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Abstract

We proposed a new approach to the existence theory for quadratic minimization problems that arise in linear shell theory. The novelty consists in considering the linearized change of metric and change of curvature tensors as the new unknowns, instead of the displacement vector field as is customary.

Such an approach naturally yields a constrained minimization problem, the constraints being ad hoc compatibility relations that these new unknowns must satisfy in order that they indeed correspond to a displacement vector field. Our major objective is thus to specify and justify such compatibility relations in appropriate function spaces. Interestingly, this result provides as a corollary a new proof of Korn's inequality on a surface. While the classical proof of this fundamental inequality essentially relies on a basic lemma of J. L. Lions, the keystone in the proposed approach is instead an appropriate weak version of a classical theorem of Poincaré.

The existence of a solution to the above constrained minimization problem is then established, also providing as a simple corollary a new existence proof for the original quadratic minimization problem.

Keywords: Linearly elastic shell theory; Korn's inequality on a surface; quadratic minimization problems

Ccode: AMS Subject Classification: 49N10, 73K15

1 Introduction

All notions and definitions used in this introduction are explained in the next section.

Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be a smooth enough immersion. Consider a *linearly elastic shell* with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ and thickness $2\varepsilon > 0$. According to Koiter[19], the *pure traction problem* for such a shell takes the form of the following *quadratic minimization problem*: The unknown $\boldsymbol{\eta}^* = (\eta_i^*)$, whose components are the covariant components $\eta_i^* : \overline{\omega} \to \mathbb{R}$ of the unknown displacement field $\eta_i^* \boldsymbol{a}^i$ of the points of the middle surface S (the vector fields $\boldsymbol{a}^i$ form the contravariant bases along S), satisfies:

$$\boldsymbol{\eta}^* = (\eta_i^*) \in \boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega) \text{ and } j(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta} \in \boldsymbol{V}(\omega)} j(\boldsymbol{\eta}),$$

where the *quadratic functional* $j : \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) \to \mathbb{R}$ is defined by

$$j(\boldsymbol{\eta}) = \frac{1}{2} \int_\omega \left\{ \varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \right\} \sqrt{a} d\omega - \int_\omega p^i \eta_i \sqrt{a} d\omega,$$

the functions $p^i \in L^2(\omega)$ account for the given applied forces, the functions $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the *elasticity tensor of the shell* (which is uniformly positive definite in $\overline{\omega}$), and $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the *linearized change of metric*, and *linearized change of curvature, tensors* associated with an arbitrary displacement field $\eta_i \boldsymbol{a}^i$ of the surface S.

Clearly, this minimization problem can have solutions only if the applied forces satisfy the *compatibility conditions*

$$l(\boldsymbol{\eta}) = 0 \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{Rig}(\omega),$$

where

$$l(\boldsymbol{\eta}) = \int_\omega p^i \eta_i \sqrt{a} dy \text{ and } \mathbf{Rig}(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{V}(\omega); \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \text{in } \omega\}.$$

It is well known that these compatibility conditions are also sufficient for the existence of solutions to the above minimization problem (such solutions are then unique up to the addition of vector fields in the space $\mathbf{Rig}(\omega)$). For the sake of later comparison with our approach (see the discussion in Section 7), we begin by briefly reviewing in Section 2 this classical existence theory, which goes back to Bernadou and Ciarlet[5]. However, we follow here the subsequent, and more illuminating, approach of Bernadou, Ciarlet and Miara[6], who established for this purpose a basic *Korn inequality on a surface "over the space* $\boldsymbol{V}(\omega)$". Like its three-dimensional counterpart (see Chapter 3 of Duvaut & Lions[16]), this inequality hinges on a fundamental *lemma of J. L. Lions* (details are provided in the proof of Theorem 2.1). From this inequality, we then derive *another Korn inequality on a surface, this time "over the quotient space* $\boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$" (see Theorem 2.2), from which the existence and uniqueness of a solution to the above minimization problem easily follow.

An inspection of the first integral occurring in the functional j suggests another approach to this minimization problem, where *the covariant components of the linearized change of metric and change of curvature tensors are considered as the primary unknowns*, instead of the customary covariant components of the displacement field. To describe and analyze such an approach constitute the *main objectives* of this paper.

To these ends, our primary aim (see Theorem 4.1) is to identify an ad hoc *Hilbert space* $\boldsymbol{X}$ and a linear and continuous operator

$$\boldsymbol{R} : ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega) \to \boldsymbol{R}((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{X}$$

with the following fundamental property: *Assume that the set ω is simply-connected. Then, if a pair $((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega)$ of symmetric matrix fields satisfies*

$$\boldsymbol{R}((c_{\alpha\beta}), (r_{\alpha\beta})) = \mathbf{0} \quad \text{in} \quad \boldsymbol{X},$$

there exists a vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ such that

$$c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}) \text{ and } r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta}) \text{ in } L^2(\omega),$$

and conversely (naturally, if $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ is a solution to these equations, all other solutions $\boldsymbol{\eta}' \in \boldsymbol{V}(\omega)$ are such that $(\boldsymbol{\eta}' - \boldsymbol{\eta}) \in \mathbf{Rig}(\omega)$).

For this purpose, we will resort to a result of Ciarlet and Ciarlet, Jr.[10] who have recently re-examined from a similar perspective the pure traction problem of *linearized three-dimensional elasticity.* Their approach consists in *considering the linearized strain tensor as the "primary" unknown instead of the displacement itself* (note that Antman[1] already proposed that the "full" strain tensor be analogously considered as the new unknown in minimization problems arising in three-dimensional nonlinear elasticity).

The main objective then consists in characterizing those symmetric 3×3 matrix fields $\boldsymbol{e} \in \boldsymbol{L}^2_{sym}(\Omega)$ that can be written as $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v})$ for some vector fields $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, where Ω is a domain in $\mathbb{R}^3$. As shown in Ciarlet & Ciarlet, Jr.[10], this is possible if the set Ω is simply-connected and the components e_{ij} of the matrix field $\boldsymbol{e}$ satisfy the following *weak form of the classical St Venant compatibility relations*:

$$\partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega) \text{ for all } i, j, k, l \in \{1, 2, 3\}.$$

Their justification crucially hinges on an H^{-2}*-version of a classical theorem of Poincaré.* For completeness, these results are briefly reviewed in Section 3 (see Theorems 3.1 and 3.2).

The seemingly natural "extension to a surface" of these "three-dimensional" compatibility relations turns out to be a not-so-simple endeavor, however (see the proof of Theorem 4.1). Suffice it to mention at this stage that it relies on results that were recently obtained (albeit for an entirely different purpose) by Ciarlet and S. Mardare[14] and on the same H^{-2}-version of a classical theorem of Poincaré alluded to above, which likewise "replaces" the lemma of J. L. Lions as the keystone to the present analysis. As a result, the sought linear and continuous operator $\boldsymbol{R}$ maps the space $\boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega)$ into the space

$$\boldsymbol{X} = (H^{-2}(\hat{\Omega}))^6,$$

where $\hat{\Omega}$ is an ad hoc open tubular neighborhood of the surface $\boldsymbol{\theta}(\omega)$.

We then use this result in the following way. Define the Hilbert space

$$\boldsymbol{T}(\omega) = \{((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega); \boldsymbol{R}((c_{\alpha\beta}), (r_{\alpha\beta})) = \mathbf{0} \text{ in } (H^{-2}(\hat{\Omega}))^6\}$$

and let

$$\boldsymbol{H} : \boldsymbol{T}(\omega) \to \dot{\boldsymbol{\eta}} \in \boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$$

be the linear mapping defined for each $(\boldsymbol{c}, \boldsymbol{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{T}(\omega)$ by $\boldsymbol{H}((c_{\alpha\beta}), (r_{\alpha\beta})) = \dot{\boldsymbol{\eta}}$ where $\dot{\boldsymbol{\eta}}$ is the unique equivalence class in the quotient space $\boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$ that satisfies $\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}) = c_{\alpha\beta}$ and $\rho_{\alpha\beta}(\dot{\boldsymbol{\eta}}) = r_{\alpha\beta}$ in $L^2(\omega)$.

We then show (Theorem 5.1) that *the mapping* $\boldsymbol{H}$ *is an isomorphism* from $\boldsymbol{T}(\omega)$ onto $\boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$, a property that in turn provides a *new proof* of the *Korn inequality on a surface "over the space* $\boldsymbol{V}(\omega)$" mentioned earlier; see Theorem 5.2.

We will then be in a position (see Theorem 6.1) to answer the main question addressed here, at least for the so-called "*pure traction problem*" for a linearly elastic shell modeled by

Koiter's equations. Recall that, in this case, the quadratic functional j is to be minimized over the *whole* space $\boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ (i.e., the unknown displacement field is not subjected to any boundary condition).

More specifically, that the mapping $\boldsymbol{H}$ is an isomorphism implies that *the following minimization problem, with the linearized change of metric and change of curvature tensors as the new unknowns, has one and only one solution*: Find

$$(\boldsymbol{c}^*, \boldsymbol{r}^*) = ((c^*_{\alpha\beta}), (r^*_{\alpha\beta})) \in \boldsymbol{T}(\omega) \text{ such that } \kappa(\boldsymbol{c}^*, \boldsymbol{r}^*) = \inf_{(\boldsymbol{c},\boldsymbol{r}) \in \boldsymbol{T}(\omega)} \kappa(\boldsymbol{c}, \boldsymbol{r}),$$

where

$$\kappa(\boldsymbol{c}, \boldsymbol{r}) = \frac{1}{2} \int_\omega \{\varepsilon a^{\alpha\beta\sigma\tau} c_{\sigma\tau} c_{\alpha\beta} + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} r_{\sigma\tau} r_{\alpha\beta}\} \sqrt{a} d\omega - \ell^{\,\flat}(\boldsymbol{c}, \boldsymbol{r}),$$

and the continuous linear form $\ell^{\,\flat} : \boldsymbol{T}(\omega) \to \mathbb{R}$ is defined by $\ell^{\,\flat} = \ell \circ \boldsymbol{H}$. Furthermore, $((c^*_{\alpha\beta}), (r^*_{\alpha\beta})) = ((\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}^*), \rho_{\alpha\beta}(\dot{\boldsymbol{\eta}}^*))$ where $\dot{\boldsymbol{\eta}}^*$ is the unique solution to the "classical" formulation of the pure traction problem as a minimization problem over the quotient space $\boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$.

Naturally, such results in turn constitute the basis for other investigations, which are briefly discussed in Section 7. We only mention two here:

First, in addition to its *mathematical* novelty, such an approach could also present a significant *practical* advantage. Since the constitutive equations of linear shell theory are invertible, the new minimization problem above can be immediately recast as a minimization problem with the *stress resultants* and *bending moments* as the only unknowns, i.e., those that are of primary interest from the mechanical and computational viewpoints.

Second, this approach could shed some light on the considerably more challenging minimization problems that arise in *fully nonlinear "intrinsic" shell theory*, where the "full" change of metric, and change of curvature, tensors appear in the energy, instead of their linearized versions considered here. Developing a similar approach in this case could provide *existence theorems* that are so far essentially lacking for nonlinear Koiter shell equations.

The results of this paper were announced in Ciarlet and Gratie[11].

2 The classical approach to linear shell theory

To begin with, we list some notations, definitions, and conventions that will be used throughout the article. Greek indices, resp. Latin indices, range in the set $\{1, 2\}$, resp. $\{1, 2, 3\}$, save when they are used for indexing sequences or when otherwise indicated. The summation convention with respect to repeated indices is used in conjunction with these rules.

The notation $\mathbb{E}^3$ designates a *three-dimensional Euclidean space*, with vectors $\hat{\boldsymbol{e}}^i$ forming an orthonormal basis. The Euclidean norm of $\boldsymbol{a} \in \mathbb{E}^3$ is denoted $|\boldsymbol{a}|$ and the Euclidean and exterior products of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3$ are denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$.

A generic point in $\mathbb{R}^2$ will be denoted $y = (y_\alpha)$; then $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. A generic point in $\mathbb{R}^3$ will be denoted $x = (x_i)$; then $\partial_i := \partial/\partial x_i$ and

$\partial_{ij} := \partial^2/\partial x_i \partial x_j$. Given a smooth enough vector field $\boldsymbol{v} = (v_i)$ defined on a subset of $\mathbb{R}^3$, the 3×3 matrix field with $\partial_j v_i$ as its element at the i-th row and j-th column is denoted $\boldsymbol{\nabla v}$. A generic point in $\mathbb{E}^3$ will be denoted $\hat{x} = (\hat{x}_i)$; then $\hat{\partial}_i := \partial/\partial \hat{x}_i$ and $\hat{\partial}_{ij} := \partial^2/\partial \hat{x}_i \partial \hat{x}_j$; the notation $\hat{\boldsymbol{\nabla}} \hat{\boldsymbol{v}}$ should be self-explanatory. The coordinates $\hat{x}_i$ of a point $\hat{x} \in \mathbb{E}^3$ will be referred to as *Cartesian coordinates*.

A *domain* U in $\mathbb{R}^n, n \geq 2$, or in $\mathbb{E}^3$, is an open, bounded connected subset with a Lipschitz continuous boundary, the set U being locally on the same side of its boundary, in the sense of Nečas[20] or Adams[1]. Spaces of vector-valued, or matrix-valued, functions over U are denoted by boldface letters, and the norms of the spaces $L^2(U)$ or $\boldsymbol{L}^2(U)$, and $H^m(U)$ or $\boldsymbol{H}^m(U), m \geq 1$, are denoted $\|\cdot\|_{0,U}$, and $\|\cdot\|_{m,U}$.

If V is a vector space and R a subspace of V, the quotient space of V *modulo* R is denoted V/R and the equivalence class of $v \in V$ *modulo* R is denoted $\dot{v}$. The space of all continuous linear mappings from a normed vector space X into a normed vector space Y is denoted $\mathcal{L}(X;Y)$.

Given a domain $\omega \subset \mathbb{R}^2$, a mapping $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$ is an *immersion* if the vectors $\partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \overline{\omega}$. Given a domain $\Omega \subset \mathbb{R}^3$, a mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^3)$ is an *immersion* if the vectors $\partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all points $x \in \overline{\Omega}$ (equivalently, the matrix $\boldsymbol{\nabla\Theta}(x)$ is invertible at all points $x \in \overline{\Omega}$).

Given a domain ω in $\mathbb{R}^2$ and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, define the *surface* $S := \boldsymbol{\theta}(\overline{\omega})$. The covariant components $a_{\alpha\beta} = a_{\beta\alpha} \in \mathcal{C}^2(\overline{\omega})$ and $b_{\alpha\beta} = b_{\beta\alpha} \in \mathcal{C}^1(\overline{\omega})$ of the *first* and *second fundamental forms* of the surface S are then respectively given by

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \text{ and } b_{\alpha\beta} := \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta,$$

where the vector fields

$$\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$$

form the *covariant bases* along S. We also let

$$a := \det(a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}),$$

so that $\sqrt{a} dy$ is the *area element* along S. Note that there clearly exist constants a_0 and a_1 such that $0 < a_0 \leq a(y) \leq a_1$ for all $y \in \overline{\omega}$.

Two other fundamental tensors play a key rôle in the two-dimensional theory of linearly elastic shells, the *linearized change of metric tensor* and the *linearized change of curvature tensor*, each one being associated with a displacement vector field

$$\widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$$

of the surface S, where

$$\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)$$

and the vector fields $\boldsymbol{a}^i$, which form the *contravariant bases* along S, are defined by the relations $\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$. The covariant components of these tensors are given by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}]^{\text{lin}} = \frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta),$$

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := [b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}]^{\rm lin} = (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}})\cdot \boldsymbol{a}_3$$

where $a_{\alpha\beta}(\boldsymbol{\eta})$ and $b_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the first and second fundamental forms of the deformed surface $(\boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}})(\overline{\omega})$, the notation $[\cdots]^{\rm lin}$ represents the linear part with respect to $\boldsymbol{\eta} = (\eta_i)$ in the expression $[\cdots]$, and $\Gamma^{\sigma}_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta$ are the *Christoffel symbols* of S. Note that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ if $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$.

Remark 2.1 More details about the elementary notions of differential geometry of surfaces needed here are found in, e.g., Chapter 2 of Ciarlet[8]. □

Let $\varepsilon > 0$ be such that the mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\omega} \times [-\varepsilon, \varepsilon]; \mathbb{E}^3)$ defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \overline{\omega} \times [-\varepsilon, \varepsilon]$$

is an immersion (for a proof that this is indeed the case if $\varepsilon > 0$ is small enough, see Theorem 3.1-1 in *ibid.*). Assume that the set $\boldsymbol{\Theta}(\overline{\omega} \times [-\varepsilon, \varepsilon])$ is the *reference configuration* occupied in the absence of applied forces by a *linearly elastic shell* with *middle surface* S and *thickness* 2ε, with a constituting material that is *homogeneous* and *isotropic*, hence characterized by its two *Lamé constants* λ and $\mu > 0$. The functions

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}) \in \mathcal{C}^2(\overline{\omega}), \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1},$$

then denote the contravariant components of the *two-dimensional elasticity tensor of the shell*. This tensor is *uniformly positive definite*, in the sense that there exists a constant $b_0 = b_0(\omega, \boldsymbol{\theta}, \mu) > 0$ such that (see Theorem 3.3-2 in *ibid.*):

$$b_0 \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq a^{\alpha\beta\sigma\tau}(y)t_{\sigma\tau}t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ of order two.

Assume that the shell is subjected to *applied forces* acting only in its interior and on its upper and lower faces (there are thus no applied forces acting on its lateral face), whose resultant after integration across the thickness of the shell has contravariant components $p^i \in L^2(\omega)$ (this means that each area element of the shell is subjected to the elementary force $p^i\boldsymbol{a}_i\sqrt{a}dy$). Assume, finally, that the lateral face of the shell is free, i.e., the displacement is not subjected to any boundary condition there. In other words, we are considering a *pure traction problem for a linearly elastic shell*.

As a mathematical model for this problem, we select the well-known *two-dimensional Koiter equations* (so named after Koiter[19]), in the form of the following *quadratic minimization problem*: The unknowns are the three covariant components $\eta_i^* : \overline{\omega} \to \mathbb{R}$ of the displacement field $\eta_i^*\boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface S of the shell and the vector field $\boldsymbol{\eta}^* := (\eta_i^*)$ satisfies

$$\boldsymbol{\eta}^* \in \boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega) \text{ and } j(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta} \in \boldsymbol{V}(\omega)} j(\boldsymbol{\eta}),$$

where

$$j(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \{\varepsilon \boldsymbol{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}\boldsymbol{A}\boldsymbol{\rho}(\boldsymbol{\eta}) : \boldsymbol{\rho}(\boldsymbol{\eta})\}\sqrt{a}dy - l(\boldsymbol{\eta}),$$

and

$$\boldsymbol{A}\boldsymbol{t} : \boldsymbol{t} := a^{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \in L^1(\omega) \text{ for all } \boldsymbol{t} = (t_{\alpha\beta}) \in \boldsymbol{L}^2_{\text{sym}}(\omega),$$

$$\boldsymbol{\gamma}(\boldsymbol{\eta}) := (\gamma_{\alpha\beta}(\boldsymbol{\eta})) \in \boldsymbol{L}^2_{\text{sym}}(\omega) := \{\boldsymbol{c} = (c_{\alpha\beta}) \in (L^2(\omega))^4; c_{\alpha\beta} = c_{\beta\alpha}\} \text{ for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega),$$
$$\boldsymbol{\rho}(\boldsymbol{\eta}) := (\rho_{\alpha\beta}(\boldsymbol{\eta})) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \text{ for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega),$$

$$l(\boldsymbol{\eta}) := \int_\omega p^i \eta_i \sqrt{a} d\omega \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega).$$

Remark 2.2 A detailed mathematical analysis of Koiter's equations, together with their justification from three-dimensional linearized elasticity and numerous references, are found in Section 2.6 and Chapter 7 of Ciarlet[8]. See also Blouza and Le Dret[7] who showed how to handle shells whose middle surface has "little regularity", e.g., when the mapping $\boldsymbol{\theta}$ is only in the space $W^{2,\infty}(\omega; \mathbb{E}^3)$, instead of $\mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ as here. □

Define the Hilbert space

$$\mathbf{Rig}(\omega) := \{\boldsymbol{\eta} \in \boldsymbol{V}(\omega);\ \boldsymbol{\gamma}(\boldsymbol{\eta}) = \rho(\boldsymbol{\eta}) = \mathbf{0} \text{ in } \boldsymbol{L}^2_{\text{sym}}(\omega)\}.$$

Then one can show that (see, e.g., Theorem 2.6-3 of *ibid.*):

$$\mathbf{Rig}(\omega) = \left\{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)\,; \eta_i \boldsymbol{a}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}, \boldsymbol{a} \in \mathbb{R}^3, b \in \mathbb{R}^3\right\}.$$

In other words, the components η_i of any vector field $\boldsymbol{\eta} = (\eta_i) \in \mathbf{Rig}\ (\omega)$ are the covariant components of an *infinitesimal rigid displacement* $\eta_i \boldsymbol{a}^i$ *of the surface* S.

We will henceforth assume that the linear form l associated with the applied forces satisfies the *compatibility conditions*

$$l(\boldsymbol{\eta}) = 0 \text{ for all } \boldsymbol{\eta} \in \mathbf{Rig}(\omega),$$

since these are clearly necessary for the existence of a minimizer of the functional j over the space $\boldsymbol{V}(\omega)$. This being the case, the above minimization problem thus amounts to finding an equivalence class $\dot{\boldsymbol{\eta}}^*$ that satisfies

$$\dot{\boldsymbol{\eta}}^* \in \dot{\boldsymbol{V}}(\omega) := \boldsymbol{V}(\omega)/\mathbf{Rig}(\omega) \quad \text{and} \quad j(\dot{\boldsymbol{\eta}}^*) = \inf_{\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)} j(\dot{\boldsymbol{\eta}}).$$

Before analyzing this problem, let us define several norms, which will be of constant use in the sequel:

$$\|(\boldsymbol{c}, \boldsymbol{r})\|_{0,\omega} := \{\sum_{\alpha,\beta} \|c_{\alpha\beta}\|^2_{0,\omega} + \sum_{\alpha,\beta} \|r_{\alpha\beta}\|^2_{0,\omega}\}^{1/2}$$

for all $(\boldsymbol{c},\boldsymbol{r}) = ((c_{\alpha\beta}),(r_{\alpha\beta})) \in \boldsymbol{L}^2_{\rm sym}(\omega)\times\boldsymbol{L}^2_{\rm sym}(\omega)$,

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} := \Big\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\Big\}^{1/2} \text{ for all } \boldsymbol{\eta}=(\eta_i)\in \boldsymbol{V}(\omega),$$
$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} := \inf_{\boldsymbol{\xi}\in\,\mathbf{Rig}\,(\omega)} \|\boldsymbol{\eta}+\boldsymbol{\xi}\|_{\boldsymbol{V}(\omega)} \text{ for all } \dot{\boldsymbol{\eta}}\in\dot{\boldsymbol{V}}(\omega).$$

In order to establish the existence and uniqueness of a minimizer of the functional j over the space $\dot{\boldsymbol{V}}(\omega)$, it suffices, thanks to the positive definiteness of the two-dimensional elasticity tensor of the shell, to show that *the mapping*

$$\dot{\boldsymbol{\eta}}\in\dot{\boldsymbol{V}}(\omega)\to\|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}),\boldsymbol{\rho}(\dot{\boldsymbol{\eta}})\|_{0,\omega}$$

is a norm over the quotient space $\dot{\boldsymbol{V}}(\omega)$ *equivalent to the quotient norm* $\|\cdot\|_{\dot{\boldsymbol{V}}(\omega)}$.

To prove that this is indeed the case will be achieved in two stages, which constitute Theorems 2.1 and 2.2 below. The first stage, which is due to Bernadou, Ciarlet and Miara[6], consists in establishing a first basic *Korn inequality on a surface, "over the space* $\boldsymbol{V}(\omega)$". Although its proof is thus known, we nevertheless summarize it, for the sake of a later comparison with the present approach.

Theorem 2.1 *Let there be given a domain* ω *in* $\mathbb{R}^2$ *and an immersion* $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{E}^3)$. *Then there exists a constant* $c=c(\omega,\boldsymbol{\theta})$ *such that*

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} \le c\Big\{\sum_\alpha \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \|(\boldsymbol{\gamma}(\boldsymbol{\eta}),\boldsymbol{\rho}(\boldsymbol{\eta}))\|^2_{0,\omega}\Big\}^{1/2}$$

for all $\boldsymbol{\eta}\in\boldsymbol{V}(\omega) = H^1(\omega)\times H^1(\omega)\times H^2(\omega)$.

Proof. The essence of this inequality is that *the two Hilbert spaces* $\boldsymbol{V}(\omega)$ *and*

$$\boldsymbol{W}(\omega) := \big\{\boldsymbol{\eta}=(\eta_i)\in L^2(\omega)\times L^2(\omega)\times H^1(\omega);\boldsymbol{\gamma}(\boldsymbol{\eta})\in\boldsymbol{L}^2_{\rm sym}(\omega),\boldsymbol{\rho}(\boldsymbol{\eta})\in\boldsymbol{L}^2_{\rm sym}(\omega)\big\}$$

coincide. Korn's inequality on a surface then becomes an immediate consequence of the *closed graph theorem* applied to the identity mapping from $\boldsymbol{V}(\omega)$ into $\boldsymbol{W}(\omega)$, which is thus surjective (and otherwise clearly continuous.)

That these two spaces are identical hinges on a fundamental *lemma of J. L. Lions*: Let U be a domain in $\mathbb{R}^n$. If a distribution $v\in H^{-1}(U)$ has its n first partial derivatives also in $H^{-1}(U)$, then $v\in L^2(\Omega)$ (see Theorem 3.2, Chapter 3 of Duvaut and Lions[16] for domains with smooth boundaries and Amrouche and Girault[3] for domains with Lipschitz-continuous boundaries).

To establish the inclusion $\boldsymbol{W}(\omega)\subset\boldsymbol{V}(\omega)$ (the other inclusion evidently holds), let $\boldsymbol{\eta}=(\eta_i)\in\boldsymbol{W}(\omega)$. The relations

$$\hat{\varepsilon}_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \Gamma^\sigma_{\alpha\beta}\eta_\sigma + b_{\alpha\beta}\eta_3$$

then imply that $\hat{\varepsilon}_{\alpha\beta}(\boldsymbol{\eta})\in L^2(\omega)$. Hence

$$\partial_\beta(\partial_\sigma\eta_\alpha) = \{\partial_\beta\hat{\varepsilon}_{\alpha\sigma}(\boldsymbol{\eta}) + \partial_\sigma\hat{\varepsilon}_{\alpha\beta}(\boldsymbol{\eta}) - \partial_\alpha\hat{\varepsilon}_{\beta\sigma}(\boldsymbol{\eta})\}\in H^{-1}(\omega)$$

since $\chi \in L^2(\omega)$ implies $\partial_\alpha \chi \in H^{-1}(\omega)$; likewise, $\partial_\sigma \eta_\alpha \in H^{-1}(\omega)$. Hence $\partial_\sigma \eta_\alpha \in L^2(\omega)$ by the lemma of J. L. Lions and thus $\eta_\alpha \in H^1(\omega)$. This shows that

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = \partial_{\alpha\beta}\eta_3 + \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \text{ with } \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega).$$

Hence $\partial_{\alpha\beta}\eta_3 \in L^2(\omega)$ and thus $\eta_3 \in H^2(\omega)$.

Remark 2.3 The Korn inequality established in Theorem 2.1 in turn implies the more commonly used *Korn inequality "with boundary conditions" on a surface.* This inequality asserts that, given any measurable subset γ_0 of the boundary of ω that satisfies *length* $\gamma_0 > 0$, there exists a constant $c_0 = c_0(\omega, \boldsymbol{\theta}, \gamma_0)$ such that

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} \leq c_0 \|(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta}))\|_{0,\omega}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$ *that satisfy* $\eta_i = \partial_\nu \eta_3 = 0$ *on* γ_0. For more details, see, e.g., Section 2.6 in Ciarlet[8]. □

The second stage consists in establishing another basic *Korn's inequality on a surface, this time "over the quotient space* $\dot{\boldsymbol{V}}(\omega)$".

Theorem 2.2 *Let there be given a domain* ω *in* $\mathbb{R}^2$ *and an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. *Then there exists a constant* $\dot{c} = \dot{}(\boldsymbol{\varpi}, \boldsymbol{\theta})$ *such that*

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} \leq \dot{c} \|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}))\|_{0,\omega}$$

for all $\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega) = (H^1(\omega) \times H^1(\omega) \times H^2(\omega))/\mathbf{Rig}(\omega)$.

Proof. By the Hahn-Banach theorem, there exist six continuous linear forms l_α on the space $\boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega), 1 \leq \alpha \leq 6$, with the following property: A vector field $\boldsymbol{\xi} \in \mathbf{Rig}(\omega)$ is equal to $\mathbf{0}$ if and only if $l_\alpha(\boldsymbol{\xi}) = 0, 1 \leq \alpha \leq 6$. It thus suffices to show that there exists a constant $\dot{c}$ such that

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} \leq \dot{c}(\|(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta}))\|_{0,\omega} + \sum_{\alpha=1}^{6} |l_\alpha(\boldsymbol{\eta})|)$$

for all $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$. For, given any $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, let $\boldsymbol{\xi}(\boldsymbol{\eta}) \in \mathbf{Rig}(\omega)$ be such that $l_\alpha(\boldsymbol{\eta} + \boldsymbol{\xi}(\boldsymbol{\eta})) = 0, 1 \leq \alpha \leq 6$. The above inequality then implies that, for all $\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)$,

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} \leq \|\boldsymbol{\eta} + \boldsymbol{\xi}(\boldsymbol{\eta})\|_{\boldsymbol{V}(\omega)} \leq \dot{c}\|(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta}))\|_{0,\omega}.$$

Assume that there does not exist such a constant $\dot{c}$. Then there exist $\boldsymbol{\eta}^k \in \boldsymbol{V}(\omega), k \geq 1$, such that

$$\|\boldsymbol{\eta}^k\|_{\boldsymbol{V}(\omega)} = 1 \text{ for all } k \geq 1,$$

$$(\|(\boldsymbol{\gamma}(\boldsymbol{\eta}^k), \boldsymbol{\rho}(\boldsymbol{\eta}^k))\|_{0,\omega} + \sum_{\alpha=1}^{6} |l_\alpha(\boldsymbol{\eta}^k)|) \underset{k\to\infty}{\to} 0.$$

By Rellich theorem, there thus exists a subsequence $(\boldsymbol{\eta}^l)_{l=1}^{\infty}$ that converges in the space $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$ on the one hand; on the other hand, each subsequence $(\boldsymbol{\gamma}(\boldsymbol{\eta}^l))_{l=1}^{\infty}$ and $(\boldsymbol{\rho}(\boldsymbol{\eta}^l))_{l=1}^{\infty}$ converges in the space $\boldsymbol{L}^2_{\mathrm{sym}}(\omega)$. Therefore the subsequence $(\boldsymbol{\eta}^l)_{l=1}^{\infty}$ is a Cauchy sequence with respect to the norm

$$\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) \to \Big\{\sum_{\alpha} \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|_{1,\omega} + \|(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta}))\|^2_{0,\omega}\Big\}^{1/2},$$

hence also with respect to the norm $\|\cdot\|_{\boldsymbol{V}(\omega)}$ by the *Korn inequality on a surface* established in Theorem 2.1.

Consequently, there exists $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ such that $\|\boldsymbol{\eta}^l - \boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} \underset{l\to\infty}{\to} 0$. But then $\boldsymbol{\eta} = \boldsymbol{0}$ since $\boldsymbol{\gamma}(\boldsymbol{\eta}) = \boldsymbol{\rho}(\boldsymbol{\eta}) = \boldsymbol{0}$ and $l_\alpha(\boldsymbol{\eta}) = 0$, in contradiction with the relations $\|\boldsymbol{\eta}^l\|_{\boldsymbol{V}(\omega)} = 1$ for all $l \geq 1$. $\square$

We emphasize that our subsequent analysis will provide "as by-products" entirely different proofs of the above Korn inequalities on a surface.

3 Weak versions of a classical theorem of Poincaré and of St Venant's compatibility relations

Our first objective naturally consists in characterizing those pairs of symmetric 2×2 matrix fields $(c_{\alpha\beta}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ and $(r_{\alpha\beta}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$, i.e., the *new unknowns* in our approach, that have the following property: There exists a vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ such that

$$c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}) \quad \text{and} \quad r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta}),$$

where the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are those of the linearized change of metric and change of curvature tensors associated with the displacement field $\eta_i \boldsymbol{a}^i$ of the surface $\boldsymbol{\theta}(\overline{\omega})$ (these components are defined in Section 2).

To shed light on our proposed methodology for this purpose, we return to that of Ciarlet & Ciarlet, Jr.[10] for handling the three-dimensional analog, viz., the characterization of those symmetric 3×3 matrix fields $\boldsymbol{e}$ with components in $L^2(\Omega)$ that can be written as $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v})$ for some $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, where Ω is a domain in $\mathbb{R}^3$.

A classical *theorem of Poincaré* (see, e.g., Schwartz[22]) asserts that, if functions $h_k \in \mathcal{C}^1(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in a simply-connected open subset Ω of $\mathbb{R}^3$ (or $\mathbb{R}^n$ for that matter), then there exists a function $p \in \mathcal{C}^2(\Omega)$ such that $h_k = \partial_k p$ in Ω. This theorem was extended by Girault and Raviart[18] (see Theorem 2.9, Chapter 1), who showed that, if functions $h_k \in L^2(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-1}(\Omega)$ on a simply-connected domain Ω of $\mathbb{R}^3$, then there exists $p \in H^1(\Omega)$ such that $h_k = \partial_k p$ in $L^2(\Omega)$. As shown by Ciarlet and Ciarlet, Jr.[10], this extension can be carried out one step further, as follows (the proof is somewhat more delicate however):

Theorem 3.1 *Let Ω be a simply-connected domain in $\mathbb{R}^3$. Let $h_k \in H^{-1}(\Omega)$ be distributions that satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-2}(\Omega)$. Then there exists a function $p \in L^2(\Omega)$, unique* *up to an additive constant, such that* $h_k = \partial_k p$ in $H^{-1}(\Omega)$.

Idea of the proof: Given *any* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, Theorem 5.1, Chapter 1 of Girault and Raviart[18] shows that there exist $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in L^2(\Omega)$ such that (the assumptions that Ω is bounded and has a Lipschitz-continuous boundary are used here) $-\Delta \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{h}$ in $\boldsymbol{H}^{-1}(\Omega)$ and div $\boldsymbol{u} = 0$ in Ω.

It then suffices to show that, if *in addition* $\mathbf{curl}\, \boldsymbol{h} = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, then $\boldsymbol{u} = \mathbf{0}$. The proof of this implication, which is by no means trivial, relies in particular on an extension result of Girault[17] (see Theorem 3.2) and on a representation result of Girault and Raviart[18] (see Theorem 2.9, Chapter 1), the assumption of simple-connectedness of Ω playing a crucial rôle here (as in the "classical" version of Theorem 3.1). □

In 1864, A. J. C. B. de Saint Venant showed that, if functions $e_{ij} = e_{ji} \in \mathcal{C}^3(\Omega)$ satisfy in a simply-connected open subset Ω of $\mathbb{R}^3$ ad hoc *compatibility relations* that since then bear his name, then there exists a vector field $(v_i) \in \boldsymbol{\mathcal{C}}^4(\Omega)$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ in Ω. Thanks to Theorem 3.1, it can be shown that the same *St Venant compatibility relations* are also sufficient conditions *in the sense of distributions*, according to the following result (again due to Ciarlet and Ciarlet, Jr.[10]):

Theorem 3.2 *Let Ω be a simply-connected domain in $\mathbb{R}^3$. Let*

$$\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2_{sym}(\Omega) := \{\boldsymbol{e} = (e_{ij}) \in (L^2(\Omega))^9; e_{ij} = e_{ji}\}$$

be a symmetric matrix field that satisfies the following compatibility relations:

$$\mathcal{R}_{ijkl}(\boldsymbol{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega).$$

Then there exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ such that

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla} \boldsymbol{v}^T + \boldsymbol{\nabla} \boldsymbol{v}) \text{ in } \boldsymbol{L}^2_{\rm sym}(\Omega),$$

and all other solutions $\tilde{\boldsymbol{v}} = (\tilde{v}_i) \in \boldsymbol{H}^1(\Omega)$ of the equations

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla} \tilde{\boldsymbol{v}}^T + \boldsymbol{\nabla} \tilde{\boldsymbol{v}}) \text{ in } \boldsymbol{L}^2_{\rm sym}(\Omega)$$

are of the form

$$\tilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}, \text{ with } \boldsymbol{a} \in \mathbb{R}^3 \text{ and } \boldsymbol{b} \in \mathbb{R}^3.$$

Proof. The compatibility relations $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$ may be equivalently rewritten as

$$\partial_l h_{ijk} = \partial_k h_{ijl} \text{ in } H^{-2}(\Omega) \text{ with } h_{ijk} := \partial_j e_{ik} - \partial_i e_{jk} \in H^{-1}(\Omega).$$

Hence Theorem 3.1 shows that there exist functions $p_{ij} \in L^2(\Omega)$, unique up to additive constants, such that $\partial_k p_{ij} = h_{ijk} = \partial_j e_{ik} - \partial_i e_{jk}$ in $H^{-1}(\Omega)$. Besides, since $\partial_k p_{ij} = -\partial_k p_{ji}$

in $H^{-1}(\Omega)$, we have the freedom of choosing the functions p_{ij} in such a way that $p_{ij}+p_{ji}=0$ in $L^2(\Omega)$.

Noting that the functions $q_{ij} := (e_{ij}+p_{ij}) \in L^2(\Omega)$ satisfy

$$\partial_k q_{ij} = \partial_k e_{ij} + \partial_k p_{ij} = \partial_k e_{ij} + \partial_j e_{ik} - \partial_i e_{jk} = \partial_j e_{ik} + \partial_j p_{ik} = \partial_j q_{ik} \text{ in } H^{-1}(\Omega),$$

we again resort to Theorem 3.1 to assert the existence of functions $v_i \in H^1(\Omega)$, unique up to additive constants, such that $\partial_j v_i = q_{ij} = e_{ij} + p_{ij}$ in $L^2(\Omega)$. Consequently,

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} + \frac{1}{2}(p_{ij}+p_{ji}) = e_{ij} \text{ in } L^2(\Omega),$$

as required. That all other solutions are of the indicated form is well-known; see, e.g., Chapter 3 of Duvaut and Lions[16]. □

Remark 3.1 A direct computation immediately shows that the St Venant compatibility relations are also *necessary*, i.e., that $\mathcal{R}_{ijkl}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v}) = 0$ in $H^{-2}(\Omega)$ for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. □

Remark 3.2 It is easily verified that the eighty-one relations $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$ are satisfied if only *six* of them hold, provided they are suitably chosen; for instance, it suffices that they be satisfied for $(i,j,k,l) = (1,2,1,2), (1,2,1,3), (1,2,2,3), (1,3,1,3), (1,3,2,3)$, and $(2,3,2,3)$. □

Remark 3.3 Another "weak" characterization of matrix fields $\boldsymbol{e} \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ satisfying $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v})$ for some $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ is found in Ting[23]. □

Interestingly, the same "three-dimensional" St Venant compatibility relations also play a key rôle in the proof of our main result, Theorem 4.1 below.

4 A necessary and sufficient condition for matrix fields to be linearized change of metric and change of curvature tensors

Thanks to Theorem 3.2, we are now in a position to establish an analog characterization, but this time in the case of a surface.

Theorem 4.1 *Let there be given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. For each $\varepsilon > 0$, define the mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\omega} \times [-\varepsilon, \varepsilon]; \mathbb{R}^3)$ by*

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \frac{\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)}{|\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)|} \text{ for all } (y, x_3) \in \overline{\omega} \times [-\varepsilon, \varepsilon].$$

Then there exist $\varepsilon_0 > 0$ and a mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$, where

$$\widehat{\Omega} := \omega \times]-\varepsilon_0, \varepsilon_0[\text{ and } \boldsymbol{H}^{-2}(\widehat{\Omega}) := (H^{-2}(\widehat{\Omega}))^6,$$

with the following property: A pair $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega)$ *of symmetric matrix fields satisfies*

$$\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\widehat{\Omega})$$

if and only if there exists a vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ *such that*

$$\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta}) \text{ and } \boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta}) \text{ in } \boldsymbol{L}^2_{\text{sym}}(\omega).$$

In this case, all other solutions $\boldsymbol{\eta}' \in \boldsymbol{V}(\omega)$ *of the equations*

$$\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta}') \text{ and } \boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta}') \text{ in } \boldsymbol{L}^2_{\text{sym}}(\omega)$$

are such that $(\boldsymbol{\eta}' - \boldsymbol{\eta}) \in \mathbf{Rig}(\omega)$.

Proof. The uniqueness up to vector fields in the space $\mathbf{Rig}(\omega)$ is well known, as already noted.

The outline of the proof is as follows: In Part(i), we briefly review some notions about curvilinear coordinates that are needed in the sequel; in Part (ii), we "recast in curvilinear coordinates" the sufficiency of the St Venant compatibility relations; in Part (iii), we summarize those results from Ciarlet & S. Mardare[14] that will be used in our analysis (the objective in *ibid.* was to show how Korn's inequality on a surface can be obtained as a corollary to the three-dimensional Korn inequality in curvilinear coordinates; in this respect, see also Akian[2]); combining all these results, we then conclude the proof in Part (iv).

(i) For details about the results reviewed here, see, e.g., Chapter 1 of Ciarlet[8]. Let Ω be a domain in $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^3)$ be a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}$ onto its image $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbb{E}^3$. Then the coordinates of $x \in \overline{\Omega}$ are called the *curvilinear coordinates* of the point $\hat{x} := \boldsymbol{\Theta}(x) \in \boldsymbol{\Theta}(\Omega)$ and $\hat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is also a domain. The *metric tensor of* $\boldsymbol{\Theta}(\overline{\Omega})$ is defined by means of its covariant components $g_{ij} := \boldsymbol{g}_i \cdot \boldsymbol{g}_j$, where the vector fields $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$ form the *covariant bases.* The vector fields $\boldsymbol{g}^j$ defined by the relations $\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta^j_i$ form the *contravariant bases* and the functions $\Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j$ are the *Christoffel symbols.*

Given any vector field $\hat{\boldsymbol{v}} = (\hat{v}_i) \in \boldsymbol{H}^1(\hat{\Omega})$, the functions

$$\hat{\varepsilon}_{ij}(\hat{\boldsymbol{v}}) := \frac{1}{2}(\hat{\partial}_j \hat{v}_i + \hat{\partial}_i \hat{v}_j) \in L^2(\hat{\Omega})$$

are the *linearized strains in Cartesian coordiantes* associated with the displacement field $\hat{v}_i \hat{\boldsymbol{e}}^i$ of the set $\hat{\Omega}$ and

$$\hat{\boldsymbol{\varepsilon}}(\boldsymbol{v}) := (\hat{\varepsilon}_{ij}(\hat{v})) \in \boldsymbol{L}^2_{\text{sym}}(\hat{\Omega})$$

is the associated *linearized strain tensor field in Cartesian coordinates.*

Let the vector field $\boldsymbol{v} = (v_j) \in \boldsymbol{H}^1(\Omega)$ be defined by means of the relations

$$v_j(x)\boldsymbol{g}^j(x) = \hat{v}_i(\hat{x})\hat{\boldsymbol{e}}^i \text{ for almost all } x = \boldsymbol{\Theta}^{-1}(\hat{x}) \in \Omega.$$

Then the functions

$$\varepsilon_{ij}(\boldsymbol{v}) := \left\{ \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma^p_{ij} v_p \right\} \in L^2(\Omega)$$

are the *linearized strains in curvilinear coordinates* associated with the displacement field $v_j \boldsymbol{g}^i$ of the set $\boldsymbol{\Theta}(\overline{\Omega})$ and

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) := (\varepsilon_{ij}(\boldsymbol{v})) \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$$

is the associated *linearized strain tensor field in curvilinear coordinates* (here, "strain" is to be understood as "change of metric", thus reflecting that the matrix field $\boldsymbol{\varepsilon}(\boldsymbol{v})$ measures half of the linearized difference between the metric tensor in the "deformed configuration" $(\boldsymbol{\Theta} + v_i \boldsymbol{g}^i)(\overline{\Omega})$ and the metric tensor in the "reference configuration" $\boldsymbol{\Theta}(\overline{\Omega})$).

It is then easily seen that the two above linearized strain tensor fields are related by the relations

$$\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}})(\hat{x})(\boldsymbol{\nabla\Theta}^{-T}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{\nabla\Theta}^{-1})(x) \text{ for almost all } \hat{x} = \boldsymbol{\Theta}(x) \in \hat{\Omega}.$$

(ii) *Let Ω be a simply-connected domain in $\mathbb{R}^3$, let $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^3)$ be a $\mathcal{C}^2-$diffeomorphism of $\overline{\Omega}$ onto its image, let $\hat{\Omega} := \boldsymbol{\Theta}(\Omega)$, let the mapping $\boldsymbol{\mathcal{B}} \in \mathcal{L}(\boldsymbol{L}^2_{\text{sym}}(\Omega); \boldsymbol{L}^2_{\text{sym}}(\hat{\Omega}))$ be defined for any $\boldsymbol{e} \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ by*

$$(\boldsymbol{\mathcal{B}e})(\hat{x}) := (\boldsymbol{\nabla\Theta}^{-T}\boldsymbol{e}\boldsymbol{\nabla\Theta}^{-1})(x) \text{ for almost all } \hat{x} = \boldsymbol{\Theta}(x) \in \hat{\Omega},$$

and let, for any $\hat{\mathbf{e}} = (\hat{e}_{ij}) \in \boldsymbol{L}^2_{\text{sym}}(\hat{\Omega})$,

$$\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{e}}) := (\hat{\boldsymbol{\mathcal{R}}}_{ijkl}(\hat{\boldsymbol{e}})) \text{ where } \hat{\boldsymbol{\mathcal{R}}}_{ijkl}(\hat{\boldsymbol{e}}) := \hat{\partial}_{lj}\hat{e}_{ik} + \hat{\partial}_{ki}\hat{e}_{jl} - \hat{\partial}_{li}\hat{e}_{jk} - \hat{\partial}_{kj}\hat{e}_{il}.$$

Then a matrix field $\boldsymbol{e} \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ is such that there exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\omega)$ that satisfies

$$\boldsymbol{e} = \boldsymbol{\varepsilon}(\boldsymbol{v}) \text{ in } \boldsymbol{L}^2_{\text{sym}}(\Omega)$$

if and only if

$$\boldsymbol{\mathcal{R}}(\boldsymbol{e}) = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\hat{\Omega}) = (H^{-2}(\hat{\Omega}))^6,$$

where the mapping $\boldsymbol{\mathcal{R}} \in \mathcal{L}(\boldsymbol{L}^2_{\text{sym}}(\Omega); \boldsymbol{H}^{-2}(\hat{\Omega}))$ is defined by

$$\boldsymbol{\mathcal{R}} := \hat{\boldsymbol{\mathcal{R}}} \circ \boldsymbol{\mathcal{B}}.$$

First, we note that the domain $\hat{\Omega} \subset \mathbb{E}^3$ is simply-connected if the domain $\Omega \subset \mathbb{R}^3$ is simply-connected. Second, it is clear that $\boldsymbol{\mathcal{B}}$ is an isomorphism from $\boldsymbol{L}^2_{\text{sym}}(\Omega)$ onto $\boldsymbol{L}^2_{\text{sym}}(\hat{\Omega})$.

Let then $\boldsymbol{e} \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ be a matrix field that satisfies $\boldsymbol{\mathcal{R}}(\boldsymbol{e}) = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\hat{\Omega})$. By Theorem 3.2, there thus exists a vector field $\hat{\boldsymbol{v}} = (\hat{v}_i) \in \boldsymbol{H}^1(\hat{\Omega})$ such that

$$\boldsymbol{\mathcal{B}e} = \hat{\varepsilon}(\hat{\boldsymbol{v}}) \text{ in } \boldsymbol{L}^2_{\text{sym}}(\hat{\Omega}).$$

By (i), the vector field $\boldsymbol{v} = (v_j) \in \boldsymbol{H}^1(\Omega)$ defined by $v_j \boldsymbol{g}^j(x) = \hat{v}_i(\hat{x})\hat{\mathbf{e}}^i$ for almost all $x = \boldsymbol{\Theta}^{-1}(\hat{x}) \in \Omega$ satisfies

$$\hat{\varepsilon}(\hat{\boldsymbol{v}}) = \boldsymbol{\mathcal{B}}\boldsymbol{\varepsilon}(\boldsymbol{v}).$$

Hence $\boldsymbol{e} = \boldsymbol{\varepsilon}(\boldsymbol{v})$ since the mapping $\mathcal{B}$ is injective.

Conversely, let $\boldsymbol{v} = (v_j) \in \boldsymbol{H}^1(\Omega)$ be given. Then, by Part (i), $\boldsymbol{\mathcal{B}\varepsilon}(\boldsymbol{v}) = \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}})$, where the vector field $\hat{\boldsymbol{v}} = (\hat{v}_i) \in \boldsymbol{H}^1(\hat{\Omega})$ is defined by $\hat{v}_i(\hat{x})\hat{\boldsymbol{e}}^i = v_j \boldsymbol{g}^j(x)$ for almost all $\hat{x} = \boldsymbol{\Theta}(x) \in \Omega$. Hence $\hat{\boldsymbol{\mathcal{R}}}_{ijkl}(\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}})) = 0$ in $H^{-2}(\hat{\Omega})$ for all i, j, k, l since these relations are necessary (Remark 3.1). We have already noted (Remark 3.2) that six of these relations suffice, thus justifying the notation $\boldsymbol{H}^{-2}(\hat{\Omega}) = (H^{-2}(\hat{\Omega}))^6$.

(iii) The results recapitulated below are proved in Ciarlet and S. Mardare[14]. Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an injective immersion. As before, we denote by $\boldsymbol{a}_i$ and $\boldsymbol{a}^j$ the vector fields defined by the relations $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}, \boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$, and $\boldsymbol{a}^j \cdot \boldsymbol{a}_i = \delta_i^j$. Then there exists $\varepsilon_0 > 0$ such that *the mapping* $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{E}^3$ *defined by*

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \overline{\Omega}, \text{ where } \Omega := \omega \times]-\varepsilon_0, \varepsilon_0[,$$

is a $\mathcal{C}^2-$*diffeomorphism from* $\overline{\Omega}$ *onto its image* $\boldsymbol{\Theta}(\overline{\Omega})$. Let $\boldsymbol{g}_i$ and $\boldsymbol{g}^j$ denote the vector fields defined by the relations $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}$ and $\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j$. With any vector field $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, let there be associated the vector field $\boldsymbol{v} = (v_j) \in \boldsymbol{H}^1(\Omega)$ defined by

$$v_j(y, x_3)\boldsymbol{g}^j(y, x_3) = \eta_i(y)\boldsymbol{a}^i(y) - x_3(\partial_\alpha \eta_3(y) + b_\alpha^\sigma(y)\eta_\sigma(y))\boldsymbol{a}^\alpha(y)$$

for all $(y, x_3) \in \Omega$, where $b_\alpha^\sigma := a^{\beta\sigma} b_{\alpha\beta}$. Then *the linear mapping*

$$\boldsymbol{F} : \boldsymbol{\eta} \in \boldsymbol{V}(\omega) \to \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$$

defined in this fashion is an isomorphism from the space $\boldsymbol{V}(\omega)$ *onto the Hilbert space*

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \varepsilon_{i3}(\boldsymbol{v}) = 0 \text{ in } \Omega\}.$$

Besides,

$$\varepsilon_{\alpha\beta}(\boldsymbol{F}(\boldsymbol{\eta})) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) - x_3 \rho_{\alpha\beta}(\boldsymbol{\eta}) + \frac{x_3^2}{2}\big\{b_\alpha^\sigma \rho_{\beta\sigma}(\boldsymbol{\eta}) + b_\beta^\tau \rho_{\alpha\tau}(\boldsymbol{\eta}) - 2b_\alpha^\sigma b_\beta^\tau \gamma_{\sigma\tau}(\boldsymbol{\eta})\big\}.$$

(iv) *Let* Ω *be a simply-connected domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ *be an injective immersion. Let the open set* $\Omega = \omega \times]-\varepsilon_0, \varepsilon_0[$ *be defined as in* Part (iii), *let the mapping* $\boldsymbol{G} \in \mathcal{L}(\boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega); \boldsymbol{L}^2_{\rm sym}(\Omega))$ *be defined by*

$$\begin{aligned}
&(\boldsymbol{G}(\boldsymbol{c}, \boldsymbol{r}))_{\alpha\beta} = c_{\alpha\beta} - x_3 r_{\alpha\beta} + \frac{x_3^2}{2}\{b_\alpha^\sigma r_{\beta\sigma} + b_\beta^\tau r_{\alpha\tau} - 2b_\alpha^\sigma b_\beta^\tau c_{\sigma\tau}\},\\
&(\boldsymbol{G}(\boldsymbol{c}, \boldsymbol{r}))_{i3} = 0,
\end{aligned}$$

for any $(\boldsymbol{c}, \boldsymbol{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega)$, *and finally, let* $\hat{\Omega} := \boldsymbol{\Theta}(\Omega)$, *where* $\boldsymbol{\Theta}$ *is the* $\mathcal{C}^2-$*diffeomorphism from* $\overline{\Omega}$ *onto its image defined in* Part (iii).

Then a pair $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega)$ *of matrix fields is such that there exists a vector field* $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ *satisfying*

$$\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta}) \text{ and } \boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta}) \text{ in } \boldsymbol{L}^2_{\rm sym}(\omega)$$

if and only if

$$\boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\hat{\Omega}) = (H^{-2}(\hat{\Omega}))^6,$$

where the mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\rm sym}(\Omega) \times \boldsymbol{L}^2_{\rm sym}(\Omega); \boldsymbol{H}^{-2}(\hat{\Omega}))$ *is defined by*

$$\boldsymbol{R} = \boldsymbol{\mathcal{R}} \circ \boldsymbol{G},$$

the mapping $\boldsymbol{\mathcal{R}} \in \mathcal{L}(\boldsymbol{L}^2_{\rm sym}(\Omega); \boldsymbol{H}^{-2}(\hat{\Omega}))$ *being that defined in* Part (ii).

First, we note that the domain $\Omega \subset \mathbb{R}^3$ is simply-connected if the domain $\omega \subset \mathbb{R}^2$ is simply-connected. Second, we show that the mapping $\boldsymbol{G}$ is injective. To see this, let functions $f, g, h \in L^2(\omega)$ be such that $\mathcal{F} = 0$ in $L^2(\Omega)$, where the function $\mathcal{F}$ is defined by

$$\mathcal{F}(y, x_3) := f(y) + x_3 g(y) + x_3^2 h(y) \text{ for } (y, x_3) \in \Omega.$$

Then $\partial_{33}\mathcal{F} = 0$ in $\mathcal{D}'(\Omega)$ and since the distribution $\partial_{33}\mathcal{F}$ is in this case the function in $L^2(\Omega)$ defined by $\partial_{33}\mathcal{F}(y, x_3) = 2h(y)$ for almost all $(y, x_3) \in \Omega$, we conclude that $h = 0$; since $\partial_3\mathcal{F} = 0$ in $\mathcal{D}'(\Omega)$, we likewise conclude that $g = 0$; hence we are left with $f = 0$.

Let then $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega)$ be a pair of matrix fields that satisfies $\boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\hat{\Omega})$. By Part (ii), there thus exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ that satisfies

$$\boldsymbol{G}(\boldsymbol{c},\boldsymbol{r}) = \boldsymbol{\varepsilon}(\boldsymbol{v}) \text{ in } \boldsymbol{L}^2_{\rm sym}(\Omega).$$

In particular then, $\varepsilon_{i3}(\boldsymbol{v}) = (\boldsymbol{G}(\boldsymbol{c},\boldsymbol{r}))_{i3} = 0$ by definition of $\boldsymbol{G}$, which shows that $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$. Let the vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ be defined by $\boldsymbol{\eta} := \boldsymbol{F}^{-1}(\boldsymbol{v})$, where $\boldsymbol{F} : \boldsymbol{V}(\omega) \to \boldsymbol{V}(\Omega)$ is the isomorphism defined in Part (iii). By definition of $\boldsymbol{G}$ and by Part (iii),

$$\begin{aligned} &(\boldsymbol{G}(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta})))_{\alpha\beta} = \varepsilon_{\alpha\beta}(\boldsymbol{F}(\boldsymbol{\eta})) = \varepsilon_{\alpha\beta}(\boldsymbol{v}), \\ &(\boldsymbol{G}(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta})))_{i3} = 0 = \varepsilon_{i3}(\boldsymbol{v}). \end{aligned}$$

In other words, $\boldsymbol{G}(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta})) = \boldsymbol{\varepsilon}(\boldsymbol{v})$ in $\boldsymbol{L}^2_{\rm sym}(\Omega)$. We thus conclude that

$$\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta}) \text{ and } \boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta}) \text{ in } \boldsymbol{L}^2_{\rm sym}(\omega),$$

since the mapping $\boldsymbol{G}$ is injective.

Conversely, let $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ be given. Then, by definition of the mappings $\boldsymbol{F}$ and $\boldsymbol{G}$, we have $\boldsymbol{G}(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta})) = \boldsymbol{\varepsilon}(\boldsymbol{v})$, where the vector field $\boldsymbol{v} \in \boldsymbol{V}(\Omega) \subset \boldsymbol{H}^1(\Omega)$ is defined by $\boldsymbol{v} := \boldsymbol{F}(\boldsymbol{\eta})$. Hence $\boldsymbol{\mathcal{R}}(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\hat{\Omega})$ by Part (ii). Equivalently,

$$\boldsymbol{R}(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta})) = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\hat{\Omega}),$$

which shows that the relation $\boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\hat{\Omega})$ is also necessary. □

Remark 4.1 In Part (iv), we showed that the mapping $\boldsymbol{G}$ is injective. It can be easily shown that, in addition, its image $\mathrm{Im}\boldsymbol{G}$ is a closed subspace of $\boldsymbol{L}^2_{\rm sym}(\Omega)$. The closed graph theorem then shows that $\boldsymbol{G}$ is in fact an isomorphism from the Hilbert space $\boldsymbol{L}^2_{\rm sym}(\omega) \times \boldsymbol{L}^2_{\rm sym}(\omega)$ onto the Hilbert space $\mathrm{Im}\boldsymbol{G}$. □

Remark 4.2 More *explicit* necessary and sufficient conditions can be also found that take the form of relations in the space $H^{-2}(\omega)$ satisfied by the components of the tensor fields $\boldsymbol{r}$ and $\boldsymbol{c}$ and some of their partial derivatives, with coefficients involving the two fundamental forms and the Christoffel symbols of the surface $\boldsymbol{\theta}(\overline{\omega})$; see Ciarlet, Gratie and C. Mardare.[12] $\square$

5 A new proof of Korn's inequality on a surface

Thanks to Theorem 4.1, we can define in a natural way a basic isomorphism (denoted by $\boldsymbol{H}$ in the next theorem), which plays a key rôle in the rest of this paper (see Theorems 5.2 and 6.1).

Theorem 5.1 *Let there be given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. Define the space*

$$\boldsymbol{T}(\omega) := \left\{(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega); \boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\hat{\Omega})\right\},$$

where the open set $\hat{\Omega}$ and the mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega); \boldsymbol{H}^{-2}(\hat{\Omega}))$ are defined as in Theorem 4.1.

Given any element $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{T}(\omega)$, there exists, again by Theorem 4.1, *a unique equivalence class $\dot{\boldsymbol{\eta}}$ in the quotient space $\dot{\boldsymbol{V}}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega))/\mathbf{Rig}(\omega)$ that satisfies*

$$\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}) = \boldsymbol{c} \text{ and } \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}) = \boldsymbol{r} \text{ in } \boldsymbol{L}^2_{\text{sym}}(\omega).$$

Then the mapping

$$\boldsymbol{H} : \boldsymbol{T}(\omega) \to \dot{\boldsymbol{V}}(\omega)$$

defined by $\boldsymbol{H}(\boldsymbol{c}, \boldsymbol{r}) := \dot{\boldsymbol{\eta}}$ is an isomorphism between the Hilbert spaces $\boldsymbol{T}(\omega)$ and $\dot{\boldsymbol{V}}(\omega)$.

Proof. Clearly, $\boldsymbol{T}(\omega)$ is a Hilbert space as a closed subspace of $\boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega)$. The mapping $\boldsymbol{H}$ is injective, for $\boldsymbol{H}(\boldsymbol{c}, \boldsymbol{r}) = \dot{\boldsymbol{0}}$ implies that $\boldsymbol{c} = \boldsymbol{\gamma}(\dot{\boldsymbol{0}}) = \boldsymbol{0}$ and $\boldsymbol{r} = \boldsymbol{\rho}(\dot{\boldsymbol{0}}) = \boldsymbol{0}$. It is also surjective since, given any $\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)$, the pair $(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}})) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega)$ necessarily satisfies $\boldsymbol{R}(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}})) = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\hat{\Omega})$ by Theorem 5.1.

Finally, the inverse mapping

$$\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega) \to (\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}})) \in \boldsymbol{T}(\omega)$$

is clearly continuous, since there evidently exists a constant C such that, for any $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ and any $\boldsymbol{\xi} \in \mathbf{Rig}(\omega)$,

$$\|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}))\|_{0,\omega} = \|(\boldsymbol{\gamma}(\boldsymbol{\eta} + \boldsymbol{\xi}), \boldsymbol{\rho}(\boldsymbol{\eta} + \boldsymbol{\xi}))\|_{0,\omega} \leq C\|\boldsymbol{\eta} + \boldsymbol{\xi}\|_{\boldsymbol{V}(\omega)}.$$

Hence

$$\|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}))\|_{0,\omega} \leq C \inf_{\boldsymbol{\xi} \in \mathbf{Rig}(\omega)} \|\boldsymbol{\eta} + \boldsymbol{\xi}\|_{\boldsymbol{V}(\omega)} = C\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)}.$$

The conclusion then follows from the *closed graph theorem.* $\square$

Remarkably, the two Korn inequalities on a surface recalled earlier in Theorems 2.1 and 2.2, can now be recovered as simple corollaries to Theorem 5.1.

Theorem 5.2. *Let there be given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$. That the mapping $\boldsymbol{H} : \boldsymbol{T}(\omega) \to \dot{\boldsymbol{V}}(\omega)$ is an isomorphism implies both Korn's inequalities on a surface, i.e., "over the space $\boldsymbol{V}(\omega) = H^1(\omega)\times H^1(\omega)\times H^2(\omega)$"* (Theorem 2.1) *and "over the quotient space $\dot{\boldsymbol{V}}(\omega) = \boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$"* (Theorem 2.2).

Proof. (i) Since $\boldsymbol{H}$ is an isomorphism, there exists a constant $\dot{c}$ such that

$$\|\boldsymbol{H}(\boldsymbol{c},\boldsymbol{r})\|_{\dot{\boldsymbol{V}}(\omega)} \leq \dot{\boldsymbol{c}}\|(\boldsymbol{c},\boldsymbol{r})\|_{0,\omega} \text{ for all } (\boldsymbol{c},\boldsymbol{r}) \in \boldsymbol{T}(\omega),$$

or equivalently, again because $\boldsymbol{H}$ is an isomorphism, such that

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} \leq \dot{\boldsymbol{c}}\|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}),\boldsymbol{\rho}(\dot{\boldsymbol{\eta}})\|_{0,\omega} \text{ for all } \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega).$$

But this is exactly *Korn's inequality "over the quotient space $\dot{\boldsymbol{V}}(\omega)$"* of Theorem 2.2.

(ii) We now show that this Korn inequality in turn implies *Korn's inequality "over the space $\boldsymbol{V}(\omega)$"* of Theorem 2.1.

Assume that this last inequality does not hold. Then there exist $\boldsymbol{\eta}^k = (\eta_i^k) \in \boldsymbol{V}(\omega), k \geq 1$, such that

$$\|\boldsymbol{\eta}^k\|_{\boldsymbol{V}(\omega)} = 1 \text{ for all } k \geq 1,$$

$$\big(\{\sum_\alpha \|\eta_\alpha^k\|^2_{0,\omega} + \|\eta_3^k\|^2_{1,\omega}\}^{1/2} + \|((\boldsymbol{\gamma}(\boldsymbol{\eta}^k),\boldsymbol{\rho}(\boldsymbol{\eta}^k))\|_{0,\omega}\big) \underset{k\to\infty}{\longrightarrow} 0\,.$$

Let $\boldsymbol{\zeta}^k \in \mathbf{Rig}(\omega)$ denote for each $k \geq 1$ the projection of $\boldsymbol{\eta}^k$ on $\mathbf{Rig}(\omega)$ with respect to the inner product of the space $\boldsymbol{V}(\omega)$. This projection thus satisfies:

$$\|\boldsymbol{\eta}^k - \boldsymbol{\zeta}^k\|_{\boldsymbol{V}(\omega)} = \inf_{\boldsymbol{\xi}\in\,\mathbf{Rig}(\omega)} \|\boldsymbol{\eta}^k + \boldsymbol{\xi}\|_{\boldsymbol{V}(\omega)} = \|\dot{\boldsymbol{\eta}}^k\|_{\dot{\boldsymbol{V}}(\omega)},$$

$$\|\boldsymbol{\eta}^k\|^2_{\boldsymbol{V}(\omega)} = \|\boldsymbol{\eta}^k - \boldsymbol{\zeta}^k\|^2_{\boldsymbol{V}(\omega)} + \|\boldsymbol{\zeta}^k\|^2_{\boldsymbol{V}(\omega)}.$$

The space $\mathbf{Rig}(\omega)$ being finite-dimensional, the inequalities $\|\boldsymbol{\zeta}^k\|_{\boldsymbol{V}(\omega)} \leq 1$ for all $k \geq 1$ imply the existence of a subsequence $(\boldsymbol{\zeta}^l)_{l=1}^\infty$ that converges in the space $\boldsymbol{V}(\omega)$ to an element $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_i) \in\mathbf{Rig}(\omega)$. Besides, Korn's inequality in the quotient space $\dot{\boldsymbol{V}}(\omega)$ obtained in Part (i) implies that

$$\|\boldsymbol{\eta}^l - \boldsymbol{\zeta}^l\|_{\boldsymbol{V}(\omega)} = \|\dot{\boldsymbol{\eta}}^l\|_{\dot{\boldsymbol{V}}(\omega)} \underset{l\to\infty}{\longrightarrow} 0,$$

since $\|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^l),\boldsymbol{\rho}(\dot{\boldsymbol{\eta}}^l))\|_{0,\omega} \underset{l\to\infty}{\longrightarrow} 0$. Consequently,

$$\|\boldsymbol{\eta}^l - \boldsymbol{\zeta}\|_{\boldsymbol{V}(\omega)} \underset{l\to\infty}{\longrightarrow} 0\,.$$

Hence $\{\sum_\alpha \|\eta_\alpha^l - \zeta_\alpha\|^2_{0,\omega} + \|\eta_3^l - \zeta_3\|^2_{1,\omega}\}^{1/2} \underset{l\to\infty}{\longrightarrow} 0$ *a fortiori*, which shows that $\boldsymbol{\zeta} = \boldsymbol{0}$ since $\{\sum_\alpha \|\eta_\alpha^l\|^2_{0,\omega} + \|\eta_3^l\|^2_{1,\omega}\}^{1/2} \underset{l\to\infty}{\longrightarrow} 0$ on the other hand. We thus reach the conclusion that $\|\boldsymbol{\eta}^l\|_{\boldsymbol{V}(\omega)} \underset{l\to\infty}{\longrightarrow} 0$, a contradiction. □

Remark 5.1 Together, Theorem 2.1 and Part (ii) of the above proof thus show that both Korn's inequalities on a surface, viz., on the space $\boldsymbol{V}(\omega)$ and on the quotient space $\dot{\boldsymbol{V}}(\omega)$, are *equivalent.* $\square$

6 A new approach to existence theory for Koiter's linear shell equations

Thanks again to the isomorphism $\boldsymbol{H}$ introduced in Theorem 5.1, we are now in a position to recast the quadratic minimization problem that models the pure traction problem of a linearly elastic shell (see Section 2) as *another quadratic minimization problem*, this time over the space $\boldsymbol{T}(\omega)$ introduced in Theorem 5.1. The various notations found in the functional κ below are all defined in Section 2.

Theorem 6.1 *Given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, define the Hilbert space $\boldsymbol{T}(\omega)$ as in* Theorem 5.1, *viz.,*

$$\boldsymbol{T}(\omega) = \{(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega); \boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\hat{\Omega})\}.$$

Furthermore, define the quadratic functional $\kappa : \boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega) \to \mathbb{R}$ by

$$\kappa(\boldsymbol{c}, \boldsymbol{r}) := \frac{1}{2} \int_\omega \{\varepsilon \boldsymbol{A}\boldsymbol{c} : \boldsymbol{c} + \frac{\varepsilon^3}{3} \boldsymbol{A}\boldsymbol{r} : \boldsymbol{r}\} \sqrt{a} dy - l^\flat(\boldsymbol{c}, \boldsymbol{r})$$

for all $(\boldsymbol{c}, \boldsymbol{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\text{sym}} \times \boldsymbol{L}^2_{\text{sym}}(\omega)$, where $l^\flat := l \circ \boldsymbol{H}$.

Then the minimization problem : Find $(\boldsymbol{c}^*, \boldsymbol{r}^*) \in \boldsymbol{T}(\omega)$ such that

$$\kappa(\boldsymbol{c}^*, \boldsymbol{r}^*) = \inf\{\kappa(\boldsymbol{c}, \boldsymbol{r}); (\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{T}(\omega)\}$$

has one and only one solution $(\boldsymbol{c}^*, \boldsymbol{r}^*)$. Besides,

$$(\boldsymbol{c}^*, \boldsymbol{r}^*) = (\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^*), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}^*)),$$

where $\dot{\boldsymbol{\eta}}^*$ is the unique solution to the "classical" minimization problem $\inf_{\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)} j(\dot{\boldsymbol{\eta}})$ described in Section 2.

Proof. Thanks to the uniform positive definiteness of the two-dimensional elasticity tensor of the shell, there exists a constant $b_1 > 0$ such that

$$\int_\omega \big\{\varepsilon \boldsymbol{A}\boldsymbol{c} : \boldsymbol{c} + \frac{\varepsilon^3}{3} \boldsymbol{A}\boldsymbol{r} : \boldsymbol{r}\big\} \sqrt{a} dy \geq b_1 \|(\boldsymbol{c}, \boldsymbol{r})\|^2_{0,\omega}$$

for all $(\boldsymbol{c}, \boldsymbol{r})$ in the space $\boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega)$. Besides, the linear form $l^\flat$ is continuous over the space $\boldsymbol{T}(\omega)$ since the mapping $\boldsymbol{H} : \boldsymbol{T}(\omega) \to \dot{\boldsymbol{V}}(\omega)$ and the linear form $l : \dot{\boldsymbol{V}}(\omega) \to \mathbb{R}$ are both continuous. Finally, $\boldsymbol{T}(\omega)$ is a Hilbert space. Hence there exists one, and only one, minimizer $(\boldsymbol{c}^*, \boldsymbol{r}^*)$ of the functional κ over the space $\boldsymbol{T}(\omega)$.

That $\dot{\boldsymbol{\eta}}^*$ minimizes the functional j over the quotient space $\dot{\boldsymbol{V}}(\omega)$ implies that $(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^*)$, $\boldsymbol{\rho}(\dot{\boldsymbol{\eta}}^*))$ minimizes the functional κ over the space $\boldsymbol{T}(\omega)$. Hence $(\boldsymbol{c}^*, \boldsymbol{r}^*) = ((\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^*), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}^*))$ since the minimizer of κ is unique. $\square$

7 Concluding remarks

(a) While the original minimization problem over the space $H^1(\omega)\times H^1(\omega)\times H^2(\omega)/\mathbf{Rig}(\omega)$ is an *unconstrained* one with three unknowns (see Section 2), that found in Theorem 6.1 is a *constrained minimization problem* over the space $\boldsymbol{L}^2_{\text{sym}}(\omega) \times \boldsymbol{L}^2_{\text{sym}}(\omega)$ with six unknowns. The *constraints* (in the sense of optimization theory) are the compatibility relations $\boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\hat{\Omega})$ that the pairs of matrix fields $(\boldsymbol{r},\boldsymbol{c}) \in \boldsymbol{T}(\omega)$ must satisfy (as shown in Ciarlet, Gratie and C. Mardare[12], in fact these relations ultimately reduce to only three independent ones, which can be set in the space $H^{-2}(\omega)$).

(b) In linear shell theory, the contravariant components of the *stress resultant* tensor field $(n^{\alpha\beta}) \in \boldsymbol{L}^2_{\text{sym}}(\omega)$ and the *bending moment* tensor field $(m^{\alpha\beta}) \in \boldsymbol{L}^2_{\text{sym}}(\omega)$ are given in terms of the displacement vector field by

$$n^{\alpha\beta} = \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{\eta}) \quad \text{and} \quad m^{\alpha\beta} = \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta}),$$

where the functions $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the two-dimensional elasticity tensor of the shell. Since this tensor is uniformly positive definite (see Section 2), the above formulas are invertible and thus the minimization problem of Theorem 6.1 can be immediately recast as a *minimization problem with the stress resultants and bending moments* (the "unknowns of choice" in engineering!) *as the primary unknowns.*

(c) There remains the task of devising efficient *numerical schemes* for approaching such a constrained minimization problem. Most likely, these approximate methods will be similar in their principle to those proposed by Ciarlet and Sauter[15] for approximating the constrained minimization problem that similarly arises in linearized three-dimensional elasticity (see Ciarlet and Ciarlet, Jr.[10]).

(d) The most daunting task consists in extending the present approach to *nonlinearly elastic shells*, where the "full" differences $(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta})$ and $(b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta})$ (see Section 2) are considered as the new unknowns, instead of their linearizations $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ as here. Various attempts in this direction have been recently undertaken, by Ciarlet[9] and Ciarlet and C. Mardare[13]. These attempts have met only partial success, however, since nonlinearity creates considerable difficulties, as expected. Nevertheless and quite interestingly, the soundness of this kind of approach is also corroborated in the mechanics literature, where such an approach bears the befitting name of "*intrinsic equations of shell theory*"; in this direction, see the key paper of Opoka and Pietraszkiewicz[21].

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