns, the constraints (in the sense of optimization theory) being the compatibility relations $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$ that the matrix fields $\boldsymbol{e} \in \boldsymbol{E}(\Omega)$ satisfy (it is easily seen that these compatibility relations reduce in fact to six independent ones).

(b) As shown in the proof of Theorem 2.1, the *lemma of J. L. Lions* recalled there is the keystone of the classical proof of Korn's inequality. In a sense, the same rôle is played in the present approach by the "*$H^{-2}-$version of a classical theorem of Poincaré* " established in Theorem 3.1.

(c) In linearized elasticity, the *stress tensor field* $\boldsymbol{\sigma} \in \boldsymbol{L}^2_{sym}(\Omega)$ is given in terms of the displacement field by $\boldsymbol{\sigma} = \boldsymbol{A}\boldsymbol{e}(v)$. Since the elasticity tensor $\boldsymbol{A}$ is assumed to be uniformly positive-definite a.e. in Ω, the minimization problem of Theorem 5.1 can be immediately recast as a *constrained minimization problem with the stress tensor as the primary unknown.*

(d) Various attempts to consider the "fully nonlinear" *Green-St Venant strain tensor* $\boldsymbol{E}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v} + \boldsymbol{\nabla v}^T\boldsymbol{\nabla v})$, or equivalently the *Cauchy-Green tensor* $\boldsymbol{I} + 2\boldsymbol{E}(\boldsymbol{v})$, as the "primary" unknown in *three-dimensional nonlinear elasticity* have been recently undertaken in the same spirit; see Refs. 5, 6, 7, 8 and 15. These attempts have met only partial success, however, since nonlinearity *per se* creates specific challenging difficulties.

For instance, since the mapping $\boldsymbol{\mathcal{F}}$ introduced here in Theorem 4.1 is linear, its continuity at $\boldsymbol{e} = \boldsymbol{0}$ automatically implies its continuity at any $\boldsymbol{e} \in \boldsymbol{E}(\Omega)$. By contrast, considerable extra work is needed to extend the continuity of the "nonlinear analog of the mapping $\boldsymbol{\mathcal{F}}$ " at the particular Cauchy-Green tensor $\boldsymbol{I}$ to its continuity at any Cauchy-Green tensor. Besides, the choice of the "right" function spaces, i.e., between which such a nonlinear mapping is continuous, is a much trickier issue than in the linearized case considered here.

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