Global Solutions to the Boltzmann Equation with External Forces

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Abstract

For the Boltzmann equation with an external potential force depending only on the space variables, there is a family of stationary solutions which are local Maxwellians with space dependent density, zero velocity and constant temperature. In this paper, we will study the nonlinear stability of these stationary solutions by using energy method. The analysis combines the analytic techniques used for the conservation laws using the fluid-type system derived from the Boltzmann equation, cf. [14], and the dissipative effects on the fluid and non-fluid components of the Boltzmann equation through the celebrated H-theorem. To our knowledge, this is the first result on the global classical solutions to the Boltzmann equation with external force and non-trivial large time behavior in the whole space.

1 Introduction

Recently, there are some progress on the nonlinear stability of three basic wave patterns for the Boltzmann equation with "slab symmetry", cf. [16, 15, 7] for the shock, rarefaction wave and contact discontinuity respectively. However, for the Boltzmann equation with external forces, to our knowledge, there are few results on the stability of non-trivial solution profiles. In this paper, we will study the nonlinear stability of a family of nontrivial profiles, i.e. the stationary solutions, to the Boltzmann equation with a potential force in the whole space. Consider

$$f_t + \xi\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_\xi f = Q(f,f), \tag{1.1}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi), \tag{1.2}$$

where $f(t,x,\xi)$ is the distribution function of the particles at time $t\geq 0$ located at $x = (x_1,x_2,x_3)\in\mathbf{R}^3$ with velocity $\xi = (\xi_1,\xi_2,\xi_3)\in\mathbf{R}^3$, and Φ denotes the potential of the external force. The short-range interaction between particles is given by the standard Boltzmann collision operator $Q(f,g)$ for the hard-sphere model

$$Q(f,g)(\xi)\equiv\frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}^2_+}\Big(f(\xi')g(\xi'_*)+f(\xi'_*)g(\xi')-f(\xi)g(\xi_*)-f(\xi_*)g(\xi)\Big)\left|(\xi-\xi_*)\cdot\Omega\right|\,d\xi_*d\Omega.$$

Here $\mathbf{S}^2_+ = \{\Omega\in\mathbf{S}^2:\ (\xi-\xi_*)\cdot\Omega\geq 0\}$, and

$$\begin{cases}\xi' = \xi - [(\xi-\xi_*)\cdot\Omega]\,\Omega,\\ \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\Omega]\,\Omega,\end{cases}$$

which represents the relation between velocities ξ', ξ'_* after and the velocities ξ, ξ_* before the collision coming from the conservation of momentum and energy.

We will consider the case when Φ depends only on space variables, i.e. $\Phi = \Phi(x)$. In this case, the local *Maxwellian* given by

$$\overline{\mathbf{M}}\equiv\overline{\mathbf{M}}(x,\xi) = \frac{\rho_1}{\left(2\pi R\overline{\theta}\right)^{\frac{3}{2}}}\exp\left\{-\frac{1}{R\overline{\theta}}\left(\Phi(x)+\frac{|\xi|^2}{2}\right)\right\} = \mathbf{M}_{[\overline{\rho}(x),0,\overline{\theta}]}(x,\xi),$$

where $R>0$ is the gas constant and $\rho_1>0,\overline{\theta}>0$ are some constants, is a stationary solution to (1.1). This local Maxwellian represents the distribution of a gas in an equilibrium state with the mass density $\overline{\rho}(x) = \rho_1\exp\left(-\frac{\Phi(x)}{R\overline{\theta}}\right)$, the zero flow velocity, and the absolute temperature $\overline{\theta}$. Our goal is to show that if the initial data $f_0(x,\xi)$ is a small perturbation of the local Maxwellian $\overline{\mathbf{M}}$, then there exists a global classical solution to (1.1)-(1.2) converging to $\overline{\mathbf{M}}$ time asymptotically.

Here we will apply the micro-macro decomposition of the Boltzmann equation and its solution introduced in [14, 16], where the Boltzmann equation can be rewritten into a fluid-type system coupled with an equation for the non-fluid(kinetic) component in [14]. In fact, this decomposition is also used in the study of the Boltzmann equation with self-induced electric field, i.e. the Vlasov-Poisson-Boltzmann system [20] near a given global Maxwellian.

Decompose the solution $f(t,x,\xi)$ of the Boltzmann (1.1) with external forces $\Phi(x)$ into the macroscopic (fluid) component, i.e., the local Maxwellian $\mathbf{M} = \mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$, and the microscopic (non-fluid) component, i.e., $\mathbf{G} = \mathbf{G}(t,x,\xi)$ as follows, [18, 14]:

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi).$$

The local Maxwellian $\mathbf{M}$ is defined by the five conserved quantities, that is, the density $\rho(t,x)$, momentum $m(t,x) = \rho(t,x)u(t,x)$, and energy density $\mathbf{E}(t,x) + \frac{1}{2}|u(t,x)|^2$ given by:

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ m^i(t,x) \equiv \int_{\mathbf{R}^3} \psi_i(\xi)f(t,x,\xi)d\xi \text{ for } i = 1,2,3, \\ \left[\rho\left(\mathbf{E} + \frac{1}{2}|u|^2\right)\right](t,x) \equiv \int_{\mathbf{R}^3} \psi_4(\xi)f(t,x,\xi)d\xi, \end{cases} \tag{1.3}$$

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\left(-\frac{|\xi - u(t,x)|^2}{2R\theta(t,x)}\right). \tag{1.4}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy $\mathbf{E}$ by $\mathbf{E} = \frac{3}{2}R\theta$, and $u(t,x)$ is the fluid velocity. And $\psi_\alpha(\xi)$, $\alpha = 0,1,\cdots,4$, are the five collision invariants, cf. [2, 3]:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i \quad \text{for} \quad i = 1,2,3 \text{ or } \psi = \xi, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases} \tag{1.5}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_\alpha(\xi)Q(h,g)d\xi = 0, \text{ for } \alpha = 0,1,2,3,4,$$

which are also the basis of the sub-manifold of the fluid components, denoted by N up to a weight function.

Define an inner product in $\xi \in \mathbf{R}^3$ w.r.t. a given Maxwellian $\tilde{\mathbf{M}}$ as:

$$\langle h, g\rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} h(\xi)g(\xi)d\xi,$$

for functions h, g of ξ so that the above integral is well-defined. With respect to this inner product, a set of pairwise orthogonal basis for N can be chosen as:

$$\begin{cases} \chi_0(\xi;\rho,u,\theta) \equiv \frac{1}{\sqrt{\rho}}\mathbf{M}, \\ \chi_i(\xi;\rho,u,\theta) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}}\mathbf{M} \quad \text{for} \quad i = 1,2,3, \\ \chi_4(\xi;\rho,u,\theta) \equiv \frac{1}{\sqrt{6\rho}}\left(\frac{|\xi-u|^2}{R\theta} - 3\right)\mathbf{M}, \\ \quad \langle \chi_i, \chi_j\rangle_{\mathbf{M}} = \delta_{ij}, \quad \text{for } i,j = 0,1,2,3,4. \end{cases} \tag{1.6}$$

With this, one can define two orthogonal and self-adjoint projections $\mathbf{P}_0$ and $\mathbf{P}_1$ onto the fluid and non-fluid sub-manifolds respectively:

$$\begin{cases} \mathbf{P}_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j \rangle_{\mathbf{M}} \chi_j, \\ \mathbf{P}_1 h \equiv h - \mathbf{P}_0 h. \end{cases} \tag{1.7}$$

Using these notations, the solution $f(t,x,\xi)$ of (1.1) satisfies,

$$\mathbf{P}_0 f = \mathbf{M}, \qquad \mathbf{P}_1 f = \mathbf{G}.$$

Then by using $f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi)$, the equation (1.1) becomes:

$$(\mathbf{M}+\mathbf{G})_t + \xi \cdot \nabla_x (\mathbf{M}+\mathbf{G}) - \nabla_x \Phi \cdot \nabla_\xi (\mathbf{M}+\mathbf{G}) = \Big(2Q(\mathbf{G},\mathbf{M}) + Q(\mathbf{G},\mathbf{G}) \Big). \tag{1.8}$$

By applying $\mathbf{P}_0$ to (1.8), we have

$$\mathbf{M}_t + \mathbf{P}_0 \Big(\xi \cdot \nabla_x \mathbf{M} \Big) + \mathbf{P}_0 \Big(\xi \cdot \nabla_x \mathbf{G} \Big) - \nabla_x \Phi \cdot \nabla_\xi \mathbf{M} = 0.$$

As usual, the system of the conservation laws below can be obtained by taking the inner product of the above equation for $\mathbf{M}$ with the collision invariants $\psi_\alpha(\xi)$:

$$\begin{cases} \rho_t + \mathrm{div}_x\, m = 0, \\ m_{it} + \sum_{j=1}^{3} (u_i m_j)_{x_j} + p_{x_i} - \overline{p}_{x_i} + (\rho - \overline{\rho})\, \Phi_{x_i} = -\int_{\mathbf{R}^3} \psi_i \left(\xi \cdot \nabla_x \mathbf{G} \right) d\xi, \quad i = 1,2,3, \\ \Big[\rho (\frac{1}{2}|u|^2 + \mathbf{E}) \Big]_t + \sum_{j=1}^{3} \left(u_j \Big(\rho \left(\frac{1}{2}|u|^2 + \mathbf{E} \right) + p \Big) \right)_{x_j} + m \cdot \nabla_x \Phi = -\int_{\mathbf{R}^3} \psi_4 \left(\xi \cdot \nabla_x \mathbf{G} \right) d\xi. \end{cases} \tag{1.9}$$

Here p is the pressure for the monatomic gases:

$$p = \frac{2}{3} \rho \mathbf{E} = R\rho\theta, \quad \overline{p} = R\overline{\rho}\overline{\theta},$$

and we have used

$$\overline{p}_{x_i} + \overline{\rho} \Phi_{x_i} = 0.$$

Moreover, the microscopic equation for $\mathbf{G}$ is obtained by applying the microscopic projection $\mathbf{P}_1$ to (1.8):

$$\mathbf{G}_t + \mathbf{P}_1 \Big(\xi \cdot \nabla_x \mathbf{G} + \xi \cdot \nabla_x \mathbf{M} \Big) - \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} = L_{\mathbf{M}} \mathbf{G} + Q(\mathbf{G},\mathbf{G}), \tag{1.10}$$

i.e.,

$$\begin{aligned} \mathbf{G} &= L_{\mathbf{M}}^{-1} \Big(\mathbf{P}_1 (\xi \cdot \nabla_x \mathbf{M}) \Big) + L_{\mathbf{M}}^{-1} \Big(\mathbf{G}_t + \mathbf{P}_1 (\xi \cdot \nabla_x \mathbf{G}) - \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} - Q(\mathbf{G},\mathbf{G}) \Big) \\ &:= L_{\mathbf{M}}^{-1} \Big(\mathbf{P}_1 (\xi \cdot \nabla_x \mathbf{M}) \Big) + \Theta, \end{aligned} \tag{1.11}$$

where

$$L_{\mathbf{M}} g = L_{[\rho,u,\theta]} g \equiv Q\Big(\mathbf{M} + g, \mathbf{M} + g \Big) - Q(g,g).$$

With the Burnett functions A and B, the viscosity and heat conductivity coefficients can be represented by:

$$\begin{cases} A_j(\xi) = \frac{|\xi|^2-5}{2}\xi_j, \quad j=1,2,3, \\ B_{ij}(\xi) = \xi_i\xi_j - \frac{1}{3}\delta_{ij}|\xi|^2, \quad i,j=1,2,3, \\ \mu(\theta) = -R\theta\int_{\mathbf{R}^3} B_{ij}\left(\frac{\xi}{\sqrt{R\theta}}\right) L^{-1}_{\mathbf{M}_{[1,u,\theta]}}\left(B_{ij}\left(\frac{\xi}{\sqrt{R\theta}}\right)\mathbf{M}_{[1,u,\theta]}\right) d\xi > 0, \quad i \neq j, \\ \kappa(\theta) = -R^2\theta\int_{\mathbf{R}^3} A_l\left(\frac{\xi}{\sqrt{R\theta}}\right) L^{-1}_{\mathbf{M}_{[1,u,\theta]}}\left(A_l\left(\frac{\xi}{\sqrt{R\theta}}\right)\mathbf{M}_{[1,u,\theta]}\right) d\xi > 0. \end{cases}$$

Hence, we have (cf. [5])

$$\begin{cases} -\int_{\mathbf{R}^3}\psi_i\xi\cdot\nabla_x L^{-1}_{\mathbf{M}}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)d\xi = \sum\limits_{j=1}^3\Big[\mu(\theta)\Big(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\Big)\Big]_{x_j}, \quad i=1,2,3, \\ -\int_{\mathbf{R}^3}\psi_4\xi\cdot\nabla_x L^{-1}_{\mathbf{M}}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)d\xi = \sum\limits_{j=1}^3\Big(\kappa(\theta)\theta_{x_j}\Big)_{x_j} \\ \qquad\qquad + \sum\limits_{i,j=1}^3\Big\{\mu(\theta)u_i\Big(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\Big)\Big\}_{x_j}. \end{cases}$$

By plugging the above indentities and (1.11) into (1.9), we now have another representation of the equation (1.1) which contains a fluid-type system

$$\begin{cases} \rho_t + \mathrm{div}_x m = 0, \\ m_{it} + \sum\limits_{j=1}^3 (u_i m_j)_{x_j} + p_{x_i} - \overline{p} + (\rho-\overline{\rho})\,\Phi_{x_i} = \sum\limits_{j=1}^3\Big[\mu(\theta)\Big(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\Big)\Big]_{x_j} \\ \qquad\qquad - \int_{\mathbf{R}^3}\psi_i\,(\xi\cdot\nabla_x\Theta)\,d\xi, \quad i=1,2,3, \\ \Big[\rho(\frac{1}{2}|u|^2+\mathbf{E})\Big]_t + \sum\limits_{j=1}^3\left(u_j\Big(\rho\Big(\frac{1}{2}|u|^2+\mathbf{E}\Big)+p\Big)\right)_{x_j} + m\cdot\nabla_x\Phi \\ \qquad = \sum\limits_{i,j=1}^3\Big\{\mu(\theta)u_i\Big(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\Big)\Big\}_{x_j} \\ \qquad\quad + \sum\limits_{j=1}^3\Big(\kappa(\theta)\theta_{x_j}\Big)_{x_j} - \int_{\mathbf{R}^3}\psi_4\,(\xi\cdot\nabla_x\Theta)\,d\xi, \end{cases} \tag{1.12}$$

and the equation (1.11) for the non-fluid component $\mathbf{G}$. Notice that if one drops all the terms containing Θ, then it becomes the system of the Navier-Stokes equations with external force. Later in this paper, we will work on this reformulated system by applying the energy method as in the study of conservation laws together with the dissipative effects from the Boltzmann equation through the celebrated H-theorem.

For preparation, we now recall some basic properties of the linearized collision operator $L_{\mathbf{M}}$. By definition, $L_{\mathbf{M}}$ is self-adjoint w.r.t. the inner product $\langle h, g\rangle_{\mathbf{M}}$, i.e.,

$$\langle h, L_{\mathbf{M}}g\rangle_{\mathbf{M}} = \langle L_{\mathbf{M}}h, g\rangle_{\mathbf{M}},$$

and the null space is N.

For the hard sphere model, $L_{\mathbf{M}}$ takes the form, cf. [10]

$$\left(L_{\mathbf{M}}h\right)(\xi) = -\nu_{\mathbf{M}}(\xi)h(\xi) + \sqrt{\mathbf{M}(\xi)}K_{\mathbf{M}}\left(\left(\frac{h}{\sqrt{\mathbf{M}}}\right)(\xi)\right). \tag{1.13}$$

Here $K_{\mathbf{M}}(\cdot) = -K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2-operator. And the collision frequency $\nu_{\mathbf{M}}(\xi)$ and $K_{i\mathbf{M}}(\cdot)$ have the following expressions

$$\begin{cases} \nu_{\mathbf{M}}(\xi) = \frac{2\rho}{\sqrt{2\pi R\theta}} \left\{ \left(\frac{R\theta}{|\xi-u|} + |\xi - u| \right) \int_0^{|\xi-u|} \exp\left(-\frac{y^2}{2R\theta}\right) dy + R\theta \exp\left(-\frac{|\xi-u|^2}{2R\theta}\right) \right\}, \\ k_{1\mathbf{M}}(\xi,\xi_*) = \frac{\pi\rho}{\sqrt{(2\pi R\theta)^3}} |\xi - \xi_*| \exp\left(-\frac{|\xi-u|^2}{4R\theta} - \frac{|\xi_*-u|^2}{4R\theta}\right), \\ k_{2\mathbf{M}}(\xi,\xi_*) = \frac{2\rho}{\sqrt{2\pi R\theta}} |\xi - \xi_*|^{-1} \exp\left(-\frac{|\xi-\xi_*|^2}{8R\theta} - \frac{(|\xi|^2-|\xi_*|^2)^2}{8R\theta|\xi-\xi_*|^2}\right), \end{cases}$$

where $k_{i\mathbf{M}}(\xi,\xi_*)(i=1,2)$ is the kernel of the operator $K_{i\mathbf{M}}(i=1,2)$ respectively, and $\nu_{\mathbf{M}}(\xi) \sim (1+|\xi|)$ as $|\xi| \to +\infty$. Furthermore, there exists $\sigma_0(u,\theta) > 0$ such that for any function $h(\xi) \in N^{\perp}$

$$\langle h, L_{\mathbf{M}} h\rangle_{\mathbf{M}} \leq -\sigma_0(u,\theta)\langle h, h\rangle_{\mathbf{M}},$$

which implies cf. [10]

$$\langle h, L_{\mathbf{M}} h\rangle_{\mathbf{M}} \leq -\sigma(u,\theta)\langle \nu_{\mathbf{M}}(\xi) h, h\rangle_{\mathbf{M}}, \tag{1.14}$$

with some constant $\sigma(u,\theta) > 0$.

Since the time asymptotic state is non-trivial(not a global Maxwellian), as in [15] and other related works, two sets of energy estimates are needed. That is, we need the energy estimates w.r.t. the local Maxwellian $\mathbf{M}_{[\rho,u,\theta]}(t,x,\xi)$ and a suitably chosen global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}(\xi)$ to close the a priori estimate. For this, a variation of the microscopic $H-$theroem is needed to relate the dissipation estimates with different weights as in Lemma 2.2 of [15]. That is, there exists a positive constant $\eta_0 = \eta_0(u,\theta;\tilde{u}.\tilde{\theta}) > 0$, which is not necessary to be small, such that if $\frac{\theta}{2} < \tilde{\theta} < \theta$ and $|u-\tilde{u}| + |\theta - \tilde{\theta}| < \eta_0$, the following microscopic $H-$therem

$$-\int_{\mathbf{R}^3} \frac{\mathbf{G} L_{\mathbf{M}} \mathbf{G}}{\tilde{\mathbf{M}}} d\xi \geq \overline{\sigma} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\mathbf{G}^2}{\tilde{\mathbf{M}}} d\xi, \tag{1.15}$$

holds for some positive constant $\overline{\sigma} = \overline{\sigma}(u,\theta;\tilde{u},\tilde{\theta}) > 0$ with $\tilde{\mathbf{M}} = \mathbf{M}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}$.

Throughout this paper, we choose positive constants ρ_- and θ_- such that

$$\begin{cases} \frac{\overline{\theta}}{2} < \theta_- < \overline{\theta}, \\ |\rho_- - \overline{\rho}| + \left|\theta_- - \overline{\theta}\right| < \eta_0. \end{cases} \tag{1.16}$$

It is easy to see that if $\mathbf{M}(t,x,\xi)$ is a small perturbation of $\overline{\mathbf{M}}(x,\xi)$, (1.15) holds for such chosen ρ_- and θ_- when $\tilde{\mathbf{M}} \equiv \mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}$.

Let $g(t,x,\xi) = f(t,x,\xi) - \overline{\mathbf{M}}(x,\xi)$, we now give the function space for the solutions considered in this paper

$$\mathbf{H}^N_{x,\xi}\left(\mathbf{R}^+\right) = \left\{ g(t,x,\xi) \;\middle|\; \begin{array}{c} \frac{\partial_t^{\gamma_0}\partial_x^{\alpha}\partial_\xi^{\beta} g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left(\mathbf{R}^+, L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right) \\ \frac{\sqrt{\nu_{\mathbf{M}}(\xi)}\partial_t^{\gamma_0}\partial_x^{\alpha}\partial_\xi^{\beta} g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in L^2_{t,x,\xi}\left(\mathbf{R}^+ \times \mathbf{R}^3 \times \mathbf{R}^3\right), \text{ for } \gamma_0 + |\alpha| > 0 \\ \gamma_0 + |\alpha| + |\beta| \leq N \end{array} \right\}.$$

Since $\frac{\overline{\theta}}{2} < \theta_- < \overline{\theta}$, we have for each $\gamma_0 + |\alpha| + |\beta| \leq N$,

$$\begin{cases} \partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(\frac{g(t,x,\xi)}{\sqrt{\overline{\mathbf{M}}(x,\xi)}}\right) \in \mathbf{BC}_t\left(\mathbf{R}^+, L^2_{x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)\right), \\ \sqrt{\nu_{\mathbf{M}}(\xi)}\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(\frac{g(t,x,\xi)}{\sqrt{\overline{\mathbf{M}}(x,\xi)}}\right) \in L^2_{t,x,\xi}\left(\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3\right), \text{ for } \gamma_0+|\alpha|>0. \end{cases}$$

By using the above notations, the main result in this paper can be stated as follows.

Theorem 1.1 *Assume that $f_0(x,\xi)\geq 0$ and $N\geq 4$. There exist two sufficiently small constants $\varepsilon>0$ and $\lambda_0>0$ such that if*

$$\begin{cases} \lambda \equiv \|\Phi(x)\|_{L^2(\mathbf{R}^3)} + \sum\limits_{1\leq|\alpha|\leq N+1} \|\partial_x^\alpha \Phi(x)\|_{L^3(\mathbf{R}^3)} < \lambda_0, \\ \mathcal{E}(f_0) = \sum\limits_{|\alpha|+|\beta|\leq N} \left\| \frac{\partial_x^\alpha\partial_\xi^\beta(f_0(x,\xi)-\overline{\mathbf{M}}(x,\xi))}{\sqrt{\mathbf{M}_-(\xi)}} \right\|_{L^2_{x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)} \leq \varepsilon, \end{cases} \tag{1.17}$$

then there exists a unique global classical solution $f(t,x,\xi)\in \mathbf{H}^N_{x,\xi}(\mathbf{R}^+)$ to the Cauchy problem (1.1), (1.2) which satisfies $f(t,x,\xi)\geq 0$ and

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\sum_{\gamma_0+|\alpha|+|\beta|\leq N-4}\int_{\mathbf{R}^3}\frac{\left|\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(f(t,x,\xi)-\overline{\mathbf{M}}(x,\xi)\right)\right|^2}{\mathbf{M}_-(\xi)}d\xi = 0. \tag{1.18}$$

Remark 1.1 *A similar result was announced in [1] on the L^∞_ξ solutions under the additional assumption that the support of $\Phi(x)$ is compact. On the other hand, the compressible Navier-Stokes equations with the potential force was solved in [17] on the same global existence and asymptotic property under the same assumption on $\Phi(x)$, as in our Theorem 1.1.*

Remark 1.2 *The assumption $(1.17)_1$ requires, among others, that $\Phi(x)\in L^2_x(\mathbf{R}^3)$, but this does not contradict to the fact that the potential $\Phi(x)$ is unique only up to an additive constant, because this constant can be absorbed into the constant ρ_1.*

The proof of Theorem 1.1 relies on the energy method based on the macro-micro (fluid dynamic-kinetic) decomposition of the Boltzmann equation developed recently in [14]. The energy estimates for marcroscopic (fluid) component of $f(t,x,\xi)$ are obtained with the $H-$theorem for the lower order derivatives and by the usual integrations by parts for the differential equations for higher order derivatives. Both estimates contain Sobolov norms of the microscopic (kinetic) component of $f(t,x,\xi)$. It should be noted that if these terms are dropped, our estimates coincide with those presented in [17] for the compressible Navier-Stokes equations.

The norm of the microscopic component can be estimated by virtue of the microscopic $H-$theorem, i.e. the negative defiteness of the linearized collision operator on the space of functions having only microscopic components, and again by the integration by parts on the differentiated microscopic equations.

This technique has been developed in [14] for the force-free case, where the energy estimates can be closed only with (t,x) derivatives of $f(t,x,\xi)$. In our case, however, ξ derivatives should be also included. Recently, in [12], another L^2 energy method has been proposed for the Boltzmann equation. Although the technique is quite different from [14], it applies also to our case, to deduce the same result.

The global existence is concluded by combining the local existence and the energy estimates. Our local solutions should be, therefore, in consistence with our energy estimates, that is, they should be L^2 solutions w.r.t. ξ as well as (t,x). Such solutions can be constructed by using the L^2 estimate of $Q(f,g)$ derived in [9].

The rest of this paper is arranged as follows. The microscopic and macroscopic versions of the $H-$theorems will be stated in Section 2. The main energy estimates are analyzed for the case when $N=4$ in Section 3. The case when $N>4$ can be discussed similarly. The proof of Theorem 1.1 will be given in Section 4, and the proofs of some technical lemmas stated in Section 3 are given in Section 5 for clear presentation.

Notation

In the rest of this paper, the generic constants (but perhaps depend on the initial values) will be denoted by $O(1)$ or C. Occasionally, we use e.g. $C(r,s)$ when we want to emphasize the dependence of C on the parameters r and s. Note that all constants may vary from line to line.

For $\gamma=(\alpha_0,\alpha),\alpha$, and β, we use $\partial^\gamma,\partial^\alpha$, and ∂^β to denote the differential operators $\partial_t^{\alpha_0}\partial_x^\alpha,\partial_x^\alpha$, and ∂_ξ^β respectively. Here α_0 is a non-negative integer and $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$ are multi-indices with length $|\alpha|$ and $|\beta|$, respectively. Finally $|\gamma|=\alpha_0+|\alpha|$ and C_b^a means $\begin{pmatrix} a \\ b \end{pmatrix}$.

2 $H-$theorem

The celebrated $H-$theorem of the Boltzmann equation is based on the special property of the bilinear structure of $Q(f,f)$ satisfying

$$\int_{\mathbf{R}^3} Q(f,f)\ln f d\xi \le 0,$$

and the equality holds only when the solution $f(t,x,\xi)$ is a Maxwellian.

Corresponding to the macroscopic and microscopic components, the H-theorem can be viewed in these two aspects. The first kind of dissipation comes from the linearized collision operator $L_{\mathbf{M}}$ acting on the microscopic components stated in (1.14) and (1.15). The second kind of dissipation comes from the nonlinear collision operator in the expression of the viscosity and heat conductivity in the macroscopic level.

In the following, we will first state some inequalities on the nonlinear and linearized collision operators $Q(f,f)$ and $L_{\mathbf{M}}$. The first lemma is from [9].

Lemma 2.1 *There exists a positive constant $C>0$ such that*

$$\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}Q(f,g)^2}{\tilde{\mathbf{M}}}d\xi \le C\left\{\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)f^2}{\tilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{g^2}{\tilde{\mathbf{M}}}d\xi+\int_{\mathbf{R}^3}\frac{f^2}{\tilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)g^2}{\tilde{\mathbf{M}}}d\xi\right\}, \tag{2.1}$$

where $\tilde{\mathbf{M}}$ is any Maxwellian such that the above integrals are well defined.

Based on Lemma 2.1, the following result was proved in [15].

Lemma 2.2 *If $\frac{\theta}{2} < \tilde{\theta} < \theta$, then there exist two positive constants $\overline{\sigma} = \overline{\sigma}(u, \theta; \tilde{u}, \tilde{\theta})$ and $\eta_0 = \eta_0(u, \theta; \tilde{u}, \tilde{\theta})$ such that if $|u - \tilde{u}| + |\theta - \tilde{\theta}| < \eta_0$, we have for $h(\xi) \in N^\perp$,*

$$-\int_{\mathbf{R}^3} \frac{h L_{\mathbf{M}} h}{\tilde{\mathbf{M}}} d\xi \geq \overline{\sigma} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi) h^2}{\tilde{\mathbf{M}}} d\xi.$$

Here $\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t, x, \xi)$ and $\tilde{\mathbf{M}}(t, x, \xi) = \tilde{\mathbf{M}}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}(t, x, \xi)$.

As a direct consequence of Lemma 2.2 and the Cauchy inequality, we have the following corollary (cf. [15]).

Corollary 2.1 *Under the assumptions in Lemma 2.2, we have for $h(\xi) \in N^\perp$,*

$$\begin{cases} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leq \sigma^{-2} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)^{-1} h^2(\xi)}{\mathbf{M}} d\xi, \\ \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}_-} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leq \overline{\sigma}^{-2} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)^{-1} h^2(\xi)}{\mathbf{M}_-} d\xi. \end{cases} \tag{2.2}$$

To construct the entropy-entropy flux pairs to (1.1), we first derive the macroscopic version of the $H-$theorem as the one in [14] for the Boltzmann equation without force. Set

$$-\frac{3}{2}\rho S \equiv \int_{\mathbf{R}^3} \mathbf{M} \ln \mathbf{M} d\xi. \tag{2.3}$$

Direct calculation yields

$$-\frac{3}{2}(\rho S)_t - \frac{3}{2}\mathrm{div}_x(\rho u S) + \nabla_x \left(\int_{\mathbf{R}^3} (\xi \ln \mathbf{M}) \mathbf{G} d\xi \right) = \int_{\mathbf{R}^3} \frac{\mathbf{G} \mathbf{P}_1 \Big(\xi \cdot \nabla_x \mathbf{M} \Big)}{\mathbf{M}} d\xi, \tag{2.4}$$

and

$$\begin{cases} S = -\frac{2}{3} \ln \rho + \ln(2\pi R \theta) + 1, \\ p = \frac{2}{3}\rho\theta = k\rho^{\frac{5}{3}} \exp(S), \\ \mathbf{E} = \theta, \quad R = \frac{2}{3}. \end{cases} \tag{2.5}$$

Remark 2.1 *Note that when the macroscopic entropy S is defined as in (2.3), the gas constant R is normalized to be $\frac{2}{3}$ and in such a case $\mathbf{E} = \theta$.*

An convex entropy-entropy flux pair (η, q) around the stationary solution $\overline{\mathbf{M}} = \mathbf{M}_{[\overline{\rho}(x),0,\overline{\theta}]}$ can be given as follows, [14]. Denote the conservation laws (1.9) by

$$\mathbf{m}_t + \mathrm{div}_x \mathbf{n} = - \begin{pmatrix} 0 \\ \int_{\mathbf{R}^3} \psi_1(\xi \cdot \nabla_x \mathbf{G}) d\xi \\ \int_{\mathbf{R}^3} \psi_2(\xi \cdot \nabla_x \mathbf{G}) d\xi \\ \int_{\mathbf{R}^3} \psi_3(\xi \cdot \nabla_x \mathbf{G}) d\xi \\ \int_{\mathbf{R}^3} \psi_4(\xi \cdot \nabla_x \mathbf{G}) d\xi \end{pmatrix} - \begin{pmatrix} 0 \\ \rho \Phi_{x_1} \\ \rho \Phi_{x_2} \\ \rho \Phi_{x_3} \\ m \cdot \nabla_x \Phi \end{pmatrix}.$$

Here

$$\begin{cases} \mathbf{m} = (m_0, m_1, m_2, m_3, m_4)^t = \left(\rho, \rho u_1, \rho u_2, \rho u_3, \rho\left(\frac{1}{2}|u|^2 + \theta\right)\right)^t, \\ \mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3), \\ \mathbf{n}_j = (n_0^j, n_1^j, n_2^j, n_3^j, n_4^j)^t \\ \quad = \left(\rho u_j, u_1 m_j + \frac{2}{3}\rho\theta, u_2 m_j + \frac{2}{3}\rho\theta, u_3 m_j + \frac{2}{3}\rho\theta, \rho u_j\left(\frac{1}{2}|u|^2 + \frac{5}{3}\theta\right)\right)^t, j = 1, 2, 3. \end{cases}$$

Then the entropy-entropy flux pair (η, q) can be defined by

$$\begin{cases} \eta = \overline{\theta}\left\{-\frac{3}{2}\rho S + \frac{3}{2}\overline{\rho}\overline{S} + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{m} - \overline{\mathbf{m}})\right\}, \\ q_j = \overline{\theta}\left\{-\frac{3}{2}\rho u_j S + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{n}_j - \overline{\mathbf{n}}_j)\right\}, \ j = 1, 2, 3. \end{cases} \tag{2.6}$$

Since

$$\begin{cases} (\rho S)_{m_0} = S + \frac{|u|^2}{2\theta} - \frac{5}{3}, \\ (\rho S)_{m_i} = -\frac{u_i}{\theta}, \ i = 1, 2, 3, \\ (\rho S)_{m_4} = \frac{1}{\theta}, \end{cases}$$

we have

$$\begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \overline{\theta}\rho S + \rho\left[\left(\overline{S} - \frac{5}{3}\right)\overline{\theta} + \frac{|u|^2}{2}\right] + \frac{2}{3}\overline{\rho}\overline{\theta}\right\}, \\ q_j = u_j\eta + u_j\left(\rho\theta - \overline{\rho}\overline{\theta}\right), j = 1, 2, 3. \end{cases} \tag{2.7}$$

Notice that for $\mathbf{m}$ in any closed bounded region $\mathcal{D} \subset \Sigma = \{\mathbf{m} : \rho > 0, \theta > 0\}$, there exists a positive constant C depending on $\mathcal{D}$ such that the entropy-entropy flux thus constructed satisfies (cf. [14, 15])

$$C^{-1}|\mathbf{m} - \overline{\mathbf{m}}|^2 \leq \eta \leq C|\mathbf{m} - \overline{\mathbf{m}}|^2. \tag{2.8}$$

And (η, q_1, q_2, q_3) solves the following partial differential equation

$$\begin{aligned} \eta_t + \mathrm{div}_x q = & -\nabla_x\left(\int_{\mathbf{R}^3}\left(\xi\mathbf{G}\ln\mathbf{M} + \frac{3}{2}\psi_4\xi\mathbf{G}\right)d\xi\right) - \frac{3}{2}m\cdot\nabla_x\Phi \\ & + \int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\mathbf{G}}{\mathbf{M}}d\xi. \end{aligned} \tag{2.9}$$

Integrating (2.9) w.r.t. x over $\mathbf{R}^3$ gives

$$\frac{d}{dt}\int_{\mathbf{R}^3}\eta(t)dx = -\frac{3}{2}\int_{\mathbf{R}^3}m\cdot\nabla_x\Phi dx + \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\left(\xi\cdot\nabla_x\mathbf{M}\right)\mathbf{G}}{\mathbf{M}}d\xi dx. \tag{2.10}$$

Since

$$\begin{aligned} \int_{\mathbf{R}^3}m\cdot\nabla_x\Phi dx = & -\int_{\mathbf{R}^3}\mathrm{div}_x m\Phi dx = \int_{\mathbf{R}^3}(\rho - \overline{\rho})_t\Phi dx \\ = & \frac{d}{dt}\int_{\mathbf{R}^3}(\rho - \overline{\rho})\Phi dx, \end{aligned} \tag{2.11}$$

we obtain the entropy estimate

$$\frac{d}{dt}\left\{\int_{\mathbf{R}^3}\left(\eta+\frac{3}{2}\left(\rho-\overline{\rho}\right)\Phi\right)dx\right\}=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\Big(\xi\cdot\nabla_x\mathbf{M}\Big)\mathbf{G}}{\mathbf{M}}d\xi dx, \tag{2.12}$$

which is crucial in the later energy estimates on the fluid components of the solutions.

Before concluding this section, we note from (2.8) that

$$\int_{\mathbf{R}^3}\left(\eta+\frac{3}{2}\left(\rho-\overline{\rho}\right)\Phi\right)(t,x)dx\geq\frac{1}{2}\int_{\mathbf{R}^3}\eta(t,x)dx-O(1)\lambda^2. \tag{2.13}$$

3 Energy estimates

In this section, we will give the entropy estimates for the proof of global existence theorem. For this, we first assume the following a priori estimate

$$\begin{aligned}N(t)^2=&\sup_{0\leq\tau\leq t}\left\{\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})(\tau,x)\right|^2dx+\sum_{|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}(\tau,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx\right\}\\&+\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}(\tau,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxd\tau\\&\leq\delta^2.\end{aligned} \tag{3.1}$$

Here $\delta>0$ is a suitably chosen sufficiently small constant whose precise range can be easily seen from the analysis below.

It is easy to see from (3.1), the conservation laws (1.9), and Sobolev's inequality that

$$N(0)\leq O(1)\mathcal{E}(f_0) \tag{3.2}$$

and

$$\begin{cases}\sup\limits_{x\in\mathbf{R}^3}\sum\limits_{|\alpha|\leq 3}|\partial^\alpha\Phi(x)|\leq O(1)\lambda,\\ \sup\limits_{(\tau,x)\in[0,t)\times\mathbf{R}^3}\sum\limits_{|\gamma|\leq 2}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})(\tau,x)\right|\leq O(1)\delta,\\ \sup\limits_{(\tau,x)\in[0,t)\times\mathbf{R}^3}\sum\limits_{|\gamma|+|\beta|\leq 2}\int_{\mathbf{R}^3}\frac{\left|\partial^\gamma\partial^\beta\mathbf{G}(\tau,x,\xi)\right|^2}{\mathbf{M}_-}d\xi\leq O(1)\delta^2.\end{cases} \tag{3.3}$$

On the other hand, if we choose $\delta>0$ sufficiently small, we have from (1.16) that

$$\begin{cases}\frac{\theta}{2}<\theta_-<\theta,\\ |\rho-\overline{\rho}|+|u|+\left|\theta-\overline{\theta}\right|<\eta_0,\end{cases}$$

and consequently the microscopic $H-$theorem (1.15) holds when $\tilde{\mathbf{M}}$ is taken as $\mathbf{M}_-=\mathbf{M}_{[\rho_-,0,\theta_-]}$.

We now give the energy estimates on the solutions $f(t,x,\xi)$ to the Cauchy problem (1.1) and (1.2) based on the a priori assumption (3.1). To do so, some basic estimates are given in Section 3.1 and then the desired energy estimates are obtained in the subsequent three subsections: The first one is on the estimates on the entropy $\eta(\rho,u,\theta)$ and the non-fluid component $\mathbf{G}$ and the other two are on the derivatives w.r.t. the weight of the local Maxwellian $\mathbf{M}$ and the derivatives w.r.t. the global Maxwellian $\mathbf{M}_-$, respectively.

3.1 Preliminary estimates

In this section we give some basic estimates related to the external forces $\Phi(x)$ and to the weighted integrals of the collision operators $Q(\mathbf{G},\mathbf{G})$ and $Q(\mathbf{M},\mathbf{G})$ w.r.t. $\mathbf{M}$ and $\mathbf{M}_-$. First we cite the following fundamental inequality (cf. [17])

Lemma 3.1 *Let Ω be the whole space $\mathbf{R}^3$, the half space $\mathbf{R}^3_+$, or the exterior domain of a bounded region with smooth boundary. Then*

$$\|g(x)\|_{L^6(\Omega)} \leq O(1)\|\nabla_x g(x)\|_{L^2(\Omega)}. \tag{3.4}$$

Based on Lemma 3.1, we have the following estimates concerning the external forces $\Phi(x)$

Lemma 3.2 *Under the assumption (3.1), we have for each $|\gamma|\leq 3, |\alpha|\leq 5$ that*

$$\int_{\mathbf{R}^3}|\partial^\alpha\Phi|^2\left|\partial^\gamma\left(\rho-\bar\rho,u,\theta-\bar\theta\right)\right|^2dx\leq O(1)\lambda^2\int_{\mathbf{R}^3}\left|\nabla_x\partial^\gamma\left(\rho-\bar\rho,u,\theta-\bar\theta\right)\right|^2dx. \tag{3.5}$$

Moreover

$$\begin{aligned}
&\sum_{\alpha>0,|\gamma|+|\beta|\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\Phi|^2\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)\sum_{\alpha>0,|\gamma|+|\beta|\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\Phi|^2\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\leq O(1)\lambda^2\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned} \tag{3.6}$$

Proof: Since (3.5) follows immediately from Holder's inequality and (3.4), we only prove (3.6) in the following. To this end, we only need to prove the second inequality in (3.6) because the first one holds trivially due to $\theta_-<\theta$.

In fact, Holder's inequality together with (3.4) imply

$$\begin{aligned}
\int_{\mathbf{R}^3}|\partial^\alpha\Phi|^2\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^2dx&\leq\left(\int_{\mathbf{R}^3}|\partial^\alpha\Phi|^3dx\right)^{\frac{2}{3}}\left(\int_{\mathbf{R}^3}\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^6dx\right)^{\frac{1}{3}}\\
&\leq O(1)\lambda^2\left\|\partial^\gamma\partial^\beta\mathbf{G}\right\|^2_{L^6(\mathbf{R}^3)}\\
&\leq O(1)\lambda^2\int_{\mathbf{R}^3}\left|\nabla_x\partial^\gamma\partial^\beta\mathbf{G}\right|^2dx.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\sum_{\alpha>0,|\gamma|+|\beta|\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\Phi|^2\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\leq O(1)\lambda^2\sum_{\alpha>0,|\gamma|+|\beta|\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\nabla_x\partial^\gamma\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\leq O(1)\lambda^2\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial^\gamma\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}$$

This is the second inequality in (3.6) and the proof of Lemma 3.2 is completed.

Now we turn to deal with the weighted integrals of the collision operators $Q(\mathbf{G},\mathbf{G})$ and $Q(\mathbf{M},\mathbf{G})$ w.r.t. $\mathbf{M}$ and $\mathbf{M}_-$.

Lemma 3.3 *Under the assumption (3.1), we have for $|\gamma|+|\beta|\leq 4$ that*

$$\begin{aligned}&\sum_{\beta'<\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|Q\left(\partial^{\beta-\beta'}\mathbf{M},\partial^{\beta'}\mathbf{G}\right)\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\leq O(1)\sum_{\beta'<\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial^{\beta'}\mathbf{G}\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau,\end{aligned}\tag{3.7}$$

$$\begin{aligned}&\sum_{0<\gamma'\leq\gamma,\beta'\leq\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|Q\left(\partial^{\gamma'}\partial^{\beta'}\mathbf{M},\partial^{\gamma-\gamma'}\partial^{\beta-\beta'}\mathbf{G}\right)\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\leq O(1)\delta^2\sum_{1\leq|\gamma'|\leq|\gamma|}\int_0^t\int_{\mathbf{R}^3}\left|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)\right|^2dxd\tau\\ &\quad+O(1)(\lambda+\delta)^2\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau,\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\sum_{\gamma'\leq\gamma,\beta'\leq\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|Q\left(\partial^{\gamma'}\partial^{\beta'}\mathbf{G},\partial^{\gamma-\gamma'}\partial^{\beta-\beta'}\mathbf{G}\right)\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\leq O(1)\delta^2\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.\end{aligned}\tag{3.9}$$

Here $\tilde{\mathbf{M}}$ can be taken as $\mathbf{M}$ or $\mathbf{M}_-$.

Since the proof of Lemma 3.1 is similar to that of Lemma 3.1 in [20], we omit the details for brevity.

As a direct corollary of Lemma 3.3, we have

Corollary 3.1 *Under the assumptions in Lemma 3.3, we have for $|\gamma|+|\beta|\leq 4$ that*

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|\partial^\gamma\partial^\beta Q(\mathbf{G},\mathbf{G})\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\leq O(1)\delta^2\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau,\\ &\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|\partial^\beta(L_{\mathbf{M}}\mathbf{G})-L_{\mathbf{M}}(\partial^\beta\mathbf{G})\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\leq O(1)\sum_{\beta'<\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial^{\beta'}\mathbf{G}\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau.\end{aligned}\tag{3.10}$$

Furthermore, if $\gamma>0$, we have

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}\left|\partial^\gamma\partial^\beta(L_{\mathbf{M}}\mathbf{G})-L_{\mathbf{M}}(\partial^\gamma\partial^\beta\mathbf{G})\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\leq O(1)\sum_{\beta'<\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial^\gamma\partial^{\beta'}\mathbf{G}\right|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\quad+O(1)\delta^2\sum_{1\leq|\gamma'|\leq|\gamma|}\int_0^t\int_{\mathbf{R}^3}\left|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)\right|^2dxd\tau\\ &\quad+O(1)(\lambda+\delta)^2\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.\end{aligned}\tag{3.11}$$

Here, as in Lemma 3.3, $\tilde{\mathbf{M}}$ *can be taken as* $\mathbf{M}$ *or* $\mathbf{M}_-$*.*

3.2 Lower order estimates

In this subsection, we will give the energy estimates on the entropy $\eta(\rho,u,\theta)$ and the non-fluid component $\mathbf{G}(t,x,\xi)$.

First, integrating (2.12) w.r.t. t over $[0,t]$ yields

$$\int_{\mathbf{R}^3}\left(\eta+\frac{3}{2}(\rho-\overline{\rho})\Phi\right)dx\Big|_0^t=\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\mathbf{G}}{\mathbf{M}}d\xi dxd\tau. \tag{3.12}$$

From (1.11) and the fact that there exists a positive constant $C>0$ such that

$$\begin{aligned}-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)}{\mathbf{M}}d\xi dxd\tau &\geq C\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})|^2}{\mathbf{M}}d\xi dxd\tau\\ &\geq C\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau,\end{aligned}$$

we have from Lemma 2.1, Corollary 2.1, and (3.1)-(3.3) that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\mathbf{G}}{\mathbf{M}}d\xi dxd\tau\leq-\frac{C}{2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau\\ &+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left(|\mathbf{G}_t|^2+|\nabla_x\mathbf{G}|^2+\lambda^2|\nabla_\xi\mathbf{G}|^2+\delta^2|\mathbf{G}|^2\right)d\xi dxd\tau.\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12) yields

$$\begin{aligned}&\int_{\mathbf{R}^3}\left(\eta+\tfrac{3}{2}(\rho-\overline{\rho})\Phi\right)(t)dx+\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau\\ &\leq O(1)\Big(\mathcal{E}(f_0)^2+\lambda^2\Big)+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left(|\mathbf{G}_t|^2+|\nabla_x\mathbf{G}|^2+\lambda^2|\nabla_\xi\mathbf{G}|^2+\delta^2|\mathbf{G}|^2\right)d\xi dxd\tau.\end{aligned} \tag{3.14}$$

For the non-fluid component $\mathbf{G}$, multiplying (1.10) by $\frac{\mathbf{G}}{\mathbf{M}}$ and integrating the result w.r.t. t,x, and ξ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$, we have from (1.14), Lemma 2.1, and Cauchy-Schwarz's inequality that

$$\begin{aligned}&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\ &\leq O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\ &+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\mathbf{G}|^2}{\mathbf{M}}\right)dxd\tau.\end{aligned} \tag{3.15}$$

Similarly, if we replace the weight $\mathbf{M}$ by the global Maxwellian $\mathbf{M}_-$, we have

$$\begin{aligned}&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\ &\leq O(1)\mathcal{E}(f_0)^2+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\mathbf{G}|^2}{\mathbf{M}_-}\right)dxd\tau.\end{aligned} \tag{3.16}$$

(3.14)-(3.16) give the complete lower order energy estimates.

3.3 Higher order estimates w.r.t. M

In this subsection, we will consider higher order energy estimates, i.e., $\partial^\gamma(\rho-\overline{\rho},u,\theta),\partial^\gamma\partial^\beta\mathbf{G}$, and $\partial^\gamma f$ for $|\gamma|\geq 1$ and $|\gamma|+|\beta|\geq 1$ w.r.t. the local Maxwellian $\mathbf{M}$.

First, for $\partial^\gamma(\rho-\overline{\rho},u,\theta)$ with $1\leq|\gamma|\leq 3$, we have the following lemma.

Lemma 3.4 *Under the assumptions in Lemma 3.2, we have for $j=1,2,3$ that*

$$
\begin{aligned}
&\sum_{|\gamma|=j}\int_{\mathbf{R}^3}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2\,dx+\sum_{|\gamma|=j}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial^\gamma(u,\theta)|^2\,dxd\tau\\
\leq\ &O(1)\mathcal{E}(f_0)^2+O(1)\sum_{|\gamma|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left(|\partial^\gamma\mathbf{G}_t|^2+|\nabla_x\partial^\gamma\mathbf{G}|^2\right)d\xi dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{1\leq|\gamma|\leq j+1}\int_0^t\int_{\mathbf{R}^3}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2\,dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{|\gamma|\leq j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left(|\partial^\gamma\mathbf{G}|^2+|\nabla_\xi\partial^\gamma\mathbf{G}|^2\right)d\xi dxd\tau.
\end{aligned}
\tag{3.17}
$$

Proof: The conservation laws (1.12) can be rewritten as

$$
\begin{cases}
\rho_t=-(\rho-\overline{\rho})\mathrm{div}_x u-\nabla_x(\rho-\overline{\rho})\cdot u-\overline{\rho}\mathrm{div}_x u-\nabla_x\overline{\rho}\cdot u,\\
u_{it}+\sum\limits_{j=1}^3 u_j u_{ix_j}+\frac{2}{3\rho}\left(\rho\theta-\overline{\rho}\overline{\theta}\right)_{x_i}+\frac{\rho-\overline{\rho}}{\rho}\Phi_{x_i}=-\int_{\mathbf{R}^3}\frac{\psi_i(\xi\cdot\nabla_x\Theta)}{\rho}d\xi\\
\qquad\qquad+\frac{1}{\rho}\sum\limits_{j=1}^3\left\{\mu(\theta)\left(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right\}_{x_j},\quad i=1,2,3,\\
\theta_t+\sum\limits_{j=1}^3\left(u_j\theta_{x_j}+\frac{2}{3}\theta u_{jx_j}\right)=-\int_{\mathbf{R}^3}\frac{\psi_4-\xi\cdot u}{\rho}(\xi\cdot\nabla_x\Theta)d\xi\\
\qquad\qquad+\frac{1}{\rho}\left\{\sum\limits_{j=1}^3\left(\kappa(\theta)\theta_{x_j}\right)_{x_j}+\frac{1}{2}\mu(\theta)\sum\limits_{i,j=1}^3\left(u_{ix_j}+u_{jx_i}\right)^2-\frac{2}{3}\mu(\theta)(\mathrm{div}_x u)^2\right\}.
\end{cases}
\tag{3.18}
$$

Once we obtained (3.18), (3.17) can be proved similar to that of [17] for the compressible Navier-Stokes equations with external forces by applying $\partial^\gamma(1\leq|\gamma|\leq 3)$ to $(3.18)_2$ and $(3.18)_3$, multiplying the resulting identities by $\rho\partial^\gamma u_i$ and $\frac{\overline{\rho}}{\theta}\partial^\gamma\theta$, taking the summation w.r.t. i from 1 to 3, and integrating the final results w.r.t. t and x over $[0,t]\times\mathbf{R}^3$. The only difference is to deal with the terms containing Θ, which can be estimated suitably by explioting Lemma 2.1, Corollary 2.1, and Lemma 3.2. We thus omit the details here for brevity. This completes the proof of Lemma 3.4.

Secondly, for $\partial^\gamma\partial^\beta\mathbf{G}$ with $|\gamma|+|\beta|\leq 4$, we have the following lemma whose proof will be given in the appendix for the brevity of presentation.

Lemma 3.5 *Under the assumptions in Lemma 3.4, we have for* $|\gamma|+|\beta|\leq 4$ *that*

$$\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
\leq\ & O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x\partial^\gamma(u,\theta)|^2+(\lambda+\delta)\sum_{1\leq|\gamma'|\leq|\gamma|}\left|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)\right|^2\right)dxd\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left(|\nabla_x\partial^\gamma\mathbf{G}|^2+\sum_{|\beta'|=|\beta|-1}\left|\nabla_x\partial^\gamma\partial^{\beta'}\mathbf{G}\right|^2\right)d\xi dx d\tau\\
&+O(1)\sum_{\beta'<\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)\left|\partial^\gamma\partial^{\beta'}\mathbf{G}\right|^2}{\mathbf{M}}d\xi dx d\tau.
\end{aligned}\tag{3.19}$$

Letting $\gamma=0$ and $\beta=0$ respectively in (3.19) yields

$$\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
\leq\ & O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}|\nabla_x\mathbf{G}|^2d\xi\right)dxd\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left(\sum_{|\beta'|=|\beta|-1}\left|\nabla_x\partial^{\beta'}\mathbf{G}\right|^2+\sum_{\beta'<\beta}\left|\partial^{\beta'}\mathbf{G}\right|^2\right)d\xi dx d\tau,\quad |\beta|\leq 4,
\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
\leq\ & O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x\partial^\gamma(u,\theta)|^2+(\lambda+\delta)\sum_{1\leq|\gamma'|\leq|\gamma|}\left|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)\right|^2\right)dxd\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left|\nabla_x\partial^\gamma\mathbf{G}\right|^2d\xi dx d\tau,\quad 1\leq|\gamma|\leq 3.
\end{aligned}\tag{3.21}$$

From the estimates (3.19)-(3.21), we can deduce, on the one hand, that we can reduce the estimates on the derivatives of the non-fluid part w.r.t. the velocity, i.e., $\partial^\gamma\partial^\beta\mathbf{G}$ with $|\beta|\geq 1, |\gamma|+|\beta|\leq 4$ to the estimates on the derivatives of the non-fluid part w.r.t. the space and time variables, i.e., $\partial^{\gamma'}\mathbf{G}$ for some $|\gamma'|\leq 4$. And on the other hand, we can get an estimate on $\partial^\gamma\mathbf{G}$. The above results are summarized in the following corollary.

Corollary 3.2 *Under the assumptions in Lemma 3.4, we have*

$$\begin{aligned}
&\sum_{\beta>0,|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{\beta>0,|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
\leq\ & O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\gamma|\leq 4}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau,
\end{aligned}\tag{3.22}$$

$$
\begin{aligned}
&\sum_{|\gamma|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\gamma|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\le O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\gamma|=j+1}|\partial^\gamma(u,\theta)|^2+(\lambda+\delta)\sum_{1\le|\gamma|\le j}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2\right)dxd\tau\\
&\quad+O(1)\sum_{|\gamma|=j+1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left|\partial^\gamma\mathbf{G}\right|^2d\xi dxd\tau,\quad j=1,2,3.
\end{aligned}\tag{3.23}
$$

A suitably linear combination of (3.17) and (3.23) yields

$$
\begin{aligned}
&\sum_{|\gamma|=j}\int_{\mathbf{R}^3}\left(|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\gamma|=j+1}|\partial^\gamma(u,\theta)|^2+\sum_{|\gamma|=j}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\
&\le O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\sum_{|\gamma|=j+1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left|\partial^\gamma\mathbf{G}\right|^2d\xi dxd\tau\\
&\quad+O(1)(\lambda+\delta)\sum_{1\le|\gamma|\le j+1}\int_0^t\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta)\right|^2dxd\tau,\quad j=1,2,3.
\end{aligned}\tag{3.24}
$$

Consequently

$$
\begin{aligned}
&\sum_{1\le|\gamma|\le 3}\int_{\mathbf{R}^3}\left(|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{2\le|\gamma|\le 4}|\partial^\gamma(u,\theta)|^2+\sum_{1\le|\gamma|\le 3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\
&\le O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\sum_{|\gamma|=4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\left|\partial^\gamma\mathbf{G}\right|^2d\xi dxd\tau\\
&\quad+O(1)(\lambda+\delta)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\le|\gamma|\le 4}|\partial^\gamma(\rho-\overline{\rho})|^2+\sum_{|\gamma|=1}|\partial^\gamma(u,\theta)|^2\right)dxd\tau.
\end{aligned}\tag{3.25}
$$

To obtain the the 4−th order derivatives w.r.t. t and x on $\mathbf{G}$, we need to work on the original system (1.1) to avoid the appearance of the 5−th order derivatives. This can be summarized in the following lemma whose proof can be found at the appendix.

Lemma 3.6 *Under the assumptions in Lemma 3.4, we have*

$$
\begin{aligned}
&\sum_{2\le|\gamma|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}}d\xi dx+\sum_{2\le|\gamma|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\le O(1)\mathcal{E}(f_0)^2+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)(\lambda+\delta)\sum_{1\le|\gamma|\le 4}\int_0^t\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta)\right|^2dxd\tau.
\end{aligned}\tag{3.26}
$$

Due to

$$\begin{aligned}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}}d\xi dx &= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{P}_0\big(\partial^\gamma(\mathbf{M}-\overline{\mathbf{M}})\big)|^2+|\mathbf{P}_1\big(\partial^\gamma(\mathbf{M}-\overline{\mathbf{M}})\big)+\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx\\ &\geq \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{P}_0\big(\partial^\gamma(\mathbf{M}-\overline{\mathbf{M}})\big)|^2}{\mathbf{M}}d\xi dx,\quad 2\leq|\gamma|\leq 4,\end{aligned}\tag{3.27}$$

we have by induction that

$$\begin{aligned}&\sum_{1\leq|\gamma|\leq 4}\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta)\right|^2dx\\ &\leq O(1)\int_{\mathbf{R}^3}\left(\sum_{|\gamma|=1}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\sum_{2\leq|\gamma|\leq 4}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}}d\xi\right)dx\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}\sum_{|\gamma|=4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx &\leq O(1)\sum_{1\leq|\gamma|\leq 4}\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta)\right|^2dx\\ &\quad+O(1)\sum_{2\leq|\gamma|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}}d\xi dx.\end{aligned}\tag{3.29}$$

Thus combining (2.13), (3.14), (3.15), (3.25), (3.26), (3.28) with (3.29) yield

$$\begin{aligned}&\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\left(\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})\right|^2+\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx\\ &+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\gamma|\leq 4}|\partial^\gamma(u,\theta)|^2+\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\ &\leq O(1)\left(\mathcal{E}(f_0)^2+\lambda^2\right)+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\ &+O(1)(\lambda+\delta)\sum_{1\leq|\gamma|\leq 4}\int_0^t\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho})\right|^2dxd\tau.\end{aligned}\tag{3.30}$$

To recover the estimates on $\partial^\gamma(\rho-\overline{\rho})$ in (3.30), we use the conservation laws (1.9) as in [15] to deduce that

$$\begin{aligned}\sum_{|\gamma|=j+1}\int_0^t\int_{\mathbf{R}^3}|\partial^\gamma(\rho-\overline{\rho})|^2dxd\tau &\leq O(1)\mathcal{E}(f_0)^2+O(1)\int_{\mathbf{R}^3}\left(\sum_{|\gamma|=j+1}|\partial^\gamma(\rho-\overline{\rho})|^2+\sum_{|\gamma|=j}|\partial^\gamma u|^2\right)dx\\ &\quad+O(1)\sum_{1\leq|\gamma|\leq j+1}\int_0^t\int_{\mathbf{R}^3}|\partial^\gamma(u,\theta)|^2dxd\tau\\ &\quad+O(1)(\lambda+\delta)\sum_{1\leq|\gamma|\leq j}\int_0^t\int_{\mathbf{R}^3}|\partial^\gamma(\rho-\overline{\rho})|^2dxd\tau\\ &\quad+O(1)\sum_{|\gamma|\leq j+1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau,\quad j=0,1,2,3.\end{aligned}\tag{3.31}$$

A suitably linear combination of (3.30) and (3.31) yields

$$\begin{aligned}&\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\left(\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})\right|^2+\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx\\ &+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\gamma|\leq 4}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\ &\leq O(1)\left(\mathcal{E}(f_0)^2+\lambda^2\right)+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.\end{aligned}\tag{3.32}$$

Putting (3.22) and (3.32) together, we finally have the following energy estimates w.r.t. $\mathbf{M}$

Corollary 3.3 *Under the assumptions of Lemma 3.4, we have*

$$
\begin{aligned}
&\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})\right|^2dx+\sum_{|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\gamma|\leq 4}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\sum_{|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau \qquad (3.33)\\
&\leq O(1)\left(\mathcal{E}(f_0)^2+\lambda^2\right)+O(1)(\lambda+\delta)\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}
$$

3.4 Higher order estimates w.r.t. $\mathbf{M}_-$

In this subsection, we will consider certain higher order energy estimates w.r.t. the global Maxwellian $\mathbf{M}_-=\mathbf{M}_{[\rho_-,0,\theta_-]}$ in order to close the a priori estimate (3.1). Compared to those w.r.t. the local Maxwellian $\mathbf{M}$, the only difference is that the fluid part and non-fluid part are no longer orthogonal w.r.t. the global Maxwellian $\mathbf{M}_-$. More precisely, from the proofs of Lemma 3.5 and Lemma 3.6, we can see that we have used the following identity

$$
\begin{cases}
\displaystyle\int_{\mathbf{R}^3}\frac{\mathbf{P}_0\big(\partial^\gamma\mathbf{M}_{[\rho-\overline{\rho},u,\theta]}\big)\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi=0,\\
\displaystyle\int_{\mathbf{R}^3}\frac{\mathbf{P}_0\Big(\partial^\gamma\Big(\overline{\rho}\Big(\mathbf{M}_{[1,u,\theta]}-\mathbf{M}_{[1,0,\overline{\theta}]}\Big)\Big)\Big)\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi=0,\quad 2\leq|\gamma|\leq 4,
\end{cases}
$$

while if the weight $\mathbf{M}$ is replaced by $\mathbf{M}_-$, the above identity does not hold any longer. As a result, there is an extra error term in the form of

$$
\sum_{1\leq|\gamma|\leq 4}\int_0^t\int_{\mathbf{R}^3}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2dxd\tau.
$$

Noticing this difference, we have by repeating the procedure to deduce (3.33) to obtain

$$
\begin{aligned}
&\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})\right|^2dx+\sum_{|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx\\
&+\sum_{|\gamma|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau \qquad (3.34)\\
&\leq O(1)\left(\mathcal{E}(f_0)^2+\lambda^2\right)+O(1)\sum_{1\leq|\gamma|\leq 4}\int_0^t\int_{\mathbf{R}^3}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2dxd\tau.
\end{aligned}
$$

Combining (3.33) with (3.34), we finally deduce that

$$
\begin{aligned}
&\sum_{|\gamma|\leq 4}\int_{\mathbf{R}^3}\left|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})\right|^2dx+\sum_{|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\gamma|\leq 4}|\partial^\gamma(\rho-\overline{\rho},u,\theta)|^2+\sum_{|\gamma|+|\beta|\leq 4}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dxd\tau \qquad (3.35)\\
&\leq O(1)\left(\mathcal{E}(f_0)^2+\lambda^2\right),
\end{aligned}
$$

which closes the a priori estimate (3.1) provided that we choose $\varepsilon>0$ and $\lambda_0>0$ sufficiently small such that

$$
\begin{cases}
\lambda<\lambda_0,\quad \mathcal{E}(f_0)<\varepsilon,\\
O(1)\left(\varepsilon^2+\lambda_0^2\right)<\delta^2.
\end{cases}
\qquad (3.36)
$$

4 The proof of Theorem 1.1

This section is devoted to the proof of the main result Theorem 1.1. The idea is to use the continuity argument to extend the local solution to all time by the closed a priori estimate (3.1). To do so, we first need to get the local existence of solutions to the Cauchy problem (1.1), (1.2) in the following energy space

$$\overline{\mathbf{H}}^4_{x,\xi}\left([0,T)\right)=\left\{g(t,x,\xi):\ \left|\ \begin{array}{c}\frac{\partial^\alpha\partial^\beta g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}}\in\mathbf{BC}_t\left([0,T),L^2_{x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)\right)\\ \|g\|_X\le M,\quad |\alpha|+|\beta|\le 4\end{array}\right.\right\},\tag{4.1}$$

which will be established in the coming subsection. Here $g(t,x,\xi)=f(t,x,\xi)-\overline{\mathbf{M}}(x,\xi)$, $M>0$ and $T>0$ are some positive constants, and $\|g\|_X$ is defined by

$$\begin{aligned}\|g\|_X=&\sup_{0\le t\le T}\left\{\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta g(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx\right\}\\&+\sum_{|\alpha|+|\beta|\le 4}\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\partial^\beta g(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxd\tau.\end{aligned}\tag{4.2}$$

4.1 Local existence

To construct local solution to the Cauchy problem (1.1), (1.2) in the energy space $\overline{\mathbf{H}}^4_{x,\xi}([0,T))$, for each given point $(t_0,x_0,\xi_0)\in\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3$, we first analysis the backward bi-characteristic curve $(X(t),\Xi(t))\equiv(X,\Xi)(t;t_0,x_0,\xi_0)$ of (1.1), (1.2) passing through (t_0,x_0,ξ_0) which is given by

$$\begin{cases}\frac{dX(t)}{dt}=\Xi(t),\\ \frac{d\Xi(t)}{dt}=-\nabla_x\Phi(X(t)),\\ (X(t),\Xi(t))|_{t=t_0}=(x_0,\xi_0).\end{cases}\tag{4.3}$$

From the assumption $(1.17)_1$, one can immediately deduce that there exists a positive constant $C(t_0,x_0,\xi_0)$ such that

$$|(X,\Xi)(t;t_0,x_0,\xi_0)|\le C(t_0,x_0,\xi_0)|t-t_0|+|x_0|.$$

Consequently the backward bi-characteristic curve $(X(t),\Xi(t))$ can be continued to the time $t=0$ and we use (X_0,Ξ_0) to denote $(X,\Xi)(0;t_0,x_0,\xi_0)$. Furthermore, it is easy to show that there exists a positive constant $C_1>0$ such that for $|t-t_0|\le T$

$$\begin{cases}1-C_1T\le\frac{\partial X_i(t;t_0,x_0,\xi_0)}{\partial x_{0i}},\frac{\partial\Xi_i(t;t_0,x_0,\xi_0)}{\partial\xi_{0i}}\le 1+C_1T,\quad i=1,2,3,\\ \left|\frac{\partial X_i(t;t_0,x_0,\xi_0)}{\partial x_{0j}}\right|+\left|\frac{\partial\Xi_i(t;t_0,x_0,\xi_0)}{\partial\xi_{0j}}\right|\le C_1T,\quad i\ne j,\ i,j=1,2,3,\\ \left|\frac{\partial X_i(t;t_0,x_0,\xi_0)}{\partial\xi_{0j}}\right|+\left|\frac{\partial\Xi_i(t;t_0,x_0,x_0j)}{\partial x_{0j}}\right|\le C_1T,\quad i,j=1,2,3,\end{cases}$$

from which we can deduce that if we choose $T>0$ sufficiently small, we have

$$\frac{1}{2}\le\left|\det\frac{\partial(X,\Xi)}{\partial(x_0,\xi_0)}\right|\le 2.\tag{4.4}$$

Note that $g(t,x,\xi)$ solves

$$\begin{cases} g_t + \xi\cdot\nabla_x g - \nabla_x\Phi\cdot\nabla_\xi g = L_{\overline{\mathbf{M}}}g + Q(g,g), \\ g(t,x,\xi)|_{t=0} = g_0(x,\xi), \end{cases} \tag{4.5}$$

where as in (1.13)

$$\left(L_{\overline{\mathbf{M}}}h\right)(\xi) = -\nu_{\overline{\mathbf{M}}}(\xi)h(\xi) + \sqrt{\overline{\mathbf{M}}(x,\xi)}K_{\overline{\mathbf{M}}}\left(\left(\frac{h}{\sqrt{\overline{\mathbf{M}}}}\right)(\xi)\right) \tag{4.6}$$

and $K_{\overline{\mathbf{M}}}(\cdot) = -K_{1\overline{\mathbf{M}}}(\cdot) + K_{2\overline{\mathbf{M}}}(\cdot)$ is a symmetric compact L^2-operator. And the collision frequency $\nu_{\overline{\mathbf{M}}}(\xi)$ and $K_{i\overline{\mathbf{M}}}(\cdot)$ have the following expressions

$$\begin{cases} \nu_{\overline{\mathbf{M}}}(\xi) = \overline{\rho}(x)\overline{\nu}(\xi), \quad \overline{\nu}(\xi) = \overline{\nu}(|\xi|) = \frac{2}{\sqrt{2\pi}}\left\{\left(\frac{1}{|\xi|}+|\xi|\right)\int_0^{|\xi|}\exp\left(-\frac{y^2}{2}\right)dy + \exp\left(-\frac{|\xi|^2}{2}\right)\right\}, \\ k_{1\overline{\mathbf{M}}}(\xi,\xi_*) = \frac{\pi\overline{\rho}(x)}{\sqrt{(2\pi)^3}}|\xi-\xi_*|\exp\left(-\frac{|\xi|^2}{4}-\frac{|\xi_*|^2}{4}\right), \\ k_{2\overline{\mathbf{M}}}(\xi,\xi_*) = \frac{2\overline{\rho}(x)}{\sqrt{2\pi}}|\xi-\xi_*|^{-1}\exp\left(-\frac{|\xi-\xi_*|^2}{8}-\frac{(|\xi|^2-|\xi_*|^2)^2}{8|\xi-\xi_*|^2}\right), \end{cases}$$

where $k_{i\overline{\mathbf{M}}}(\xi,\xi_*)(i=1,2)$ is the kernel of the operator $K_{i\overline{\mathbf{M}}}(i=1,2)$ respectively. Moreover, we have from $(4.3)_2$ that for each $0\leq s,t\leq T$

$$\begin{cases} |\Xi(t)-\Xi(s)| \leq O(1)T, \\ \frac{\overline{\rho}(X(t))}{\overline{\rho}(X(s))} = \exp(\Phi(X(s))-\Phi(X(t))) \leq 2. \end{cases} \tag{4.7}$$

Thus we have from $\left|\frac{d\overline{\nu}(r)}{dr}\right| \leq O(1)$ that there exists a positive constant $C_2\geq 2$ such that for all $0\leq s,t\leq T$

$$\begin{cases} \nu_{\overline{\mathbf{M}}}(\Xi(t)) \leq 2\nu_{\overline{\mathbf{M}}}(\Xi(s)) + C_2T, \\ 1+|\xi| \leq C_2\nu_{\overline{\mathbf{M}}}(\Xi(t)), \\ \frac{1}{\mathbf{M}_-(\Xi(t))} \leq \frac{C_2}{\mathbf{M}_-(\Xi(s))}. \end{cases} \tag{4.8}$$

By using the explicit expressions of $k_{i\overline{\mathbf{M}}}(\xi,\xi_*)(i=1,2)$, straightforward calculations yields the following lemma

Lemma 4.1 *If $0<\frac{\overline{\theta}}{2}<\theta_-<\overline{\theta}$, then for $i=1,2$*

$$\begin{cases} \sup_{\xi\in\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}|\mathcal{K}_i(\xi,\xi_*)|d\xi_*\right\} \leq O(1), \\ \sup_{\xi_*\in\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}|\mathcal{K}_i(\xi,\xi_*)|d\xi\right\} \leq O(1). \end{cases} \tag{4.9}$$

Here

$$\mathcal{K}_i(\xi,\xi_*) = \sqrt{\frac{\overline{\mathbf{M}}(x,\xi)}{\mathbf{M}_-(\xi_*)}}k_{i\overline{\mathbf{M}}}(\xi,\xi_*)\sqrt{\frac{\overline{\mathbf{M}}(x,\xi_*)}{\mathbf{M}_-(\xi)}}, \quad i=1,2. \tag{4.10}$$

Consequently

$$\left|\int_{\mathbf{R}^3}\frac{g(\xi)\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{h}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\xi)}{\mathbf{M}_-(\xi)}d\xi\right|\leq O(1)\left(\int_{\mathbf{R}^3}\frac{h^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\int_{\mathbf{R}^3}\frac{g^2}{\mathbf{M}_-}d\xi\right)^{\frac12},$$

i.e.

$$\int_{\mathbf{R}^3}\frac{\left|\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{h}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\xi)\right|^2}{\mathbf{M}_-(\xi)}d\xi\leq O(1)\int_{\mathbf{R}^3}\frac{h^2}{\mathbf{M}_-}d\xi. \tag{4.11}$$

Under the above preparations, we now turn to construct local solutions to the Cauchy problem (1.1), (1.2) in $\overline{\mathbf{H}}^4_{x,\xi}([0,T))$. For this purpose, we consider the following iterating sequence $\{g^n(t,x,\xi)\}$ $(n\geq 0)$ for solving (4.6)

$$\begin{cases} g^0(t,x,\xi)=g_0(x,\xi),\\ g^{n+1}_t+\xi\cdot\nabla_x g^{n+1}-\nabla_x\Phi\cdot\nabla_\xi g^{n+1}=L_{\overline{\mathbf{M}}}g^{n+1}+Q(g^n,g^n)\\ \qquad\qquad =-\nu_{\overline{\mathbf{M}}}(\xi)g^{n+1}+\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)+Q(g^n,g^n),\\ g^{n+1}(t,x,\xi)|_{t=0}=g_0(x,\xi). \end{cases} \tag{4.12}$$

Integrating $(4.12)_2$ along $(X,\Xi)(t;t_0,x_0,\xi_0)$, we have

$$\begin{aligned} & g^{n+1}(t_0,x_0,\xi_0)\\ ={}& \exp\left(-\int_0^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)g_0(X_0,\Xi_0)\\ &+\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\eta,X(\eta),\Xi(\eta))d\eta\\ &+\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))d\eta. \end{aligned} \tag{4.13}$$

We now show by induction that if $\|g_0\|_X\leq\frac{M}{3C_2}$, then $\|g^n\|_X\leq M$ for all n provided that M and T are chosen sufficiently small. To do so, if $\|g^n\|_X\leq M$, we have from (4.13) that

$$\begin{aligned} &\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(t_0,x_0,\xi_0)|^2}{\mathbf{M}_-(\xi_0)}d\xi_0dx_0\leq 2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(X(0),\Xi(0))|^2}{\mathbf{M}_-(\xi_0)}d\xi_0dx_0\\ &+2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\left|\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\eta,X(\eta),\Xi(\eta))d\eta\right|^2\frac{d\xi_0dx_0}{\mathbf{M}_-(\xi_0)}\\ &+2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\left|\int_0^{t_0}\exp\left(-\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))d\eta\right|^2\frac{d\xi_0dx_0}{\mathbf{M}_-(\xi_0)}\\ &=\sum_{j=1}^3 I_j. \end{aligned} \tag{4.14}$$

From $(4.8)_3$ and (4.4), we have

$$\begin{aligned} I_1&\leq 2C_2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(X(0),\Xi(0))|^2}{\mathbf{M}_-(\Xi(0))}d\xi_0dx_0\\ &\leq 4C_2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx, \end{aligned} \tag{4.15}$$

$$\begin{aligned}
I_2 &\leq 2\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\eta,X(\eta),\Xi(\eta))\right|^2}{\mathbf{M}_-(\xi_0)}d\xi_0dx_0d\eta\\
&\leq 2C_2\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\left(\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\right)(\eta,X(\eta),\Xi(\eta))\right|^2}{\mathbf{M}_-(\Xi(\eta))}d\xi_0dx_0d\eta\\
&\leq 4C_2\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(\eta,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxd\eta,
\end{aligned}\tag{4.16}$$

where in deducing (4.16), we have used (4.11).

For I_3, we have from $(4.8)_3$, Lemma 2.1, and Cauchy-Schwarz's inequality that

$$\begin{aligned}
I_3 &\leq O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\left|\int_0^{t_0}\exp\left(-2\int_\eta^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(s))ds\right)\nu_{\overline{\mathbf{M}}}(\Xi(\eta))d\eta\right|\\
&\quad\times\left|\int_0^{t_0}\nu_{\overline{\mathbf{M}}}(\Xi(\eta))^{-1}|Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))|^2d\eta\right|\frac{d\xi_0dx_0}{\mathbf{M}_-(\xi_0)}\\
&\leq O(1)\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\nu_{\overline{\mathbf{M}}}(\Xi(\eta))^{-1}\frac{|Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))|^2}{\mathbf{M}_-(\xi_0)}d\xi_0dx_0d\eta\\
&\leq O(1)\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\nu_{\overline{\mathbf{M}}}(\Xi(\eta))^{-1}\frac{|Q(g^n,g^n)(\eta,X(\eta),\Xi(\eta))|^2}{\mathbf{M}_-(\Xi(\eta))}d\xi_0dx_0d\eta\\
&\leq O(1)\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\nu_{\overline{\mathbf{M}}}(\xi)^{-1}\frac{|Q(g^n,g^n)(\eta,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxd\eta\\
&\leq O(1)M^4.
\end{aligned}\tag{4.17}$$

Inserting (4.15)-(4.17) into (4.14) deduce

$$\begin{aligned}
\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx &\leq 4C_2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx+O(1)\left(M^4+T^2\right)\\
&\quad+4C_2\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxdt.
\end{aligned}$$

By exploiting the Gronwall inequality and by choosing $T>0$ sufficiently small such that $\exp(4C_2T)<2$, we have from the above inequality that

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx\leq 8C_2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx+O(1)\left(M^4+T^2\right).\tag{4.18}$$

On the other hand, we can get by repeating the techniques used above that

$$\begin{aligned}
\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxdt &\leq C_2^2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx\\
&\quad+O(1)\left(M^4+T^2\right).
\end{aligned}\tag{4.19}$$

Combining (4.18) with (4.19) yields

$$\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx+\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxdt\\
&\quad\leq 4C_2^2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx+O(1)\left(M^4+T^2\right).
\end{aligned}\tag{4.20}$$

For the corresponding estimate for $\partial^\alpha\partial^\beta g^{n+1}(t,x,\xi)$ with $|\alpha|+|\beta|=j, 1\leq j\leq 4$, since

$$\begin{aligned}\partial^\alpha\partial^\beta\left(L_{\overline{\mathbf{M}}}g^{n+1}\right) &= L_{\overline{\mathbf{M}}}\left(\partial^\alpha\partial^\beta g^{n+1}\right)+2\sum_{0<|\alpha'|+|\beta'|\leq j}C_{\alpha,\beta}^{\alpha',\beta'}Q\left(\partial^{\alpha'}\partial^{\beta'}\overline{\mathbf{M}},\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\mathbf{G}\right)\\ &= -\nu_{\overline{\mathbf{M}}}(\xi)\partial^\alpha\partial^\beta g^{n+1}+\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{\partial^\alpha\partial^\beta g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)\\ &\quad +2\sum_{0<|\alpha'|+|\beta'|\leq j}C_{\alpha,\beta}^{\alpha',\beta'}Q\left(\partial^{\alpha'}\partial^{\beta'}\overline{\mathbf{M}},\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\mathbf{G}\right),\end{aligned}$$

it is easy to check that it solves

$$\begin{cases}\left(\partial^\alpha\partial^\beta g^{n+1}\right)_t+\xi\cdot\nabla_x\left(\partial^\alpha\partial^\beta g^{n+1}\right)-\nabla_x\Phi\cdot\nabla_\xi\left(\partial^\alpha\partial^\beta g^{n+1}\right)\\ \qquad = -\nu_{\overline{\mathbf{M}}}(\xi)\partial^\alpha\partial^\beta g^{n+1}+\sqrt{\overline{\mathbf{M}}}K_{\overline{\mathbf{M}}}\left(\frac{\partial^\alpha\partial^\beta g^{n+1}}{\sqrt{\overline{\mathbf{M}}}}\right)+2Q\left(\partial^\alpha\partial^\beta g^n,g^n\right)\\ \qquad\quad -\sum_{|\beta'|=1}C_\beta^{\beta'}\partial^{\beta'}\xi\cdot\nabla_x\left(\partial^\alpha\partial^{\beta-\beta'}g^{n+1}\right)\\ \qquad\quad +\sum_{|\alpha'|=1}C_\alpha^{\alpha'}\nabla_x\partial^{\alpha'}\Phi\cdot\nabla_\xi\left(\partial^{\alpha-\alpha'}\partial^\beta g^{n+1}\right)\\ \qquad\quad +S_{l.o.t},\\ \partial^\alpha\partial^\beta g^{n+1}\Big|_{t=0}=\partial^\alpha\partial^\beta g_0(x,\xi).\end{cases}\tag{4.21}$$

Here

$$\begin{aligned}S_{l.o.t} &= 2\sum_{0<|\alpha'|+|\beta'|\leq j}C_{\alpha,\beta}^{\alpha',\beta'}Q\left(\partial^{\alpha'}\partial^{\beta'}\overline{\mathbf{M}},\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\mathbf{G}\right)\\ &\quad +\sum_{0<|\alpha'|+|\beta'|<j}C_{\alpha,\beta}^{\alpha',\beta'}Q\left(\partial^{\alpha'}\partial^{\beta'}\mathbf{G},\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\mathbf{G}\right)\\ &\quad +\sum_{1<|\alpha'|\leq|\alpha|}C_\alpha^{\alpha'}\nabla_x\partial^{\alpha'}\Phi\cdot\nabla_\xi\partial^{\alpha-\alpha'}\partial^\beta g^{n+1}.\end{aligned}$$

Compared with the estimate on $g^{n+1}(t,x,\xi)$, the only difference is to estimate the first and the third tems in $S_{l.o.t.}$ when the order of the derivatives of the potential of the external forces $\Phi(x)$ w.r.t. x is more than 3 which can be controlled suitably by using Lemma 3.2. Hence

$$\begin{aligned}&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx+\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\partial^\beta g^{n+1}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxdt\\ &\qquad\leq 4C_2^2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta g_0(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx+O(1)\left(M^4+T^2\right).\end{aligned}\tag{4.22}$$

Thus (4.20) together with (4.23) imply

$$\begin{aligned}\|g^{n+1}\|_X^2 &\leq 4C_2^2\|g_0\|_X^2+O(1)\left(M^4+T^2\right)\\ &\leq \frac{4M^2}{9}+O(1)\left(M^4+T^2\right)\\ &\leq M^2\end{aligned}\tag{4.23}$$

provided that we choose $M>0$ and $T>0$ sufficiently small such that

$$O(1)\left(M^4+T^2\right)\leq\frac{5M^2}{9}.$$

Moreover, for each $|\alpha| + |\beta| \leq 4$, similar to that of (4.13), we can get a similar integral formula for the solution $\partial^\alpha \partial^\beta g^{n+1}(t, x, \xi)$ of (4.21), from which and (4.23) one can easily verify that this integral equation can be solved on the existence of $g^{n+1}(t, x, \xi)$ by the Neumann series for small T_1, and further that $\frac{\partial^\alpha \partial^\beta g^{n+1}(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left([0, T), L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right)$ provided that

$$\frac{\partial^\alpha \partial^\beta g^n(t, x, \xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left([0, T), L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right).$$

This observation together with (4.23) imply that $g^{n+1}(t, x, \xi) \in \overline{\mathbf{H}}^4_{x,\xi}([0, T))$.

To show that $\{g^n(t, x, \xi)\}$ is a Cauchy sequence in $\overline{\mathbf{H}}^4_{x,\xi}([0, T))$, we set

$$h^n(t, x, \xi) = g^{n+1}(t, x, \xi) - g^n(t, x, \xi), \quad n \geq 0,$$

then $h^n(t, x, \xi)(n \geq 1)$ solves

$$\begin{cases} h^n_t + \xi \cdot \nabla_x h^n - \nabla_x \Phi \cdot \nabla_\xi h^n \\ \qquad = -\nu_{\overline{\mathbf{M}}}(\xi) h^n + \sqrt{\overline{\mathbf{M}}} K_{\overline{\mathbf{M}}}\left(\frac{h^n}{\sqrt{\overline{\mathbf{M}}}}\right) + Q(g^n, h^{n-1}) + Q(g^{n-1}, h^{n-1}), \\ h^n(t, x, \xi)|_{t=0} = 0. \end{cases} \tag{4.24}$$

From (4.24), we have by repeating the argument to deduce (4.23) that

$$\|h^n\|_X \leq \frac{1}{2}\|h^{n-1}\|_X, \quad n \geq 1 \tag{4.25}$$

provided that we choose $M > 0$ and $T > 0$ sufficiently small. Thus $\{g^n(t, x, \xi)\}$ is a Cauchy sequence in $\overline{\mathbf{H}}^4_{x,\xi}([0, T))$ and we finally arrive at

Lemma 4.2 (Local existence) *For any sufficiently small constant $M > 0$, there exists a positive constant $T^*(M) > 0$ such that if*

$$\mathcal{E}(f_0) = \sum_{|\alpha|+|\beta|\leq 4} \left\| \frac{\partial^\alpha \partial^\beta (f_0(x, \xi) - \overline{\mathbf{M}}(x, \xi))}{\sqrt{\mathbf{M}_-(\xi)}} \right\|_{L^2_{x,\xi}(\mathbf{R}^3 \times \mathbf{R}^3)} \leq \frac{M}{3C_2},$$

then the Cauchy problem (1.1), (1.2) admits a unique classical solution $f(t, x, \xi) \in \overline{\mathbf{H}}^4_{x,\xi}([0, T^*(M)))$ on $[0, T^*(M)) \times \mathbf{R}^3 \times \mathbf{R}^3$ such that $f(t, x, \xi) \geq 0$ and

$$\sup_{0 \leq t \leq T^*(M)} \sum_{|\alpha|+|\beta|\leq 4} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \partial^\beta (f(t, x, \xi) - \overline{\mathbf{M}}(x, \xi))|^2}{\mathbf{M}_-(\xi)} d\xi dx \leq M.$$

4.2 Global existence

By combining the local existence result Lemma 4.2 and the energy estimates obtained in Section 3, we can conclude immediately that the Cauchy problem (1.1), (1.2) has a unique global classical solution $f(t, x, \xi) \in \mathbf{H}^4_{x,\xi}(\mathbf{R}^+)$ satisfying $f(t, x, \xi) \geq 0$.

To complete the proof of Theorem 1.1, we show that (1.18) holds. In fact, we have from (3.35) that

$$\begin{cases} \sum_{|\gamma|\leq 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\leq O(1),\\ \sum_{|\gamma|\leq 3}\int_0^\infty\left|\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx\right|d\tau\leq O(1)\sum_{|\gamma|\leq 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2+|\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dxd\tau\\ \qquad\leq O(1),\\ \sum_{1\leq|\gamma|\leq 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(\mathbf{M}-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dxd\tau\leq O(1),\\ \sum_{1\leq|\gamma|\leq 3}\int_0^\infty\left|\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(\mathbf{M}-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dx\right|d\tau\leq O(1)\sum_{|\gamma|\leq 4}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(\mathbf{M}-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dxd\tau\\ \qquad\leq O(1). \end{cases} \tag{4.26}$$

Consequently

$$\lim_{t\to\infty}\sum_{1\leq|\gamma|\leq 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\gamma(\mathbf{M}-\overline{\mathbf{M}})|^2+|\partial_x^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx=0. \tag{4.27}$$

Since

$$\begin{aligned}\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}d\xi\leq{}& O(1)\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\sum_{|\alpha|=3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha(\mathbf{M}-\overline{\mathbf{M}})|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\\ &+O(1)\left(\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha(\mathbf{M}-\overline{\mathbf{M}})|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\sum_{|\alpha|=2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha(\mathbf{M}-\overline{\mathbf{M}})|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12},\end{aligned}$$

we have from (4.27) that

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\left(\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi=0. \tag{4.28}$$

Thus

$$\begin{aligned}\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|f(t,x,\xi)-\overline{\mathbf{M}}(x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi&\leq O(1)\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}(t,x,\xi)-\overline{\mathbf{M}}(x,\xi)|^2+|\mathbf{G}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi\\ &=0,\end{aligned}$$

which is (1.18). And this completes the proof of Theorem 1.1.

5 Appendix

In the last section, we will give the proof of Lemma 3.5 and Lemma 3.6 respectively.

5.1 The proof of Lemma 3.5

Applying $\partial^\gamma\partial^\beta(|\gamma|+|\beta|\leq 4)$ to (1.10) and integrating its product with $\frac{\partial^\gamma\partial^\beta\mathbf{G}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$ yield

$$\begin{aligned}\frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_0^t &= -\frac{1}{2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}^2}\mathbf{M}_t d\xi dxd\tau\\ &\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}\cdot\partial^\gamma\partial^\beta(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))}{\mathbf{M}}d\xi dxd\tau\\ &\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}\cdot\partial^\gamma\partial^\beta(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\ &\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}\cdot\partial^\gamma\left(\nabla_x\Phi\cdot\nabla_\xi\partial^\beta\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau\\ &\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}\cdot\partial^\gamma\partial^\beta(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\ &\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}\cdot\partial^\gamma\partial^\beta(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\ &:=\sum_{j=7}^{12}I_j,\end{aligned}\tag{5.1}$$

where I_7-I_{12} are the corresponding terms in the above equation.

Due to

$$\begin{cases}\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})=\left\{\sqrt{R\theta}\sum_{j=1}^3 A_j\left(\frac{\xi-u}{\sqrt{R\theta}}\right)\frac{\theta_{x_j}}{\theta}+\sum_{i,j=1}^3 B_{jk}\left(\frac{\xi-u}{\sqrt{R\theta}}\right)u_{kx_j}\right\}\mathbf{M},\\ \partial^\gamma\partial^\beta\left(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G})\right)=\xi\cdot\nabla_x\partial^\gamma\partial^\beta\mathbf{G}+\sum_{|\beta'|=1}C_\beta^{\beta'}\partial^{\beta'}\xi\cdot\nabla_x\partial^\gamma\partial^{\beta-\beta'}\mathbf{G}\\ \qquad -\sum_{j=0}^4\partial^\gamma\left\{\langle\xi\cdot\nabla_x\mathbf{G},\chi_j\rangle_{\mathbf{M}}\partial^\beta\chi_j\right\},\end{cases}\tag{5.2}$$

we have from Cauchy-Schwarz's inequality that

$$I_7\leq O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau,\tag{5.3}$$

$$\begin{aligned}I_8&\leq\frac{\sigma}{4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}|\partial^\gamma\partial^\beta(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))|^2}{\mathbf{M}}d\xi dxd\tau\\ &\leq\frac{\sigma}{4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial^\gamma(u,\theta)|^2dxd\tau\\ &\quad+O(1)(\lambda+\delta)\sum_{1\leq|\gamma'|\leq|\gamma|}\int_0^t\int_{\mathbf{R}^3}|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)|^2dxd\tau,\end{aligned}\tag{5.4}$$

and

$$
\begin{aligned}
I_9 \le{}& \tfrac{\sigma}{4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{|\beta'|=|\beta|-1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\gamma\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{1\le|\gamma'|\le|\gamma|}\int_0^t\int_{\mathbf{R}^3}|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)|^2dxd\tau.
\end{aligned}\tag{5.5}
$$

As to I_{10}, due to

$$
\partial^\gamma\left\{\nabla_x\Phi\cdot\nabla_\xi\partial^\beta\mathbf{G}\right\}=\nabla_x\Phi\cdot\nabla_\xi\partial^\gamma\partial^\beta\mathbf{G}+\sum_{0<\gamma'\le\gamma}C_\gamma^{\gamma'}\nabla_x\partial^{\gamma'}\Phi\cdot\nabla_\xi\partial^{\gamma-\gamma'}\partial^\beta\mathbf{G},
$$

we have from Lemma 3.2 that

$$
I_{10}\le O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.\tag{5.6}
$$

Moreover, since

$$
\begin{aligned}
\partial^\gamma\partial^\beta(L_{\mathbf{M}}\mathbf{G})={}&L_{\mathbf{M}}\left(\partial^\gamma\partial^\beta\mathbf{G}\right)+2\sum_{0<\beta'\le\beta}C_\beta^{\beta'}Q\left(\partial^{\beta'}\mathbf{M},\partial^\gamma\partial^{\beta-\beta'}\mathbf{G}\right)\\
&+2\sum_{0<\gamma'\le\gamma,\beta'\le\beta}C_{\gamma,\beta}^{\gamma',\beta'}Q\left(\partial^{\gamma'}\partial^{\beta'}\mathbf{M},\partial^{\gamma-\gamma'}\partial^{\beta-\beta'}\mathbf{G}\right),
\end{aligned}
$$

we have from Lemma 2.1, Lemma 3.3, and Corollary 3.1 that

$$
\begin{aligned}
I_{11}\le{}&-\tfrac{3\sigma}{4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{\beta'<\beta}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{1\le|\gamma'|\le|\gamma|}\int_0^t\int_{\mathbf{R}^3}|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)|^2dxd\tau.
\end{aligned}\tag{5.7}
$$

Similarly, we deduce from Lemma 2.1, Lemma 3.3, and Corollary 3.1 that

$$
I_{12}\le O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.\tag{5.8}
$$

Inserting (5.3)-(5.8) into (5.1), we can get (3.19) immediately. This completes the proof of Lemma 3.5.

5.2 The proof of Lemma 3.6

For Lemma 3.6, since

$$\left(f-\overline{\mathbf{M}}\right)_t+\xi\cdot\nabla_x\left(f-\overline{\mathbf{M}}\right)-\nabla_x\Phi\cdot\nabla_\xi\left(f-\overline{\mathbf{M}}\right)=L_{\mathbf{M}}\mathbf{G}+Q(\mathbf{G},\mathbf{G}),\tag{5.9}$$

we have by applying $\partial^\gamma(2\leq|\gamma|\leq 4)$ to (5.9), multiplying it by $\frac{\partial^\gamma(f-\overline{\mathbf{M}})}{\mathbf{M}}$, and integrating the final equation w.r.t. t,x, and ξ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$ that

$$\begin{aligned}
&\tfrac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial^\gamma(f-\overline{\mathbf{M}})\right|^2}{\mathbf{M}}d\xi dx\Big|_0^t\\
=&-\tfrac{1}{2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial^\gamma(f-\overline{\mathbf{M}})\right|^2}{\mathbf{M}^2}\left(\mathbf{M}_t+\xi\cdot\nabla_x\mathbf{M}\right)d\xi dxd\tau\\
&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma(f-\overline{\mathbf{M}})\partial^\gamma(\nabla_x\Phi\cdot\nabla_\xi(f-\overline{\mathbf{M}}))}{\mathbf{M}}d\xi dxd\tau+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma(f-\overline{\mathbf{M}})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma(f-\overline{\mathbf{M}})\partial^\gamma(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau:=\sum_{j=13}^{16}I_j,
\end{aligned}\tag{5.10}$$

where $I_{13}-I_{16}$ are the corresponding terms in the above equation.

Now we estimate $I_j(j=13,14,15,16)$ term by term. First from Lemma 3.2, we have

$$\begin{aligned}
I_{13}\leq&\, O(1)(\lambda+\delta)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial^\gamma(f-\overline{\mathbf{M}})\right|^2}{\mathbf{M}_-}d\xi dxd\tau\\
\leq&\, O(1)(\lambda+\delta)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\gamma'|\leq|\gamma|}|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dxd\tau,
\end{aligned}\tag{5.11}$$

and

$$\begin{aligned}
I_{14}\leq&\, O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{1\leq|\gamma'|\leq|\gamma|}\int_0^t\int_{\mathbf{R}^3}|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)|^2dxd\tau.
\end{aligned}\tag{5.12}$$

For I_{15}, due to

$$\begin{cases}
\displaystyle\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M}_{[\rho-\overline{\rho},u,\theta]})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi=0,\\
\displaystyle\int_{\mathbf{R}^3}\frac{\mathbf{P}_0\left(\partial^\gamma\left(\overline{\rho}\left(\mathbf{M}_{[1,u,\theta]}-\mathbf{M}_{[1,0,\overline{\theta}]}\right)\right)\right)\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi=0,\quad 2\leq|\gamma|\leq 4,
\end{cases}$$

we have from Lemma 2.1, Lemma 3.2, Lemma 3.3, and Corollary 3.1 that

$$\begin{aligned}
I_{15}=&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial^\gamma\mathbf{M}_{[\rho-\overline{\rho},u,\theta]})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\left(\partial^\gamma\left(\overline{\rho}\left(\mathbf{M}_{[1,u,\theta]}-\mathbf{M}_{[1,0,\overline{\theta}]}\right)\right)\right)\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\mathbf{G}\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
\leq&-\frac{\sigma}{2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)(\lambda+\delta)\sum_{1\leq|\gamma'|\leq|\gamma|}\int_0^t\int_{\mathbf{R}^3}|\partial^{\gamma'}(\rho-\overline{\rho},u,\theta)|^2dxd\tau\\
&+O(1)(\lambda+\delta)\sum_{|\gamma'|+|\beta'|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}\tag{5.13}$$

Finally by employing the same argument as above, we can estimate I_{16} as follows

$$\begin{aligned} I_{16} \le O(1)(\lambda+\delta) & \sum_{|\gamma'|+|\beta'|\le 4} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ & + O(1)(\lambda+\delta) \sum_{1\le|\gamma'|\le|\gamma|} \int_0^t \int_{\mathbf{R}^3} |\partial^{\gamma'}(\rho-\overline{\rho}, u, \theta)|^2 dx d\tau. \end{aligned} \tag{5.14}$$

Substituting (5.11)-(5.14) into (5.10) yields

$$\begin{aligned} & \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^{\gamma}(f-\overline{\mathbf{M}})|^2}{\mathbf{M}} d\xi dx + \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma}\mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ \le & \, O(1)\mathcal{E}(f_0)^2 + O(1)(\lambda+\delta) \sum_{|\gamma'|+|\beta'|\le 4} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ & + O(1)(\lambda+\delta) \sum_{1\le|\gamma'|\le|\gamma|} \int_0^t \int_{\mathbf{R}^3} |\partial^{\gamma'}(\rho-\overline{\rho}, u, \theta)|^2 dx d\tau. \end{aligned} \tag{5.15}$$

Having obtained (5.15), (3.26) follows immediately. This completes the proof of Lemma 3.6.

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