RECOVERY OF A SURFACE WITH BOUNDARY AND ITS CONTINUITY AS A FUNCTION OF ITS TWO FUNDAMENTAL FORMS

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Abstract. If a field $\boldsymbol{A}$ of class $\mathcal{C}^2$ of positive-definite symmetric matrices of order two and a field $\boldsymbol{B}$ of class $\mathcal{C}^1$ of symmetric matrices of order two satisfy together the Gauss and Codazzi-Mainardi equations in a connected and simply-connected open subset ω of $\mathbb{R}^2$, then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$, uniquely determined up to proper isometries in $\mathbb{R}^3$, such that $\boldsymbol{A}$ and $\boldsymbol{B}$ are the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$. Let $\dot{\boldsymbol{\theta}}$ denote the equivalence class of $\boldsymbol{\theta}$ modulo proper isometries in $\mathbb{R}^3$ and let $\mathcal{F} : (\boldsymbol{A},\boldsymbol{B}) \to \dot{\boldsymbol{\theta}}$ denote the mapping determined in this fashion.

The first objective of this paper is to show that, if ω satisfies a certain "geodesic property" (in effect a mild regularity assumption on the boundary of ω) and if the fields $\boldsymbol{A}$ and $\boldsymbol{B}$ and their partial derivatives of order ≤ 2, resp. ≤ 1, have continuous extensions to $\overline{\omega}$, the extension of the field $\boldsymbol{A}$ remaining positive-definite on $\overline{\omega}$, then the immersion $\boldsymbol{\theta}$ and its partial derivatives of order ≤ 3 also have continuous extensions to $\overline{\omega}$.

The second objective is to show that, if ω satisfies the geodesic property and is bounded, the mapping $\mathcal{F}$ can be extended to a mapping that is locally Lipschitz-continuous with respect to the topologies of the Banach spaces $\mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^1(\overline{\omega})$ for the continuous extensions of the matrix fields $(\boldsymbol{A},\boldsymbol{B})$, and $\mathcal{C}^3(\overline{\omega})$ for the continuous extensions of the immersions $\boldsymbol{\theta}$.

1. Introduction

All notations used, but not defined, here are defined in the next sections. Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$, let $\mathbb{S}^2$, resp. $\mathbb{S}^2_>$, denote the set of symmetric, resp. positive-definite symmetric, matrices of order 2, and let matrix fields $\boldsymbol{A} = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $\boldsymbol{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2)$ be given that satisfies the Gauss and Codazzi-Mainardi equations in ω, i.e.,

$$C^p{}_{\alpha\beta\sigma} = 0 \text{ in } \omega \text{ for all } \alpha,\beta,\sigma \in \{1,2\} \text{ and } p \in \{1,2,3\},$$

where

$$C^\tau{}_{\alpha\beta\sigma} = \partial_\sigma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\sigma} + \Gamma^\gamma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\gamma_{\alpha\sigma}\Gamma^\tau_{\beta\gamma} - b_{\alpha\beta}b^\tau_\sigma + b_{\alpha\sigma}b^\tau_\beta,$$
$$C^3{}_{\alpha\beta\sigma} = \partial_\sigma b_{\alpha\beta} - \partial_\beta b_{\alpha\sigma} + \Gamma^\gamma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\gamma_{\alpha\sigma} b_{\beta\gamma},$$

and the Christoffel symbols of the second kind $\Gamma^\tau_{\alpha\beta}$ associated with the matrix field $\boldsymbol{A}$ are defined by

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2} a^{\tau\sigma} \left(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta} \right),$$

where $(a^{\alpha\beta})$ denotes the inverse of the matrix $(a_{\alpha\beta})$, and $b^\tau_\sigma = a^{\beta\tau} b_{\beta\sigma}$.

Then the *fundamental theorem of surface theory* (recalled in Theorem 1 for convenience) asserts that there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$, uniquely determined up to proper isometries in $\mathbb{R}^3$, such that

$$a_{\alpha\beta}(y) = \partial_\alpha \boldsymbol{\theta}(y) \cdot \partial_\beta \boldsymbol{\theta}(y) \text{ for all } y \in \omega,$$
$$b_{\alpha\beta}(y) = \partial_{\alpha\beta} \boldsymbol{\theta}(y) \cdot \frac{\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)}{|\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)|} \text{ for all } y \in \omega,$$

i.e., such that the fields $\boldsymbol{A}$ and $\boldsymbol{B}$ are the first and second fundamental forms of the immersed surface $\boldsymbol{\theta}(\omega)$.

Hence there exists a well-defined mapping $\mathcal{F}$ that associates with any pair of matrix fields $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^1(\omega; \mathbb{S}^2)$ satisfying the Gauss and Codazzi-Mainardi equations a well-defined equivalence class in the quotient set $\mathcal{C}^3(\omega; \mathbb{R}^3)/\mathcal{R}$, where $(\boldsymbol{\phi}; \boldsymbol{\theta}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{c} \in \mathbb{R}^3$ and a proper orthogonal matrix $\boldsymbol{Q}$ of order 3 such that $\boldsymbol{\phi}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y)$ for all $y \in \omega$.

Our first objective is to extend this classical existence and uniqueness result "*up to the boundary*" of the set ω. More specifically, we assume that the set ω satisfies what we call the "*geodesic property*" (in effect, a mild smoothness assumption on the boundary $\partial\omega$; cf. Definition 2) and that the functions $a_{\alpha\beta}$, resp. $b_{\alpha\beta}$, and their partial derivatives of order ≤ 2, resp. ≤ 1, can be extended by continuity to the closure $\overline{\omega}$, the extension of $\boldsymbol{A}$ defined in this fashion remaining positive-definite over the set $\overline{\omega}$. Then we show that the immersion $\boldsymbol{\theta}$ and its partial derivatives of order ≤ 3 can be also extended by continuity to $\overline{\omega}$ (cf. Theorem 2 and Corollary 1).

Let $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>)$ denote the set formed by the positive-definite symmetric matrix fields that, together with their partial derivatives of order ≤ 2 admit such continuous extensions, the extensions remaining positive-definite on $\overline{\omega}$. Let $\mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$, resp. $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$, denote the space formed by the symmetric matrix fields, resp. the space formed by the vector fields, that, together with their partial derivatives of order ≤ 1, resp. ≤ 3, admit such continuous extensions. Then the above result shows that there exists a mapping $\overline{\mathcal{F}}$ that associates with any pair of matrix fields $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ satisfying the Gauss and Codazzi-Mainardi equations in ω a well-defined element in the quotient set $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)/\mathcal{R}$. The mapping $\overline{\mathcal{F}}$ thus maps pairs of matrix fields defined "*up to the boundary*" into equivalence classes of vector fields also defined "*up to the boundary*".

Our second objective is to study the continuity of the mapping $\overline{\mathcal{F}}$. In this direction, we show that, if the set ω is *bounded* and again satisfies the "*geodesic property*", the mapping $\overline{\mathcal{F}}_0$ is *locally Lipschitz-continuous* when the vector spaces $\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$, $\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$, and $\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ are equipped with their natural norms of Banach space (cf. Theorem 3 and Corollary 2).

Note that the issue of continuity of the mapping $\mathcal{F}$ defined earlier, i.e., "when the boundary of the open set ω is ignored", was recently addressed by Ciarlet [2] who showed, *albeit* by means of a completely different approach, that $\mathcal{F}$ is a continuous mapping when the spaces $\mathcal{C}^2(\omega;\mathbb{S}^2)$, $\mathcal{C}^2(\omega;\mathbb{S}^2)$, and $\mathcal{C}^3(\omega;\mathbb{R}^3)/\mathcal{R}$ are equipped with their natural Fréchet topologies.

The results of this paper have been extended by Szopos [11] to "multi-dimensional differential geometry", i.e., to the isometric immersion in $\mathbb{R}^{p+q}$ of a connected and simply-connected open subset ω of $\mathbb{R}^p$ equipped with a Riemannian metric.

2. Preliminaries

This section gathers the main conventions, notations, and definitions used in this article, as well as various preliminary results that will be subsequently needed.

Throughout this article, Greek indices and exponents vary in the set $\{1,2\}$ while Latin indices and exponents vary in the set $\{1,2,3\}$, save when they are used for indexing sequences. The summation convention with respect to repeated indices and exponents is systematically used in conjunction with this rule.

The Euclidean inner product of $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ and the Euclidean norm of $\boldsymbol{a}\in\mathbb{R}^3$ are denoted by $\boldsymbol{a}\cdot\boldsymbol{b}$ and $|\boldsymbol{a}|$. For any integer $n>1$, the notations $\mathbb{M}^n,\mathbb{S}^n,\mathbb{S}^n_>$, and $\mathbb{O}^n_+$, respectively designate the sets of all square matrices, of all symmetric matrices, of all positive-definite symmetric matrices, and of all proper orthogonal matrices, of order n. The notation $(t_{\alpha\beta})$ designates the matrix of $\mathbb{M}^2$ with $t_{\alpha\beta}$ as its elements, the first index α being the row index. The spectral norm of a matrix $\boldsymbol{A}\in\mathbb{M}^n$ is

$$|\boldsymbol{A}| := \sup\{|\boldsymbol{A}\boldsymbol{v}|;\, \boldsymbol{v}\in\mathbb{R}^n,\, |\boldsymbol{v}|\le 1\}.$$

In any metric space, the open ball with center x and radius $\delta>0$ is denoted $B(x;\delta)$. The notation $f|_U$ designates the restriction to a set U of a function f.

The coordinates of a point $y\in\mathbb{R}^2$ are denoted y_α. Partial derivative operators of order $\ell\ge 1$ are denoted $\partial^{\boldsymbol{p}}$, where $\boldsymbol{p}=(p_\alpha)\in\mathbb{N}^2$ is a multi-index satisfying $|\boldsymbol{p}|:=\sum_\alpha p_\alpha=\ell$. Partial derivative operators of the first, second, and third order are also denoted $\partial_\alpha:=\partial/\partial y_\alpha$, $\partial_{\alpha\beta}:=\partial^2/\partial y_\alpha\partial y_\beta$, and $\partial_{\alpha\beta\gamma}:=\partial^3/\partial y_\alpha\partial y_\beta\partial y_\gamma$.

The space of all continuous functions from a normed space X into a normed space Y is denoted $\mathcal{C}^0(X;Y)$, or simply $\mathcal{C}^0(X)$ if $Y=\mathbb{R}$.

Let ω be an open subset of $\mathbb{R}^2$. For any integer $\ell\ge 1$, the space of all real-valued functions that are ℓ times continuously differentiable in ω is denoted $\mathcal{C}^\ell(\omega)$. Similar definitions hold for the spaces $\mathcal{C}^\ell(\omega;\mathbb{R}^3)$, $\mathcal{C}^\ell(\omega;\mathbb{M}^2)$, and $\mathcal{C}^\ell(\omega;\mathbb{S}^2)$.

We recall that a mapping $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbb{R}^3)$ is an *immersion* if the vectors $\partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \omega$. We also define the set

$$\mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^1(\omega; \mathbb{S}^2) := \{(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\omega; \mathbb{S}^2) \times \mathcal{C}^1(\omega; \mathbb{S}^2); \\ \boldsymbol{A}(y) \in \mathbb{S}^2_> \text{ for all } y \in \omega\}.$$

Central to this paper is the following notion of spaces of functions, vector fields, or matrix fields, "of class $\mathcal{C}^\ell$ up to the boundary of ω".

Definition 1. Let ω be an open subset of $\mathbb{R}^2$. For any integer $\ell \geq 1$, we define the *space* $\mathcal{C}^\ell(\overline{\omega})$ as the subspace of the space $\mathcal{C}^\ell(\omega)$ that consists of all functions $f \in \mathcal{C}^\ell(\omega)$ that, together with all their partial derivatives $\partial^{\boldsymbol{p}} f$, $1 \leq |\boldsymbol{p}| \leq \ell$, possess continuous extensions to the closure $\overline{\omega}$ of ω. Equivalently, a function $f : \omega \to \mathbb{R}$ belongs to $\mathcal{C}^\ell(\overline{\omega})$ if $f \in \mathcal{C}^\ell(\omega)$ and, at each point y_0 of the boundary $\partial\omega$ of ω, $\lim_{y \in \omega \to y_0} f(y)$ and $\lim_{y \in \omega \to y_0} \partial^{\boldsymbol{p}} f(y)$ for all $1 \leq |\boldsymbol{p}| \leq \ell$ exist. Analogous definitions hold for the *spaces* $\mathcal{C}^\ell(\overline{\omega}; \mathbb{R}^n), \mathcal{C}^\ell(\overline{\omega}; \mathbb{M}^n)$, and $\mathcal{C}^\ell(\overline{\omega}; \mathbb{S}^n)$.

All the continuous extensions appearing in such spaces will be identified by a bar. Thus for instance, we shall denote by $\overline{f} \in \mathcal{C}^0(\overline{\omega})$ and $\overline{\partial^{\boldsymbol{p}} f} \in \mathcal{C}^0(\overline{\omega})$, $1 \leq |\boldsymbol{p}| \leq \ell$, the continuous extensions to $\overline{\omega}$ of the functions f and $\partial^{\boldsymbol{p}} f$ if $f \in \mathcal{C}^\ell(\overline{\omega})$; similarly, we shall denote by $\overline{\partial_\alpha \boldsymbol{\theta}} \in \mathcal{C}^0(\overline{\omega}; \mathbb{R}^3)$ the continuous extensions to $\overline{\omega}$ of the field $\partial_\alpha \boldsymbol{\theta} \in \mathcal{C}^0(\omega; \mathbb{R}^3)$ if $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$; etc.

Finally, we also define the set

$$\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) := \{(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2); \\ \overline{\boldsymbol{A}}(y) \in \mathbb{S}^2_> \text{ for all } y \in \overline{\omega}\}.$$

Remark. The above definition of the space $\mathcal{C}^\ell(\overline{\omega})$, which in fact holds *verbatim* for an open subset of $\mathbb{R}^n$ for any $n \geq 2$, coincides with that given in Adams [1, Definition 1.26] when the set ω is bounded. □

If the open set ω is bounded, the spaces $\mathcal{C}^\ell(\overline{\omega}; \mathbb{R}^n)$ and $\mathcal{C}^\ell(\overline{\omega}; \mathbb{M}^n)$, $\ell \geq 0$, become Banach spaces when they are equipped with their usual norms, defined by

$$\|\boldsymbol{\theta}\|_{\ell,\overline{\omega}} := \sup_{\substack{y \in \overline{\omega} \\ |\boldsymbol{p}| \leq \ell}} |\overline{\partial^{\boldsymbol{p}} \boldsymbol{\theta}}(y)| \text{ for all } \boldsymbol{\theta} \in \mathcal{C}^\ell(\overline{\omega}; \mathbb{R}^n),$$

$$\|\boldsymbol{F}\|_{\ell,\overline{\omega}} := \sup_{\substack{y \in \overline{\omega} \\ |\boldsymbol{p}| \leq \ell}} |\overline{\partial^{\boldsymbol{p}} \boldsymbol{F}}(y)| \text{ for all } \boldsymbol{F} \in \mathcal{C}^\ell(\overline{\omega}; \mathbb{M}^n)$$

Given a differentiable real-valued, vector-valued, or matrix-valued, function of a single variable, its first-order derivative is indicated by a prime.

Thus for instance

$$\gamma'_\alpha(t) := \frac{d\gamma_\alpha}{dt}(t) \text{ and } \boldsymbol{\gamma}'(t) := \frac{d\boldsymbol{\gamma}}{dt}(t),\ 0 \le t \le 1,\ \text{if } \boldsymbol{\gamma} = (\gamma_\alpha) \in \mathcal{C}^1([0,1];\mathbb{R}^2),$$

$$\boldsymbol{Z}'(t) := \frac{d\boldsymbol{Z}}{dt}(t),\ 0 \le t \le 1,\ \text{if } \boldsymbol{Z} \in \mathcal{C}^1([0,1];\mathbb{M}^3),\ \text{etc.}$$

Let ω be a connected open subset of $\mathbb{R}^2$. Given two points $x, y \in \omega$, a *path joining x to y in ω* is any mapping $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{R}^2)$ that satisfies $\boldsymbol{\gamma}(t) \in \omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$. Note that there always exist such paths. Given a path $\boldsymbol{\gamma}$ joining x to y in ω, its *length* is defined by

$$L(\boldsymbol{\gamma}) := \int_0^1 |\boldsymbol{\gamma}'(t)|\, dt.$$

Let ω be a connected open subset of $\mathbb{R}^2$. The *geodesic distance in ω* between two points $x, y \in \omega$ is defined by

$$d_\omega(x, y) := \inf\{L(\boldsymbol{\gamma}); \boldsymbol{\gamma} \text{ is a path joining } x \text{ to } y \text{ in } \omega\}.$$

Remark. The function d_ω defines a *distance* on any connected open subset ω of $\mathbb{R}^2$. □

Most results of this paper will be established under a specific, but mild, regularity assumption on the boundary of an open subset of $\mathbb{R}^2$, according to the following definition:

Definition 2. An open subset ω of $\mathbb{R}^n$ satisfies the *geodesic property* if it is connected and, given any point $y_0 \in \partial\omega$ and any $\varepsilon > 0$, there exists $\delta = \delta(y_0, \varepsilon) > 0$ such that

$$d_\omega(x, y) < \varepsilon \text{ for all } x, y \in \omega \cap B(y_0; \delta).$$

Remarks. a) Any connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary, in the sense of Adams [1, Definition 4.5] or Nečas [8, pp. 14–15] satisfies the geodesic property.

(b) Let $I = \{(y_1, y_2) \in \mathbb{R}^2; 0 \le y_1 \le 1, y_2 = 0\}$. Then $\mathbb{R}^2 \setminus I$ is an instance of a connected open subset of $\mathbb{R}^2$ that does not satisfy the geodesic property. □

Definition 3. Let ω be a connected open subset of $\mathbb{R}^n$. The *geodesic diameter* of ω is defined by

$$D_\omega := \sup_{x,y \in \omega} d_\omega(x, y).$$

Note that $D_\omega = +\infty$ is not excluded.

The following lemma, whose proof can be found in Ciarlet and Mardare [5, Lemma 2.3], gives a useful characterization of boundedness in terms of the geodesic diameter.

Lemma 1. *An open subset ω of $\mathbb{R}^n$ that satisfies the geodesic property is bounded if and only if $D_\omega < +\infty$.*

The next lemma records a well-known property of the mapping that associates with any symmetric positive-definite matrix $\boldsymbol{C}$ its *square root* $\boldsymbol{C}^{1/2}$. For a proof, see, e.g., Gurtin [6, Sect. 3] or Ciarlet and Mardare [5, Lemma 2.4].

Lemma 2. *Given any matrix $\boldsymbol{C} \in \mathbb{S}^n_>$, there exists a unique matrix $\boldsymbol{C}^{1/2} \in \mathbb{S}^n_>$ such that $(\boldsymbol{C}^{1/2})^2 = \boldsymbol{C}$, and the mapping*

$$\boldsymbol{C} \in \mathbb{S}^n_> \to \boldsymbol{C}^{1/2} \in \mathbb{S}^n_>$$

defined in this fashion is of class $\mathcal{C}^\infty$.

We conclude these preliminaries by a useful estimate.

Lemma 3. *Let there be given matrix fields $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{C}^0([0,1], \mathbb{M}^n)$ and $\boldsymbol{Z} \in \mathcal{C}^1([0,1]; \mathbb{M}^n)$ that satisfy*

$$\boldsymbol{Z}'(t) = \boldsymbol{Z}(t)\boldsymbol{A}(t) + \boldsymbol{B}(t),\ 0 \le t \le 1.$$

Then

$$|\boldsymbol{Z}(t)| \le |\boldsymbol{Z}(0)| \exp\Big(\int_0^t |\boldsymbol{A}(\tau)|\,\mathrm{d}\tau\Big) + \int_0^t |\boldsymbol{B}(s)| \exp\Big(\int_s^t |\boldsymbol{A}(\tau)|\,\mathrm{d}\tau\Big)\,\mathrm{d}s,\ 0 \le t \le 1.$$

Proof. Since

$$|\boldsymbol{Z}'(t)| \le |\boldsymbol{Z}(t)||\boldsymbol{A}(t)| + |\boldsymbol{B}(t)|,\ 0 \le t \le 1,$$

it suffices to apply Gronwall's lemma for vector fields (see, e.g., Schatzman [9, Lemma 15.2.6]). ☐

3. Recovery of a surface with boundary with prescribed fundamental forms

Let a Riemannian metric $\boldsymbol{A} = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and a symmetric matrix field $\boldsymbol{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ be given over a simply-connected open subset ω of $\mathbb{R}^2$ and assume that the pair of matrix fields $(\boldsymbol{A}, \boldsymbol{B})$ satisfies the *Gauss and Codazzi-Mainardi equations* in ω, i.e.,

$$C^p{}_{\alpha\beta\sigma} = 0 \text{ in } \omega \text{ for all } \alpha, \beta, \sigma \in \{1,2\} \text{ and } p \in \{1,2,3\}, \tag{1}$$

where

$$\begin{aligned} C^\tau{}_{\alpha\beta\sigma} &:= \partial_\sigma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\sigma} + \Gamma^\gamma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\gamma_{\alpha\sigma}\Gamma^\tau_{\beta\gamma} - b_{\alpha\beta} b^\tau_\sigma + b_{\alpha\sigma} b^\tau_\beta, \\ C^3{}_{\alpha\beta\sigma} &:= \partial_\sigma b_{\alpha\beta} - \partial_\beta b_{\alpha\sigma} + \Gamma^\gamma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\gamma_{\alpha\sigma} b_{\beta\gamma}, \end{aligned} \tag{2}$$

and the Christoffel symbols of the second kind $\Gamma^\tau_{\alpha\beta}$ associated with the matrix field $\boldsymbol{A}$ are defined by

$$\Gamma^\tau_{\alpha\beta} := \frac{1}{2} a^{\sigma\tau} \left(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta}\right), \tag{3}$$

where $(a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}$ and $b^\tau_\sigma := a^{\beta\tau} b_{\beta\sigma}$.

Then the *fundamental theorem of surface theory* asserts that $\boldsymbol{A}$ and $\boldsymbol{B}$ are the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega)$ that is isometrically immersed in $\mathbb{R}^3$ and, if ω is connected, the immersion $\boldsymbol{\theta}$ is unique up to proper isometries in $\mathbb{R}^3$. More specifically, we have (see, e.g., Spivak [10] for the local version of this theorem, and, e.g., Klingenberg [7] or Ciarlet and Larsonneur [3] for its global version):

Theorem 1. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Let a pair of matrix fields $(\boldsymbol{A}, \boldsymbol{B}) = ((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^1(\omega; \mathbb{S}^2)$ be given that satisfies the Gauss and Codazzi-Mainardi equations (1) in ω. Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ such that*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \text{ in } \omega. \tag{4}$$

Moreover, the mapping $\boldsymbol{\theta}$ is unique up to proper isometries in $\mathbb{R}^3$, that is, if a mapping $\boldsymbol{\phi} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ also satisfies relations (4), then there exists a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that $\boldsymbol{\phi}(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y)$ for all $y \in \omega$.

Remark. The uniqueness up to proper isometries in $\mathbb{R}^3$ of the mapping $\boldsymbol{\theta}$ holds under substantially weaker regularity assumptions (see Ciarlet and Mardare [4]). However this is not relevant for our present purposes. □

We now show that this result can be extended "up to the boundary". In what follows, sets such as $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ and extensions such as $\overline{a_{\alpha\beta}}$ or $\overline{\partial_\alpha \boldsymbol{\theta}}$ are meant according to Definition 1. The "geodesic property" is defined in Definition 2.

Theorem 2. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ that satisfies the geodesic property. Let a pair of matrix fields $(\boldsymbol{A}, \boldsymbol{B}) = ((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ be given that satisfies the Gauss and Codazzi-Mainardi equations (1) in ω. Then there exists a mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ such that*

$$\overline{a_{\alpha\beta}} = \overline{\partial_\alpha \boldsymbol{\theta}} \cdot \overline{\partial_\beta \boldsymbol{\theta}} \text{ and } \overline{b_{\alpha\beta}} = \overline{\partial_{\alpha\beta} \boldsymbol{\theta}} \cdot \frac{\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}}}{|\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}}|} \text{ in } \overline{\omega}.$$

Moreover, the mapping $\boldsymbol{\theta}$ is unique up to proper isometries in $\mathbb{R}^3$. This means that, if a mapping $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ satisfies the same relations, then there exists a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\overline{\boldsymbol{\phi}}(y) = \boldsymbol{a} + \boldsymbol{Q}\overline{\boldsymbol{\theta}}(y) \text{ for all } y \in \overline{\omega}.$$

Proof. Thanks to Theorem 1, there exists a mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$ such that relations (4) are satisfied. It thus remains to prove that this mapping belongs in fact to the space $\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ (see Definition 1). The proof of this assertion is broken into five steps, numbered (i) to (v).

(i) *Preliminaries.* The Christoffel symbols $\Gamma^\tau_{\alpha\beta}$ associated with the metric $\boldsymbol{A}$ being defined by (3), let the matrix field $\boldsymbol{\Gamma}_\alpha : \omega \to \mathbb{M}^3$ be defined by

$$\boldsymbol{\Gamma}_\alpha := \begin{pmatrix} \Gamma^1_{\alpha 1} & \Gamma^1_{\alpha 2} & -b^1_\alpha \\ \Gamma^2_{\alpha 1} & \Gamma^2_{\alpha 2} & -b^2_\alpha \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix}$$

where $b^\sigma_\alpha = a^{\beta\sigma} b_{\alpha\beta}$ as before. Let

$$\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}, \tag{5}$$

and let $\boldsymbol{F}$ denote the matrix field whose i-th column is $\boldsymbol{a}_i$, $i \in \{1,2,3\}$. It is well known that the partial derivatives of the vector fields $\boldsymbol{a}_i$ satisfy the Gauss and Weingarten formulas, viz.,

$$\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3 \text{ and } \partial_\alpha \boldsymbol{a}_3 = -b^\sigma_\alpha \boldsymbol{a}_\sigma.$$

These relations can be conveniently re-written in matrix form as

$$\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha \text{ in } \omega. \tag{6}$$

(ii) *The matrix field $\boldsymbol{\Gamma}_\alpha$ belongs to the space $\mathcal{C}^1(\overline{\omega},\mathbb{M}^3)$.* Since $\boldsymbol{A} \in \mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ and $\overline{\boldsymbol{A}}(y) \in \mathbb{S}^2_>$ for all $y \in \overline{\omega}$, we deduce that $\det \boldsymbol{A} \in \mathcal{C}^2(\overline{\omega})$ and $\det \overline{\boldsymbol{A}}(y)) > 0$ for all $y \in \overline{\omega}$. This implies that the function $\det(\boldsymbol{A}^{-1})$ belongs to the space $C^2(\overline{\omega})$. Since the inverse of the matrix $\boldsymbol{A}(y)$ is given by

$$\boldsymbol{A}^{-1}(y) = (a^{\alpha\beta}(y)) = \frac{1}{\det(\boldsymbol{A}(y))} \begin{pmatrix} a_{22}(y) & -a_{12}(y) \\ -a_{21}(y) & a_{11}(y) \end{pmatrix},$$

we deduce that the field $\boldsymbol{A}^{-1}$ belongs to the space $\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$. This implies that the functions b^σ_α and $\Gamma^\sigma_{\alpha\beta}$ belong to the space $\mathcal{C}^1(\overline{\omega})$. Consequently, $\boldsymbol{\Gamma}_\alpha \in \mathcal{C}^1(\overline{\omega},\mathbb{M}^3)$.

(iii) *The matrix field $\boldsymbol{F}$ belongs to the space $\mathcal{C}^2(\overline{\omega};\mathbb{M}^3)$.* Let K be any compact subset of $\mathbb{R}^2$. Since

$$|\boldsymbol{F}(y)| = |\boldsymbol{F}(y)^T \boldsymbol{F}(y)|^{1/2} = \max\{1, |\boldsymbol{A}(y)|^{1/2}\} \text{ for all } y \in \omega,$$

we have

$$\sup_{y\in K\cap\omega} |\boldsymbol{F}(y)| = \max\left\{1, \sup_{y\in K\cap\overline{\omega}} |\overline{\boldsymbol{A}}(y)|^{1/2}\right\} < +\infty, \tag{7}$$

since the field $\boldsymbol{A}$ belongs to the space $\mathcal{C}^0(\overline{\omega};\mathbb{S}^2)$ by assumption.

Fix a point $y_0 \in \partial\omega$ and let $K_0 = \overline{B(y_0;1)}$. Then the properties established in part (ii) and relation (7) together imply that

$$c_0 := \Big(\sup_{y\in K_0\cap\omega} |\boldsymbol{F}(y)|\Big)\Big(\sup_{y\in K_0\cap\omega}\big(\sum_\alpha |\boldsymbol{\Gamma}_\alpha(y)|^2\big)^{1/2}\Big) < +\infty.$$

Let $\varepsilon > 0$ be given. Because ω satisfies the geodesic property (Definition 2), there exists $\delta(\varepsilon) > 0$ such that, given any two points $x, y \in B(y_0;\delta(\varepsilon)) \cap \omega$, there exists a path $\boldsymbol{\gamma} = (\gamma_\alpha)$ joining x to y in ω whose length satisfies $L(\boldsymbol{\gamma}) \leq \dfrac{\varepsilon}{\max\{c_0, 2\}}$. To ensure that the set $\boldsymbol{\gamma}([0,1])$ is contained in the set K_0, we assume, without loss of generality, that $\varepsilon \leq 1$ and $\delta(\varepsilon) \leq \dfrac{1}{2}$.

We infer from equation (6) that the matrix field $\boldsymbol{Y} := \boldsymbol{F}\circ\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{M}^3)$ associated with any such path $\boldsymbol{\gamma}$ satisfies

$$\boldsymbol{Y}'(t) = \gamma_\alpha{}'(t)\boldsymbol{Y}(t)\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t)) \text{ for all } t \in [0,1].$$

Expressing that $\boldsymbol{Y}(1) = \boldsymbol{Y}(0) + \int_0^1 \boldsymbol{Y}'(t)\,\mathrm{d}t$, we thus have, for any two points $x, y \in B(y_0;\delta) \cap \omega$,

$$|\boldsymbol{F}(y) - \boldsymbol{F}(x)| = |\boldsymbol{Y}(1) - \boldsymbol{Y}(0)| \leq \int_0^1 |\gamma_\alpha{}'(t)||\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))||\boldsymbol{Y}(t)|dt$$

$$\leq \sup_{t\in[0,1]} |\boldsymbol{Y}(t)| \left\{\int_0^1 |\gamma_\alpha{}'(t)|^2 dt\right\}^{1/2} \left\{\int_0^1 |\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))|^2 dt\right\}^{1/2} \leq c_0 L(\boldsymbol{\gamma}) \leq \varepsilon$$

It is easy to see that this inequality implies that $\lim_{y\in\omega\to y_0} \boldsymbol{F}(y)$ exists in $\mathbb{M}^3$. Consequently, the field $\boldsymbol{F}$ can be extended to a field that is continuous on $\overline{\omega}$.

Since $\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha$ in ω and the fields $\boldsymbol{\Gamma}_\alpha$ belong to the space $\mathcal{C}^1(\overline{\omega};\mathbb{M}^3)$ by part (ii), each field $\partial_\alpha \boldsymbol{F}$ can be extended to a field that is continuous on $\overline{\omega}$; hence $\boldsymbol{F} \in \mathcal{C}^1(\overline{\omega};\mathbb{M}^3)$. Differentiating the relations $\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha$ in ω further shows that $\boldsymbol{F} \in \mathcal{C}^2(\overline{\omega};\mathbb{M}^3)$.

(iv) *The vector field $\boldsymbol{\theta}$ belong to the space $\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$.* Given $y_0 \in \partial\omega$, we proceed as in (iii), the number $\delta(\varepsilon) > 0$ being now chosen in such a way that $L(\boldsymbol{\gamma}) \leq \dfrac{\varepsilon}{\max\{c_1, 2\}}$, where

$$c_1 := \sqrt{2} \sup_{y\in K_0\cap\omega} |\boldsymbol{F}(y)| < \infty.$$

Again without loss of generality, we assume that $\varepsilon \leq 1$ and $\delta(\varepsilon) \leq \dfrac{1}{2}$.

We infer from equations (5) that the vector field $\mathbf{y} = \boldsymbol{\theta}\circ\boldsymbol{\gamma} \in C^1([0,1];\mathbb{R}^3)$ associated with the path $\boldsymbol{\gamma}$ joining x to y in ω satisfies the equation

$$\mathbf{y}'(t) = \gamma_\alpha{}'(t)\boldsymbol{a}_\alpha(\boldsymbol{\gamma}(t)) \text{ for all } t \in [0,1],$$

so that, for any two points $x, y \in B(y_0; \delta(\varepsilon)) \cap \omega$,

$$|\boldsymbol{\theta}(y) - \boldsymbol{\theta}(x)| = |\mathbf{y}(1) - \mathbf{y}(0)| \leq \int_0^1 |\gamma_\alpha{}'(t)||\boldsymbol{a}_\alpha(\boldsymbol{\gamma}(t))|dt$$

$$\leq L(\boldsymbol{\gamma}) \sup_{y \in K_0 \cap \omega} \left\{ \sum_\alpha |\boldsymbol{a}_\alpha(y)|^2 \right\}^{1/2} \leq \sqrt{2}\, L(\boldsymbol{\gamma}) \sup_{y \in K_0 \cap \omega} |\boldsymbol{F}(y)| \leq c_1 L(\boldsymbol{\gamma}) \leq \varepsilon.$$

This inequality shows that $\lim_{y \in \omega \to y_0} \boldsymbol{\theta}(y)$ exists in $\mathbb{R}^3$. Consequently, the field $\boldsymbol{\theta}$ can be extended to a field that is continuous on $\overline{\omega}$.

Since $\boldsymbol{a}_\alpha \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ and $\partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha$, we conclude that the fields $\partial^{\boldsymbol{p}} \boldsymbol{\theta}$, $1 \leq |\boldsymbol{p}| \leq 3$, can be extended to fields that are continuous on $\overline{\omega}$. Hence $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$.

(v) *Uniqueness up to proper isometries in* $\mathbb{R}^3$. If $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ satisfies

$$\overline{a_{\alpha\beta}} = \overline{\partial_\alpha \boldsymbol{\phi}} \cdot \overline{\partial_\beta \boldsymbol{\phi}} \ \text{ and } \overline{b_{\alpha\beta}} = \overline{\partial_{\alpha\beta} \boldsymbol{\phi}} \cdot \frac{\overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}}}{|\overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}}|} \ \text{ in } \overline{\omega},$$

then by Theorem 1, there exists $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\boldsymbol{\phi}(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y) \text{ for all } y \in \omega.$$

Consequently, the continuous extensions $\overline{\boldsymbol{\phi}}$ and $\overline{\boldsymbol{\theta}}$ satisfy

$$\overline{\boldsymbol{\phi}}(y) = \boldsymbol{a} + \boldsymbol{Q}\overline{\boldsymbol{\theta}}(y) \text{ for all } y \in \overline{\omega}.$$

□

While the immersions $\boldsymbol{\theta}$ found in Theorem 2 are only defined up to proper isometries in $\mathbb{R}^3$, they become uniquely determined if they are required to satisfy *ad hoc* additional conditions, according to the following corollary to Theorem 2.

Corollary 1. *Let the assumptions on the set* ω *and on the matrix fields* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be as in Theorem 2, let a point* $y_0 \in \omega$ *be given, and let the matrix* $\mathcal{a}_0 \in \mathbb{S}^3_>$ *be defined by*

$$\mathcal{a}_0 := \begin{pmatrix} \boldsymbol{A}(y_0) & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix},$$

where $\boldsymbol{A}(y_0) = (a_{\alpha\beta}(y_0)) \in \mathbb{S}^2_>$. *Then there exists one and only one mapping* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ *that satisfies*

$$\overline{a_{\alpha\beta}} = \overline{\partial_\alpha \boldsymbol{\theta}} \cdot \overline{\partial_\beta \boldsymbol{\theta}} \ \text{ and } \overline{b_{\alpha\beta}} = \overline{\partial_{\alpha\beta} \boldsymbol{\theta}} \cdot \frac{\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}}}{|\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}}|} \ \text{ in } \overline{\omega},$$

$$\boldsymbol{\theta}(y_0) = \boldsymbol{0} \text{ and } \partial_\alpha \boldsymbol{\theta}(y_0) = \boldsymbol{b}^0_\alpha,$$

where $\boldsymbol{b}^0_\alpha$ *denotes the* α*-th column vector of the matrix* $\mathcal{a}_0^{1/2} \in \mathbb{S}^3_>$.

Proof. Theorem 2 shows that there exists a mapping $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ such that

$$\overline{a_{\alpha\beta}} = \overline{\partial_\alpha \boldsymbol{\phi}} \cdot \overline{\partial_\beta \boldsymbol{\phi}} \text{ and } \overline{b_{\alpha\beta}} = \overline{\partial_{\alpha\beta} \boldsymbol{\phi}} \cdot \frac{\overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}}}{|\overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}}|} \text{ in } \overline{\omega}.$$

Let $\boldsymbol{F}_0 \in \mathbb{M}^3$ denote the matrix whose columns are $\partial_1 \boldsymbol{\phi}(y_0), \partial_2 \boldsymbol{\phi}(y_0)$ and $\dfrac{\partial_1 \boldsymbol{\phi}(y_0) \wedge \partial_2 \boldsymbol{\phi}(y_0)}{|\partial_1 \boldsymbol{\phi}(y_0) \wedge \partial_2 \boldsymbol{\phi}(y_0)|}$. Let the mapping $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\theta}(y) := a_0^{1/2} \boldsymbol{F}_0^{-1} (\boldsymbol{\phi}(y) - \boldsymbol{\phi}(y_0)) \text{ for all } y \in \omega.$$

Then the field $\boldsymbol{\theta}$ belongs to the space $\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ and satisfies

$$\boldsymbol{\theta}(y_0) = \mathbf{0} \text{ and } \partial_\alpha \boldsymbol{\theta}(y_0) = a_0^{1/2} \boldsymbol{F}_0^{-1} \partial_\alpha \boldsymbol{\phi}(y_0) = \boldsymbol{b}_\alpha^0.$$

Since the matrix $a_0^{1/2} \boldsymbol{F}_0^{-1}$ is proper orthogonal, we have

$$\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}} = (a_0^{1/2} \boldsymbol{F}_0^{-1}) \overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}} \text{ in } \overline{\omega},$$

so that we also have

$$\overline{\partial_\alpha \boldsymbol{\theta}} \cdot \overline{\partial_\beta \boldsymbol{\theta}} = \overline{\partial_\alpha \boldsymbol{\phi}} \cdot \overline{\partial_\beta \boldsymbol{\phi}} \text{ in } \overline{\omega}$$

and

$$\overline{\partial_{\alpha\beta} \boldsymbol{\theta}} \cdot \frac{\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}}}{|\overline{\partial_1 \boldsymbol{\theta}} \wedge \overline{\partial_2 \boldsymbol{\theta}}|} = \overline{\partial_{\alpha\beta} \boldsymbol{\phi}} \cdot \frac{\overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}}}{|\overline{\partial_1 \boldsymbol{\phi}} \wedge \overline{\partial_2 \boldsymbol{\phi}}|} \text{ in } \overline{\omega}.$$

This shows that the above mapping $\boldsymbol{\theta}$ possesses the announced properties.

Besides, it is uniquely determined. To see this, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ and $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be two immersions in ω that satisfy the last two relations in $\overline{\omega}$, together with

$$\boldsymbol{\theta}(y_0) = \boldsymbol{\phi}(y_0) \text{ and } \partial_\alpha \boldsymbol{\theta}(y_0) = \partial_\alpha \boldsymbol{\phi}(y_0) = \boldsymbol{b}_\alpha^0.$$

Then Theorem 2 shows that there exists a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that $\overline{\boldsymbol{\phi}}(y) = \boldsymbol{a} + \boldsymbol{Q}\overline{\boldsymbol{\theta}}(y)$ for all $y \in \overline{\omega}$. By differentiating this relation, we find that $\partial_\alpha \boldsymbol{\phi}(y) = \boldsymbol{Q} \partial_\alpha \boldsymbol{\theta}(y)$ for all $y \in \omega$. In particular, for $y = y_0$, we have $\boldsymbol{b}_\alpha^0 = \boldsymbol{Q}\boldsymbol{b}_\alpha^0$. Since $\boldsymbol{b}_1^0$ and $\boldsymbol{b}_2^0$ are linearly independent, the matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ must be the identity $\boldsymbol{I}$. Since $\boldsymbol{\phi}(y_0) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y_0)$, we also have $\boldsymbol{a} = \mathbf{0}$. Hence the two mappings $\overline{\boldsymbol{\phi}}$ and $\overline{\boldsymbol{\theta}}$ coincide in $\overline{\omega}$. □

4. Continuity of a surface as a function of its fundamental forms

The following property of mappings between metric spaces is central to this section:

Definition 4. Let (E, d_E) and (F, d_F) be two metric spaces. A mapping $f : E \to F$ is *locally Lipschitz-continuous* if, given any $\tilde{e} \in E$, there exist two constants $C(\tilde{e}) > 0$ and $\delta(\tilde{e}) > 0$ such that

$$d_F(f(e), f(\hat{e})) \le C(\tilde{e}) d_E(e, \hat{e}) \text{ for all } e, \hat{e} \in B(\tilde{e}; \delta(\tilde{e})) \subset E.$$

Let ω be a simply-connected open subset of $\mathbb{R}^2$ that satisfies the geodesic property. Define the subset

$$\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2) := \{(\boldsymbol{A},\boldsymbol{B})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\times\mathcal{C}^1(\overline{\omega};\mathbb{S}^2);C^p{}_{\alpha\beta\sigma}=0 \text{ in } \omega\}$$

of the space $\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)\times\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ and the quotient space

$$\dot{\mathcal{C}}^3(\overline{\omega};\mathbb{R}^3) := \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)/\mathcal{R},$$

where $(\boldsymbol{\phi},\boldsymbol{\theta})\in\mathcal{R}$ means that there exist a vector $\boldsymbol{c}\in\mathbb{R}^3$ and a matrix $\boldsymbol{Q}\in\mathbb{O}^3_+$ such that $\boldsymbol{\phi}(y)=\boldsymbol{c}+\boldsymbol{Q}\boldsymbol{\theta}(y)$ for all $y\in\omega$. Then Theorem 2 shows that there exists a well-defined mapping

$$\overline{\mathcal{F}}:\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)\to\dot{\mathcal{C}}^3(\overline{\omega};\mathbb{R}^3),$$

that associates with any pair of matrix fields $(\boldsymbol{A},\boldsymbol{B})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ the unique equivalence class $\dot{\boldsymbol{\theta}}\in\dot{\mathcal{C}}^3(\overline{\omega};\mathbb{R}^3)$ that satisfies

$$\overline{a_{\alpha\beta}}=\overline{\partial_\alpha\boldsymbol{\theta}}\cdot\overline{\partial_\beta\boldsymbol{\theta}} \text{ and } \overline{b_{\alpha\beta}}=\overline{\partial_{\alpha\beta}\boldsymbol{\theta}}\cdot\frac{\overline{\partial_1\boldsymbol{\theta}}\wedge\overline{\partial_2\boldsymbol{\theta}}}{|\overline{\partial_1\boldsymbol{\theta}}\wedge\overline{\partial_2\boldsymbol{\theta}}|} \text{ in } \overline{\omega}.$$

If in addition the set ω is bounded, the spaces $\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$, $\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$, and $\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ endowed with their natural norms (defined in Section 2) become Banach spaces and thus in this case the sets $\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ and $\dot{\mathcal{C}}^3(\overline{\omega};\mathbb{R}^3)$ become metric spaces when they are equipped with the induced topologies. Then a natural question arises:

Is the mapping $\overline{\mathcal{F}}$ continuous when the sets $\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ and $\dot{\mathcal{C}}^3(\overline{\omega};\mathbb{R}^3)$ are equipped with these topologies?

The second objective of this paper is to provide an affirmative answer to this question (see Corollary 2). To this end, we will first answer the same question for a mapping $\overline{\mathcal{F}}_0$ that associates with any pair of matrix fields $(\boldsymbol{A},\boldsymbol{B})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ a specific element in the equivalence class $\overline{\mathcal{F}}(\boldsymbol{A},\boldsymbol{B})$, by showing that this mapping is locally Lipschitz-continuous.

Theorem 3. *Let ω be a bounded, connected and simply-connected open subset of $\mathbb{R}^2$ that satisfies the geodesic property. Let a point $y_0\in\omega$ be given. By Corollary 1 there exists a well-defined mapping*

$$\overline{\mathcal{F}}_0:\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)\to\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$$

that associates with any pair of matrix fields $(\boldsymbol{A},\boldsymbol{B})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2_>)\mathbin{\underset{\circ}{\times}}\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ the unique mapping $\boldsymbol{\theta}\in\overline{\mathcal{F}}(\boldsymbol{A},\boldsymbol{B})$ that satisfies

$$\boldsymbol{\theta}(y_0)=\mathbf{0} \text{ and } \partial_\alpha\boldsymbol{\theta}(y_0)=\boldsymbol{b}^0_\alpha,$$

where the vectors $\boldsymbol{b}^0_\alpha$ are defined as in Corollary 1.

Then the mapping $\overline{\mathcal{F}}_0$ is locally Lipschitz-continuous.

Proof. The proof is broken into six steps, numbered (i) to (vi).

(i) *Preliminaries.* The image $\boldsymbol{\theta} = \overline{\mathcal{F}}_0(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ of a pair $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ is constructed as follows (see the proof of Theorem 2 and Corollary 1): Let

$$\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \quad \text{in } \omega,$$

let $\boldsymbol{F} \in \mathcal{C}^2(\overline{\omega}; \mathbb{M}^3)$ denote the matrix field whose i-th column is $\boldsymbol{a}_i$ and let $\boldsymbol{\Gamma}_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathbb{M}^3)$ denote the matrix field defined by

$$\boldsymbol{\Gamma}_\alpha = \begin{pmatrix} \Gamma^1_{\alpha 1} & \Gamma^1_{\alpha 2} & -b^1_\alpha \\ \Gamma^2_{\alpha 1} & \Gamma^2_{\alpha 2} & -b^2_\alpha \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix}.$$

Let a point $y_0 \in \omega$ be given once and for all. Then the matrix field $\boldsymbol{F} \in \mathcal{C}^2(\overline{\omega}; \mathbb{M}^3)$ is defined as the unique one that satisfies

$$\begin{aligned} &\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha \quad \text{in } \omega, \\ &\boldsymbol{F}(y_0) = \boldsymbol{F}_0, \end{aligned}$$

where the matrix $\boldsymbol{F}_0 \in \mathbb{M}^3$ is defined by

$$\boldsymbol{F}_0 := \mathcal{a}_0^{1/2} = \begin{pmatrix} \boldsymbol{A}(y_0)^{1/2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}. \tag{8}$$

Then the vector field $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ is defined as the unique one that satisfies

$$\begin{aligned} &\partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha \quad \text{in } \omega, \\ &\boldsymbol{\theta}(y_0) = \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{a}_\alpha$ is the α-th column of the matrix $\boldsymbol{F}$.

The mapping $\overline{\mathcal{F}}_0$ can thus be written as a composite mapping. In order to prove that $\overline{\mathcal{F}}_0$ is locally Lipschitz-continuous, is suffices to prove that each one of its component mappings is locally Lipschitz-continuous (a composite mapping is locally Lipschitz-continuous if all its component mappings share this property).

Let $r > 0$ and $R > 0$ be fixed, but otherwise arbitrary, numbers and define the set

$$\begin{aligned} \boldsymbol{K}(r, R) := \{&(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2); \\ &\inf_{y \in \omega} \det \boldsymbol{A}(y) > r \text{ and } \|\boldsymbol{A}\|_{2,\overline{\omega}} + \|\boldsymbol{B}\|_{1,\overline{\omega}} < R\}. \end{aligned}$$

Note that $\boldsymbol{K}(r, R)$ is a relatively open subset of $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$.

In what follows, the symbols $C_1, C_2, ...$ designate positive numbers depending only on r and R and we use notations, such as $\hat{\Gamma}^\tau_{\alpha\beta}$, $\hat{a}^{\sigma\tau}$, $(\hat{\boldsymbol{\Gamma}}_\alpha)$, etc., that should be self-explanatory for designating specific functions, matrix fields, etc., corresponding to a pair of matrix fields $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$.

From the expressions of the inverse of the matrix $\boldsymbol{A}(y)$ (see part (ii) of the proof of Theorem 2), of the Christoffel symbols $\Gamma^\tau_{\alpha\beta}$ (see relation (3)), and of the functions $b^\sigma_\alpha = a^{\beta\sigma} b_{\alpha\beta}$, we easily infer that there exist constants C_1 and C_2 such that

$$\|\boldsymbol{A}^{-1}\|_{1,\overline{\omega}} \le C_1 \text{ and } \left(\sum_\alpha \|\boldsymbol{\Gamma}_\alpha\|^2_{1,\overline{\omega}}\right)^{1/2} \le C_2 \text{ for all } (\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{K}(r, R).$$

Noting that the matrix field $\boldsymbol{F}$ associated with a pair of matrix fields $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\diamond}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ satisfies $|\boldsymbol{F}| = \max\{1, |\boldsymbol{A}|^{1/2}\}$ and $\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha$ in ω, we next conclude that there exists a constant C_3 such that

$$\|\boldsymbol{F}\|_{2,\overline{\omega}} \le C_3 \text{ for all } (\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{K}(r, R).$$

(ii) *The mapping*

$$(\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{K}(r, R) \mapsto (\boldsymbol{\Gamma}_\alpha) \in \left(\mathcal{C}^1(\overline{\omega}; \mathbb{M}^3)\right)^2$$

is Lipschitz-continuous. Let $(\boldsymbol{A}, \boldsymbol{B}), (\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}) \in \boldsymbol{K}(r, R)$. On the one hand, we have

$$\begin{aligned}
&\|\Gamma^\tau_{\alpha\beta} - \hat{\Gamma}^\tau_{\alpha\beta}\|_{1,\overline{\omega}} \\
&\quad \le \frac{1}{2}\|\hat{a}^{\sigma\tau}\left(\partial_\alpha(a_{\beta\sigma} - \hat{a}_{\beta\sigma}) + \partial_\beta(a_{\alpha\sigma} - \hat{a}_{\alpha\sigma}) - \partial_\sigma(a_{\alpha\beta} - \hat{a}_{\alpha\beta})\right)\|_{1,\overline{\omega}} \\
&\qquad + \frac{1}{2}\|(a^{\sigma\tau} - \hat{a}^{\sigma\tau})(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta})\|_{1,\overline{\omega}} \\
&\qquad\qquad \le 3\|\hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}}\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + 3\|\boldsymbol{A}^{-1} - \hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}}\|\boldsymbol{A}\|_{2,\overline{\omega}}
\end{aligned}$$

and

$$\begin{aligned}
\|b^\sigma_\alpha - \hat{b}^\sigma_\alpha\|_{1,\overline{\omega}} &= \|a^{\beta\sigma} b_{\alpha\beta} - \hat{a}^{\beta\sigma}\hat{b}_{\alpha\beta}\|_{1,\overline{\omega}} \\
&\le 2\|\hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}}\|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}} + 2\|\boldsymbol{A}^{-1} - \hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}}\|\boldsymbol{B}\|_{1,\overline{\omega}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\boldsymbol{A}^{-1} - \hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}} &= \|\boldsymbol{A}^{-1}(\hat{\boldsymbol{A}} - \boldsymbol{A})\hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}} \\
&\le 3\|\boldsymbol{A}^{-1}\|_{1,\overline{\omega}}\|\hat{\boldsymbol{A}} - \boldsymbol{A}\|_{1,\overline{\omega}}\|\hat{\boldsymbol{A}}^{-1}\|_{1,\overline{\omega}}.
\end{aligned}$$

Combining the above inequalities, we thus obtain

$$\|\Gamma^\tau_{\alpha\beta} - \hat{\Gamma}^\tau_{\alpha\beta}\|_{1,\overline{\omega}} \le 3C_1(1 + 3RC_1)\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}}$$

and

$$\|b^\sigma_\alpha - \hat{b}^\sigma_\alpha\|_{1,\overline{\omega}} \le 2C_1(1 + 3RC_1)\left(\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{1,\overline{\omega}} + \|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}}\right).$$

The last two inequalities together imply that there exists a constant $C_4 > 0$ such that

$$\|(\boldsymbol{\Gamma}_\alpha) - (\hat{\boldsymbol{\Gamma}}_\alpha)\|_{(\mathcal{C}^1(\overline{\omega}; \mathbb{M}^3))^2} \le C_4\left(\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + \|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}}\right).$$

(iii) *The mapping*

$$(\boldsymbol{A},\boldsymbol{B})\in\boldsymbol{K}(r,R)\mapsto\boldsymbol{F}_0\in\mathbb{M}^3$$

is Lipschitz-continuous. Since the mapping $\boldsymbol{C}\in\mathbb{S}^2_>\mapsto\boldsymbol{C}^{1/2}\in\mathbb{S}^2_>$ is of class $\mathcal{C}^\infty$ (Lemma 2), there exists a constant C_5 such that

$$|\boldsymbol{A}(y_0)^{1/2}-\hat{\boldsymbol{A}}(y_0)^{1/2}|\le C_5|\boldsymbol{A}(y_0)-\hat{\boldsymbol{A}}(y_0)|\le C_5\|\boldsymbol{A}-\hat{\boldsymbol{A}}\|_{0,\overline{\omega}}$$

for all $(\boldsymbol{A},\boldsymbol{B}),(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}})\in\boldsymbol{K}(r,R)$. We then infer from defition (8) that

$$|\boldsymbol{F}_0-\hat{\boldsymbol{F}}_0|\le C_5\|\boldsymbol{A}-\hat{\boldsymbol{A}}\|_{0,\overline{\omega}}\le C_5\left(\|\boldsymbol{A}-\hat{\boldsymbol{A}}\|_{2,\overline{\omega}}+\|\boldsymbol{B}-\hat{\boldsymbol{B}}\|_{1,\overline{\omega}}\right)$$

for any pairs $(\boldsymbol{A},\boldsymbol{B}),(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}})\in\boldsymbol{K}(r,R)$.

(iv) *The mapping*

$$(\boldsymbol{A},\boldsymbol{B})\in\boldsymbol{K}(r,R)\mapsto\boldsymbol{F}\in\mathcal{C}^2(\overline{\omega};\mathbb{M}^3),$$

where $\boldsymbol{F}$ *satisfies the system*

$$\begin{aligned}&\partial_\alpha\boldsymbol{F}=\boldsymbol{F}\boldsymbol{\Gamma}_\alpha\quad\text{in }\omega\\&\boldsymbol{F}(y_0)=\boldsymbol{F}_0,\end{aligned}\tag{9}$$

is Lipschitz-continuous. Given $(\boldsymbol{A},\boldsymbol{B}),(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}})\in\boldsymbol{K}(r,R)$, the matrix field $\boldsymbol{Q}:=\boldsymbol{F}-\hat{\boldsymbol{F}}$ satisfies

$$\begin{aligned}&\partial_\alpha\boldsymbol{Q}=\boldsymbol{Q}\boldsymbol{\Gamma}_\alpha+\hat{\boldsymbol{F}}(\boldsymbol{\Gamma}_\alpha-\hat{\boldsymbol{\Gamma}}_\alpha)\ \text{ in }\omega,\\&\boldsymbol{Q}(y_0)=\boldsymbol{F}_0-\hat{\boldsymbol{F}}_0.\end{aligned}\tag{10}$$

Let y be a fixed, but otherwise arbitrary, point in ω. Since ω is bounded and satisfies the geodesic property, its geodesic diameter D_ω is finite (see Definition 3 and Lemma 1). There thus exists a path $\boldsymbol{\gamma}\in\mathcal{C}^1([0,1];\omega)$ joining y_0 to y in ω such that its length satisfies $L(\boldsymbol{\gamma})<2D_\omega$. Then the field $\boldsymbol{Y}:=\boldsymbol{Q}\circ\boldsymbol{\gamma}\in\mathcal{C}^1([0,1];\mathbb{M}^3)$ satisfies

$$\boldsymbol{Y}'(t)=\gamma'_\alpha(t)\boldsymbol{Y}(t)\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))+\gamma'_\alpha(t)\hat{\boldsymbol{F}}(\boldsymbol{\gamma}(t))(\boldsymbol{\Gamma}_\alpha-\hat{\boldsymbol{\Gamma}}_\alpha)(\boldsymbol{\gamma}(t))\text{ for all }t\in[0,1],$$
$$\boldsymbol{Y}(0)=\boldsymbol{F}_0-\hat{\boldsymbol{F}}_0,$$

so that, by Lemma 3,

$$\begin{aligned}|\boldsymbol{Y}(1)|\le|\boldsymbol{F}_0-\hat{\boldsymbol{F}}_0|\exp\left(\int_0^1|\gamma'_\alpha(t)||\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))|dt\right)\\+\int_0^1|\gamma'_\alpha(t)|\exp\left(\int_t^1|\gamma'_\alpha(s)||\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(s))|ds\right)|\hat{\boldsymbol{F}}(\boldsymbol{\gamma}(t))||(\boldsymbol{\Gamma}_\alpha-\hat{\boldsymbol{\Gamma}}_\alpha)(\boldsymbol{\gamma}(t))|dt.\end{aligned}\tag{11}$$

Since

$$|\gamma'_\alpha(t)||\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))|\le|\boldsymbol{\gamma}'(t)|\left(\sum_\alpha|\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))|^2\right)^{1/2}\le|\boldsymbol{\gamma}'(t)|\|(\boldsymbol{\Gamma}_\alpha)\|_{0,\overline{\omega}},$$

it follows from (i) that

$$\int_0^1 |\gamma'_\alpha(t)||\boldsymbol{\Gamma}_\alpha(\boldsymbol{\gamma}(t))|dt \leq 2D_\omega \|(\boldsymbol{\Gamma}_\alpha)\|_{0,\overline{\omega}} \leq C_6,$$

where $C_6 := 2D_\omega C_2$. We thus infer from (11) that

$$\begin{aligned}
|\boldsymbol{Q}(y)| &= |\boldsymbol{Y}(1)| \\
&\leq e^{C_6}\left(|\boldsymbol{F}_0 - \hat{\boldsymbol{F}}_0| + 2D_\omega \|\hat{\boldsymbol{F}}\|_{0,\overline{\omega}} \sup_{y\in\omega}\Big(\sum_\alpha |(\boldsymbol{\Gamma}_\alpha - \hat{\boldsymbol{\Gamma}}_\alpha)(y)|^2\Big)^{1/2}\right) \\
&\qquad\qquad \leq e^{C_6}\Big(|\boldsymbol{F}_0 - \hat{\boldsymbol{F}}_0| + 2D_\omega \|\hat{\boldsymbol{F}}\|_{0,\overline{\omega}} \|(\boldsymbol{\Gamma}_\alpha) - (\hat{\boldsymbol{\Gamma}}_\alpha)\|_{0,\overline{\omega}}\Big),
\end{aligned}$$

and then from (i)-(iii) that

$$\|\boldsymbol{Q}\|_{0,\overline{\omega}} \leq C_7(\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + \|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}})$$

where $C_7 := e^{C_6}(C_5 + 2D_\omega C_3 C_4)$. Relations (10) also yield

$$\begin{aligned}
\|\partial_\alpha \boldsymbol{Q}\|_{0,\overline{\omega}} &\leq \|\boldsymbol{Q}\|_{0,\overline{\omega}}\|\boldsymbol{\Gamma}_\alpha\|_{0,\overline{\omega}} + \|\hat{\boldsymbol{F}}\|_{0,\overline{\omega}}\|\boldsymbol{\Gamma}_\alpha - \hat{\boldsymbol{\Gamma}}_\alpha\|_{0,\overline{\omega}} \\
&\leq C_8(\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + \|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}})
\end{aligned}$$

where $C_8 := C_2 C_7 + C_3 C_4$. Likewise,

$$\begin{aligned}
\|\partial_{\alpha\beta} \boldsymbol{Q}\|_{0,\overline{\omega}} &\leq 2\|\boldsymbol{Q}\|_{1,\overline{\omega}}\|\boldsymbol{\Gamma}_\alpha\|_{1,\overline{\omega}} + 2\|\boldsymbol{\Gamma}_\alpha - \hat{\boldsymbol{\Gamma}}_\alpha\|_{1,\overline{\omega}}\|\hat{\boldsymbol{F}}\|_{1,\overline{\omega}} \\
&\leq C_9(\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + \|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}}),
\end{aligned}$$

where $C_9 := 2(C_2 C_8 + C_3 C_4)$. The last three relations thus show that

$$\|\boldsymbol{F} - \hat{\boldsymbol{F}}\|_{2,\overline{\omega}} \leq C_{10}\Big(\|\boldsymbol{A} - \hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + \|\boldsymbol{B} - \hat{\boldsymbol{B}}\|_{1,\overline{\omega}}\Big)$$

for all $(\boldsymbol{A}, \boldsymbol{B}), (\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}) \in \boldsymbol{K}(r, R)$, where $C_{10} := \max(C_7, C_8, C_9)$.

(v) *The mapping*

$$\overline{\mathcal{F}}_0 : (\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{K}(r, R) \mapsto \boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$$

is Lipschitz-continuous. Recall that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ is the solution to the system

$$\begin{aligned}
\partial_\alpha \boldsymbol{\theta} &= \boldsymbol{a}_\alpha \text{ in } \omega, \\
\boldsymbol{\theta}(y_0) &= \boldsymbol{0},
\end{aligned}$$

where $\boldsymbol{a}_\alpha$ is the α-th column of the matrix field $\boldsymbol{F} \in \mathcal{C}^2(\overline{\omega}; \mathbb{M}^3)$ constructed as in part (iv) from the pair $(\boldsymbol{A}, \boldsymbol{B})$.

Let y be a fixed, but otherwise arbitrary, point in ω. As in (iv), let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be a path joining y_0 to y in ω such that its length satisfies $L(\boldsymbol{\gamma}) < 2D_\omega$. Then the mapping $\mathbf{y} := (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \circ \boldsymbol{\gamma}$ belongs to the space $\mathcal{C}^1([0, 1]; \mathbb{R}^3)$ and its derivative satisfies

$$\mathbf{y}'(t) = \gamma'_\alpha(t)(\boldsymbol{a}_\alpha - \hat{\boldsymbol{a}}_\alpha)(\boldsymbol{\gamma}(t)) \text{ for all } t \in [0, 1].$$

This implies that

$$|(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})(y)| = \left|\int_0^1 \mathbf{y}'(t)\,dt\right| \leq \int_0^1 |\gamma'_\alpha(t)(\boldsymbol{a}_\alpha - \hat{\boldsymbol{a}}_\alpha)(\boldsymbol{\gamma}(t))|dt$$
$$\leq \left(\sum_\alpha \|\boldsymbol{a}_\alpha - \hat{\boldsymbol{a}}_\alpha\|^2_{0,\overline{\omega}}\right)^{1/2} \int_0^1 |\gamma'(t)|dt \leq 2\sqrt{2}D_\omega \|\boldsymbol{F}-\hat{\boldsymbol{F}}\|_{0,\overline{\omega}},$$

so that

$$\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|_{0,\overline{\omega}} \leq 2\sqrt{2}D_\omega \|\boldsymbol{F}-\hat{\boldsymbol{F}}\|_{0,\overline{\omega}}.$$

Moreover,

$$\|\partial_\alpha(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})\|_{2,\overline{\omega}} = \|\boldsymbol{a}_\alpha - \hat{\boldsymbol{a}}_\alpha\|_{2,\overline{\omega}} \leq \|\boldsymbol{F}-\hat{\boldsymbol{F}}\|_{2,\overline{\omega}}.$$

The last two inequalities show that

$$\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|_{3,\overline{\omega}} \leq \max(1, 2\sqrt{2}D_\omega)\|\boldsymbol{F}-\hat{\boldsymbol{F}}\|_{2,\overline{\omega}}.$$

We then infer from part (iv) that

$$\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|_{3,\overline{\omega}} \leq C_{11}\left(\|\boldsymbol{A}-\hat{\boldsymbol{A}}\|_{2,\overline{\omega}} + \|\boldsymbol{B}-\hat{\boldsymbol{B}}\|_{1,\overline{\omega}}\right),$$

where $C_{11} := \max(1, 2\sqrt{2}D_\omega)C_{10}$.

(vi) *The mapping* $\overline{\mathcal{F}}_0 : \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \to \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ *is locally Lipschitz-continuous.* We have seen in (v) that, for any $r > 0$ and $R > 0$, the mapping $\overline{\mathcal{F}}_0$ is Lipschitz-continuous over the set $\boldsymbol{K}(r, R)$. We thus infer from the relation

$$\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) = \cup_{n=1}^\infty \boldsymbol{K}(\frac{1}{n}, n)$$

that the mapping $\overline{\mathcal{F}}_0$ is locally Lipschitz-continuous according to Definition 4. □

We are now in a position to prove the announced continuity of the mapping $\overline{\mathcal{F}}$ defined at the beginning of this section:

Corollary 2. *Let ω be a bounded, connected and simply-connected open subset of $\mathbb{R}^2$ that satisfies the geodesic property. Then the mapping $\overline{\mathcal{F}}$: $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \to \dot{\mathcal{C}}^3(\overline{\omega}; \mathbb{R}^3)$ is locally Lipschitz continuous.*

Proof. Since

$$\|\mathcal{F}(\boldsymbol{A}, \boldsymbol{B}) - \mathcal{F}(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}})\|_{\dot{\mathcal{C}}^3(\overline{\omega};\mathbb{R}^3)} \leq \|\overline{\mathcal{F}}_0(\boldsymbol{A}, \boldsymbol{B}) - \overline{\mathcal{F}}_0(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}})\|_{3,\overline{\omega}}$$

for all $(\boldsymbol{A}, \boldsymbol{B}), (\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$, the local Lipschitz-continuity of $\overline{\mathcal{F}}$ in a consequence of the local Lipschitz-continuity of $\overline{\mathcal{F}}_0$. □

Let now ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Define the subset

$$\mathcal{C}^2(\omega; \mathbb{S}^2_>) \underset{\circ}{\times} \mathcal{C}^1(\omega; \mathbb{S}^2) := \{(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^1(\omega; \mathbb{S}^2); C^p{}_{\alpha\beta\sigma} = 0 \text{ in } \omega\}$$

of the space $\mathcal{C}^2(\omega;\mathbb{S}^2)\times\mathcal{C}^1(\omega;\mathbb{S}^2)$ (recall that the relations $C^p{}_{\alpha\beta\sigma}=0$ mean that the pair of matrix fields $(\boldsymbol{A},\boldsymbol{B})$ satisfies the Gauss and Codazzi-Mainardi equations). Let

$$\dot{\mathcal{C}}^3(\omega;\mathbb{R}^3) := \mathcal{C}^3(\omega;\mathbb{R}^3)/\mathcal{R},$$

be the quotient space of $\mathcal{C}^3(\omega;\mathbb{R}^3)$ by the equivalence relation $\mathcal{R}$, where $(\boldsymbol{\theta},\boldsymbol{\phi})\in\mathcal{R}$ means that there exists a vector $\boldsymbol{a}\in\mathbb{R}^3$ and a matrix $\boldsymbol{Q}\in\mathbb{O}^3_+$ such that $\boldsymbol{\phi}(y)=\boldsymbol{a}+\boldsymbol{Q}\boldsymbol{\theta}(y)$ for all $y\in\omega$. Then Theorem 1 shows that there exists a well-defined mapping

$$\mathcal{F}:\mathcal{C}^2(\omega;\mathbb{S}^2_>)\underset{\circ}{\times}\mathcal{C}^1(\omega;\mathbb{S}^2)\to\dot{\mathcal{C}}^3(\omega;\mathbb{R}^3),$$

that associates with any pair of matrix fields $(\boldsymbol{A}=(a_{\alpha\beta}),\boldsymbol{B}=(b_{\alpha\beta}))\in\mathcal{C}^2(\omega;\mathbb{S}^2_>)\underset{\circ}{\times}\mathcal{C}^1(\omega;\mathbb{S}^2)$ the unique equivalence class $\dot{\boldsymbol{\theta}}\in\dot{\mathcal{C}}^3(\omega;\mathbb{R}^3)$ that satisfies

$$a_{\alpha\beta}=\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta}\ \text{ and } b_{\alpha\beta}=\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}\ \text{ in }\omega.$$

The continuity of the mapping $\mathcal{F}$ can also be established as a corollary to Theorem 3. More specifically, let the spaces $\mathcal{C}^2(\omega;\mathbb{S}^2)\times\mathcal{C}^1(\omega;\mathbb{S}^2)$ and $\mathcal{C}^3(\omega;\mathbb{R}^3)$ be endowed with their usual Fréchet topologies. The subset $\mathcal{C}^2(\omega;\mathbb{S}^2_>)\underset{\circ}{\times}\mathcal{C}^1(\omega;\mathbb{S}^2)$ of $\mathcal{C}^2(\omega;\mathbb{S}^2)\times\mathcal{C}^1(\omega;\mathbb{S}^2)$ and the quotient space $\dot{\mathcal{C}}^3(\omega;\mathbb{R}^3)$ are then equipped with the induced topologies. The following result then improves upon an earlier result due to Ciarlet [2]:

Corollary 3. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Then the mapping $\mathcal{F}:\mathcal{C}^2(\omega;\mathbb{S}^2_>)\underset{\circ}{\times}\mathcal{C}^1(\omega;\mathbb{S}^2)\to\dot{\mathcal{C}}^3(\omega;\mathbb{R}^3)$ is continuous.*

Proof. It suffices to write the set ω as a countable union $\omega=\cup_{i=1}^\infty B_i$ of open balls B_i satisfying $\overline{B}_i\subset\omega$. ☐

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