ON THE CONTINUITY OF A DEFORMATION AS A FUNCTION OF ITS CAUCHY–GREEN TENSOR

Philippe G. Ciarlet* Florian Laurent†

Abstract. If the Riemann–Christoffel tensor associated with a field of class $\mathcal{C}^2$ of positive definite symmetric matrices of order three vanishes in a connected and simply connected open subset $\Omega \subset \mathbb{R}^3$, then this field is the Cauchy–Green tensor field associated with a deformation of class $\mathcal{C}^3$ of the set Ω, uniquely determined up to isometries of $\mathbb{R}^3$. We establish here that the mapping defined in this fashion is continuous, for *ad hoc* metrizable topologies.

SUR LA CONTINUITÉ D'UNE DEFORMATION EN FONCTION DE SON TENSEUR DE CAUCHY–GREEN

Résumé. Si le tenseur de Riemann–Christoffel associé à un champ de classe $\mathcal{C}^2$ de matrices symétriques définies positives d'ordre trois s'annule sur un ouvert connexe et simplement connexe $\Omega \subset \mathbb{R}^3$, alors ce champ est celui du tenseur de Cauchy–Green associé à une déformation de classe $\mathcal{C}^3$ de l'ensemble Ω, déterminée de façon unique à une isométrie de $\mathbb{R}^3$ près. On établit ici la continuité de l'application ainsi définie, pour des topologies métrisables convenables.

*Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, e–mail: `mapgc@cityu.edu.hk`

†Radial Soft, 12 rue de la Faisanderie, 67381 INGELSHEIM, France, e–mail: `f.laurent@noos.fr`

INTRODUCTION

During the past decades, considerable attention has been devoted to the mathematical analysis of *deformations in a three–dimensional Euclidean space* $\boldsymbol{E}^3$ and to the (right) *Cauchy–Green tensor field* that they engender. Most of these analyses found their motivation in nonlinear three–dimensional elasticity.

After the earlier fundamental contributions of John [1961, 1972] and Kohn [1982], these works culminated with the existence theory of Ball [1977] and with the recent solution of the long–standing question of how to rigorously identify and justify nonlinear two–dimensional plate and shell theories from three–dimensional elasticity, by Le Dret & Raoult [1995, 1996] and Friesecke, Müller & James [2002a, 2002b, 2002c].

Having likewise in mind applications to nonlinear three–dimensional elasticity, we consider here the following question: Given an open, connected and simply connected subset Ω of $\boldsymbol{E}^3$, let $\mathcal{F}$ denote the mapping that associates a deformation of Ω to any field of positive definite matrices whose Riemann–Christoffel tensor vanishes in Ω (naturally, $\mathcal{F}$ is defined up to isometries of $\boldsymbol{E}^3$ only). Then, *do there exist topologies such that $\mathcal{F}$ is continuous*? The object of this paper is to provide an affirmative answer to this question (see Theorem 4).

It is worth emphasizing that our continuity result holds "at any Cauchy–Green tensor", i.e., not only at the identity, which is the Cauchy–Green tensor corresponding to an isometry in $\boldsymbol{E}^3$, also known in elasticity as a "rigid body motion".

In Ciarlet [2002b], we likewise establish, albeit by a different method, the *continuity of a surface in $\boldsymbol{E}^3$, considered as a function of its two fundamental forms.*

These results have been announced in Ciarlet & Laurent [2002] and Ciarlet [2002a].

1 FORMULATION OF THE PROBLEM

To begin with, we list some notations and conventions that will be consistently used throughout the article.

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^3$, $\mathbb{O}^3$, $\mathbb{S}^3$, and $\mathbb{S}^3_>$ respectively designate the sets of all square matrices of order three, of all orthogonal matrices of order three, of all symmetric matrices of order three, and of all symmetric and positive definite matrices of order three.

Latin indices and exponents vary in the set $\{1, 2, 3\}$, except when they are used for indexing sequences or when otherwise indicated, and the summation convention with respect to repeated indices or exponents is used in conjunction with this rule. Kronecker's symbols are designated by δ_{ij} or δ_i^j according to the context.

Let $\boldsymbol{E}^3$ denote a three–dimensional Euclidean space, let $\boldsymbol{a} \cdot \boldsymbol{b}$ denote the Euclidean inner product of $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{E}^3$, and let $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a} \in \boldsymbol{E}^3$. Let $\rho(\boldsymbol{A})$ denote the spectral radius and let $|\boldsymbol{A}| := \{\rho(\boldsymbol{A}^T\boldsymbol{A})\}^{1/2}$ denote the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^3$. Finally, $\boldsymbol{id}$ denotes the identity mapping of $\boldsymbol{E}^3$.

Let there be also given a three–dimensional vector space, identified with $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

Let Ω be an open subset of $\mathbb{R}^3$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If $g \in \mathcal{C}^\ell(\Omega; \mathbb{R})$, $\ell \geq 0$, and $K \Subset \Omega$, we let

$$|g|_{\ell,K} = \sup_{\begin{cases} x \in K \\ |\alpha| = \ell \end{cases}} |\partial^\alpha g(x)| \quad \text{and} \quad \|g\|_{\ell,K} = \sup_{\begin{cases} x \in K \\ |\alpha| \leq \ell \end{cases}} |\partial^\alpha g(x)|,$$

where ∂^α stands for the standard multi–index notation for partial derivatives. If $\boldsymbol{\Theta} \in \mathcal{C}^\ell(\Omega; \boldsymbol{E}^3)$ or $\boldsymbol{A} \in \mathcal{C}^\ell(\Omega; \mathbb{M}^3)$ and $K \Subset \Omega$, we likewise let

$$|\boldsymbol{\Theta}|_{\ell,K} = \sup_{\begin{cases} x \in K \\ |\alpha| = \ell \end{cases}} |\partial^\alpha \boldsymbol{\Theta}(x)| \quad \text{and} \quad \|\boldsymbol{\Theta}\|_{\ell,K} = \sup_{\begin{cases} x \in K \\ |\alpha| \leq \ell \end{cases}} |\partial^\alpha \boldsymbol{\Theta}(x)|,$$

$$|\boldsymbol{A}|_{\ell,K} = \sup_{\begin{cases} x \in K \\ |\alpha| = \ell \end{cases}} |\partial^\alpha \boldsymbol{A}(x)| \quad \text{and} \quad \|\boldsymbol{A}\|_{\ell,K} = \sup_{\begin{cases} x \in K \\ |\alpha| \leq \ell \end{cases}} |\partial^\alpha \boldsymbol{A}(x)|,$$

where $|\cdot|$ denotes the Euclidean vector norm or the matrix spectral norm, respectively.

Let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ be an *immersion*, i.e., a mapping such that the three vectors $\partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all points $x \in \Omega$. Then the *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{S}^3_>)$ of the set $\boldsymbol{\Theta}(\Omega)$ (which is open in $\boldsymbol{E}^3$ since $\boldsymbol{\Theta}$ is an immersion) is defined by means of its *covariant components*

$$g_{ij}(x) := \partial_i \boldsymbol{\Theta}(x) \cdot \partial_j \boldsymbol{\Theta}(x), \; x \in \Omega,$$

which are used in particular for computing *lengths of curves inside the set* $\boldsymbol{\Theta}(\Omega)$, considered as being *isometrically imbedded in* $\boldsymbol{E}^3$. This means that their length is precisely that induced by the Euclidean metric of the Euclidean space $\boldsymbol{E}^3$.

When $\mathbb{R}^3$ is identified with $\boldsymbol{E}^3$, immersions such as $\boldsymbol{\Theta} = (\Theta_i) \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ may be thought of as *deformations* of the set Ω viewed as a *reference configuration*, in the sense of *geometrically exact three–dimensional elasticity* (although they should then be in addition injective and orientation–preserving in order to qualify for this definition; for details, see, e.g., Ciarlet [1988, Sect. 1.4] or Antman [1995, Chap. XII, Sect. 1]).

In this context, the matrix $(g_{ij}(x))$ is usually denoted

$$\boldsymbol{C}(x) := (g_{ij}(x)),$$

and is called the (right) *Cauchy–Green tensor at* x. Note that one also has

$$(g_{ij}(x)) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x),$$

where

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) := (\partial_j \Theta_i(x)) \in \mathbb{M}^3,$$

denotes the *deformation gradient at* x (j denotes the column index in the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$).

The Cauchy–Green tensor field $\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} : \Omega \to \mathbb{S}^3_>$ associated with a deformation $\boldsymbol{\Theta} : \Omega \to \boldsymbol{E}^3$ plays a major role in *nonlinear three–dimensional elasticity*, since the response function, or the stored energy function, of a frame–indifferent elastic, or hyperelastic, material necessarily depends on the deformation gradient through the Cauchy–Green tensor (see, e.g., Ciarlet [1988, Chapters 3 and 4]).

It is well known that the matrix field $\boldsymbol{C} = (g_{ij}) : \Omega \to \mathbb{S}^3_>$ cannot be arbitrary, in that its components g_{ij} and some of their partial derivatives must satisfy *necessary conditions* taking the form of the relations $R_{qijk} = 0$ in Ω shown below (according to our rule governing Latin indices, these relations are meant to hold for all $i, j, k, q \in \{1, 2, 3\}$).

More specifically, given an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, let the functions $\Gamma_{ijq} \in \mathcal{C}^1(\Omega)$ and $\Gamma^p_{ij} \in \mathcal{C}^1(\Omega)$ be defined by

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}, \text{ where } (g^{pq}) := (g_{ij})^{-1}.$$

Then, necessarily,

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega.$$

To see this, let $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$. As is easily verified, the necessary conditions $R_{qijk} = 0$ then simply amount to re–writing the relations $\partial_{ik}\boldsymbol{g}_j = \partial_{ij}\boldsymbol{g}_k$ in the form of the equivalent relations $\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_{ij}\boldsymbol{g}_k \cdot \boldsymbol{g}_q$.

The vectors $\boldsymbol{g}_i$ introduced above form the *covariant bases*, the function g^{ij} are the *contravariant* components of the metric tensor, the functions Γ^p_{ij} and Γ_{ijq} are the *Christoffel symbols of the first*, and *second, kind* and finally, the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$$

are the covariant components of the *Riemann–Christoffel curvature tensor*, of the set $\boldsymbol{\Theta}(\Omega)$. The relations $R_{qijk} = 0$ thus express that the *Riemann–Christoffel tensor of the set* $\boldsymbol{\Theta}(\Omega)$ (equipped with the metric tensor with covariant components g_{ij}) *vanishes.* For details, see, e.g., Choquet–Bruhat, Dewitt–Morette & Dillard–Bleick [1977, p. 303].

It is remarkable that, *conversely*, given a smooth enough matrix field $(g_{ij}) : \Omega \to \mathbb{S}^3_>$ under the additional assumptions that Ω is connected and simply connected, the necessary conditions $R_{qijk} = 0$ in Ω are also *sufficient* for the *existence* of an immersion $\boldsymbol{\Theta} : \Omega \to \boldsymbol{E}^3$ such that $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω. Besides, this immersion is *unique up to isometries in* $\mathbb{R}^3$ (see Theorems 1 and 2).

A self–contained, complete, and essentially elementary, proof of this well–known result from differential geometry, whose outline follows with some modifications and simplifications that of Blume [1989], is found in Ciarlet & Larsonneur [2002]. “Local” versions of the existence result, based on the theory of locally integrable Pfaff systems and on the Frobenius theorem, are found in, e.g., Malliavin [1972, p. 133] and Choquet–Bruhat, Dewitt–Morette & Dillard–Bleick [1977, p. 303].

This result comprises two essentially distinct parts, a *global existence result* (Theorem 1) and a uniqueness result (Theorem 2), the latter being called a *rigidity theorem.* Note that these two results are established under *different assumptions* on the set Ω and on the smoothness of the field (g_{ij}).

Theorem 1 (global existence theorem) *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega,$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma_{ij}^p := g^{pq}\Gamma_{ijq}, \text{ where } (g^{pq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ *such that*

$$\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} \text{ in } \Omega.$$

$\square$

Theorem 2 (rigidity theorem) *Let* Ω *be a connected open subset of* $\mathbb{R}^3$ *and let* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ *and* $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ *be two immersions whose associated Cauchy–Green tensors* $\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ *and* $\widetilde{\boldsymbol{C}} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}$ *satisfy*

$$\boldsymbol{C} = \widetilde{\boldsymbol{C}} \text{ in } \Omega.$$

Then there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ *and a matrix* $\boldsymbol{Q} \in \mathbb{O}^3$ *such that*

$$\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

$\square$

Together, Theorems 1 and 2 establish the existence of a mapping $\mathcal{F}$ that associates to any matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfying $R_{qijk} = 0$ in Ω (the functions R_{qijk} being defined in terms of the functions g_{ij} as in Theorem 1) a well–defined element $\mathcal{F}(\boldsymbol{C})$ in the quotient set $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exists a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$.

A natural question thus arises as to whether there exist *ad hoc* topologies on the space $\mathcal{C}^2(\Omega; \mathbb{S}^3)$ and on the quotient set $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$ such that the mapping $\mathcal{F}$ defined in this fashion is *continuous.*

2 A KEY PRELIMINARY RESULT

The next theorem constitutes the key step toward establishing the continuity of the mapping $\mathcal{F}$ (see Theorem 4 in Section 3).

Theorem 3 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, and $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields respectively satisfying $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$* (with self–explanatory notations)*, such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \boldsymbol{C}$ in Ω (such mappings exist by Theorem 1)*. Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T\boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all} K \Subset \Omega.$$

$\square$

For clarity, the proof of Theorem 3 is broken into those of three lemmas. Lemma 1 deals with the special case where $\boldsymbol{C} = \boldsymbol{I}$; Lemma 2 deals with the special case where the mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ is injective; finally, Lemma 3 deals with the general case.

Lemma 1 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{I}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T\boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{id}\|_{3,K} = 0 \text{ for all } K \Subset \Omega$$

where $\boldsymbol{id}$ denotes the identity mapping of $\mathbb{R}^3$, identified here with $\boldsymbol{E}^3$.

Proof. The proof of Lemma 1 is broken into four parts, numbered (i) to (iv). The first part is a preliminary result about matrices (for convenience, it is stated here for matrices of order three, but it holds as well for matrices of arbitrary order).

(i) *Let there be given matrices* $\boldsymbol{A}^n \in \mathbb{M}^3$, $n \geq 0$, *that satisfy*

$$\lim_{n\to\infty} (\boldsymbol{A}^n)^T \boldsymbol{A}^n = \boldsymbol{I}.$$

Then there exist matrices $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geq 0$, *that satisfy*

$$\lim_{n\to\infty} \boldsymbol{Q}^n \boldsymbol{A}^n = \boldsymbol{I}.$$

Since the set $\mathbb{O}^3$ is compact, there exist matrices $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$|\boldsymbol{Q}^n \boldsymbol{A}^n - \boldsymbol{I}| = \inf_{\boldsymbol{R}\in\mathbb{O}^3} |\boldsymbol{R}\boldsymbol{A}^n - \boldsymbol{I}|.$$

Then the matrices $\boldsymbol{Q}^n$ defined in this fashion satisfy $\lim_{n\to\infty} \boldsymbol{Q}^n \boldsymbol{A}^n = \boldsymbol{I}$. For otherwise, there exist a subsequence $(\boldsymbol{Q}^p)_{p\geq 0}$ of the sequence $(\boldsymbol{Q}^n)_{n\geq 0}$ and $\delta > 0$ such that

$$|\boldsymbol{Q}^p \boldsymbol{A}^p - \boldsymbol{I}| = \inf_{\boldsymbol{R}\in\mathbb{O}^3} |\boldsymbol{R}\boldsymbol{A}^p - \boldsymbol{I}| \geq \delta \text{ for all } p \geq 0.$$

Since

$$\lim_{p\to\infty} |\boldsymbol{A}^p| = \lim_{p\to\infty} \sqrt{\rho((\boldsymbol{A}^p)^T\boldsymbol{A}^p)} = \sqrt{\rho(\boldsymbol{I})} = 1,$$

the sequence $(\boldsymbol{A}^p)_{p\geq 0}$ is bounded. Therefore there exists a further subsequence $(\boldsymbol{A}^q)_{q\geq 0}$ that converges to a matrix $\boldsymbol{S}$. Besides, $\boldsymbol{S}$ is orthogonal since

$$\boldsymbol{S}^T\boldsymbol{S} = \lim_{q\to\infty} (\boldsymbol{A}^q)^T \boldsymbol{A}^q = \boldsymbol{I}.$$

But then

$$\lim_{q\to\infty} \boldsymbol{S}^T \boldsymbol{A}^q = \boldsymbol{S}^T\boldsymbol{S} = \boldsymbol{I},$$

which contradicts $\inf_{\boldsymbol{R}\in\mathbb{O}^3} |\boldsymbol{R}\boldsymbol{A}^q - \boldsymbol{I}| \geq \delta$ for all $q \geq 0$. This proves (i). In the remainder of the proof, the matrix fields $\boldsymbol{C}^n, n \geq 0$, are meant to be those appearing in the statement of Lemma 1.

(ii) *Let there be given mappings* $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, $n \geq 0$, *that satisfy* $(\boldsymbol{\nabla\Theta}^n)^T \boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ *in* Ω (such mappings exist by Theorem 1). *Then*

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \text{ and for } \ell = 2, 3.$$

Given any immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, let $\boldsymbol{g}_i := \partial_i\boldsymbol{\Theta}$, let the vectors $\boldsymbol{g}^q$ be defined by means of the relations $\boldsymbol{g}_i \cdot \boldsymbol{g}^q = \delta_i^q$ (the vectors $\boldsymbol{g}^q$ form the *contravariant bases* in the set $\boldsymbol{\Theta}(\Omega)$), and let the Christoffel symbols of the second kind be defined by $\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})$, as in Section 1. It is then immediately verified that the same Christoffel symbols are also given by $\Gamma_{ijq} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q$. Consequently,

$$\partial_{ij}\boldsymbol{\Theta} = \partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})\boldsymbol{g}^q.$$

Applying this relation to the given mappings $\boldsymbol{\Theta}^n$ thus gives (with self–explanatory notations):

$$\partial_{ij}\boldsymbol{\Theta}^n = \frac{1}{2}(\partial_j g^n_{iq} + \partial_i g^n_{jq} - \partial_q g^n_{ij})(\boldsymbol{g}^q)^n, \; n \geq 0.$$

Let K denote an arbitrary compact subset of Ω. On the one hand,

$$\lim_{n\to\infty} |\partial_j g^n_{iq} + \partial_i g^n_{jq} - \partial_q g^n_{ij}|_{0,K} = 0,$$

since $\lim_{n\to\infty} |g^n_{ij}|_{1,K} = \lim_{n\to\infty} |g^n_{ij} - \delta_{ij}|_{1,K} = 0$ by assumption. On the other, the norms $|(\boldsymbol{g}^q)^n|_{0,K}$ are bounded independently of $n \geq 0$; to see this, observe that $(\boldsymbol{g}^q)^n$ is the q–th column vector of the matrix $(\boldsymbol{\nabla\Theta}^n)^{-1}$, then that

$$\begin{aligned} |(\boldsymbol{\nabla\Theta}^n)^{-1}|_{0,K} &= |\{\rho((\boldsymbol{\nabla\Theta}^n)^{-T}(\boldsymbol{\nabla\Theta}^n)^{-1})\}^{1/2}|_{0,K} \\ &= |\{\rho((g^n_{ij})^{-1})\}^{1/2}|_{0,K} \leq \{|(g^n_{ij})^{-1}|_{0,K}\}^{1/2}, \end{aligned}$$

and, finally, that

$$\lim_{n\to\infty} |(g^n_{ij}) - \boldsymbol{I}|_{0,K} = 0 \Longrightarrow \lim_{n\to\infty} |(g^n_{ij})^{-1} - \boldsymbol{I}|_{0,K} = 0.$$

These two properties thus imply that

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{2,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Differentiating the relations $\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})$ yields

$$\begin{aligned} \partial_{ijp}\boldsymbol{\Theta} &= \partial_{ip}\boldsymbol{g}_j = (\partial_{ip}\boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q \\ &= \left(\frac{1}{2}(\partial_{jp} g_{iq} + \partial_{ip} g_{jq} - \partial_{pq} g_{ij}) - \partial_i \boldsymbol{g}_j \cdot \partial_p \boldsymbol{g}_q\right)\boldsymbol{g}^q. \end{aligned}$$

Observing that $\lim_{n\to\infty} |g_{ij}^n|_{\ell,K} = \lim_{n\to\infty} |g_{ij}^n - \delta_{ij}|_{\ell,K} = 0$ for $\ell = 1, 2$ by assumption and recalling that the norms $|(\boldsymbol{g}^q)^n|_{0,K}$ are bounded independently of $n \geq 0$, we likewise conclude that

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{3,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

(iii) *There exist mappings* $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ *that satisfy* $(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{C}^n$ *in* Ω, $n \geq 0$, *and*

$$\lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{id}|_{1,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be mappings that satisfy $(\boldsymbol{\nabla}\boldsymbol{\psi}^n)^T \boldsymbol{\nabla}\boldsymbol{\psi}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$ (such mappings exist by Theorem 1) and let x_0 denote a point in the set Ω. Since $\lim_{n\to\infty} \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0)^T \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \boldsymbol{I}$ by assumption, Part (i) implies that there exist orthogonal matrices $\boldsymbol{Q}^n$, $n \geq 0$, such that

$$\lim_{n\to\infty} \boldsymbol{Q}^n \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \boldsymbol{I}.$$

Then the mappings

$$\widetilde{\boldsymbol{\Theta}}^n := \boldsymbol{Q}^n \boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3),\ n \geq 0,$$

satisfy

$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{C}^n \text{ in } \Omega,$$

so that their gradients $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfy

$$\lim_{n\to\infty} |\partial_i \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n|_{0,K} = \lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n|_{2,K} = 0 \text{ for all } K \Subset \Omega,$$

by Part (ii). In addition,

$$\lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x_0) = \lim_{n\to\infty} \boldsymbol{Q}^n \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \boldsymbol{I}.$$

Hence a classical theorem about the differentiability of the limit of a sequence of mappings that are continuously differentiable on a connected open set and that take their values in a Banach space (see, e.g., Schwartz [1992, Thm. 3.5.12]) shows that *the mappings* $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n$ *uniformly converge on every compact subset of* Ω *toward a limit* $\boldsymbol{R} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ *that satisfies*

$$\partial_i \boldsymbol{R}(x) = \lim_{n\to\infty} \partial_i \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x) = \boldsymbol{0} \text{ for all } x \in \Omega.$$

This shows that $\boldsymbol{R}$ is a constant mapping since Ω is connected. Consequently, $\boldsymbol{R} = \boldsymbol{I}$ since in particular $\boldsymbol{R}(x_0) = \lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x_0) = \boldsymbol{I}$. We have therefore established that

$$\lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{id}|_{1,K} = \lim_{n\to\infty} |\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{I}|_{0,K} = 0 \text{ for all } K \Subset \Omega.$$

(iv) *There exist mappings* $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ *satisfying* $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ *in* Ω, $n \geq 0$*, and*

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \text{ and for } \ell = 0, 1.$$

The mappings

$$\boldsymbol{\Theta}^n := \left(\widetilde{\boldsymbol{\Theta}}^n - \{\widetilde{\boldsymbol{\Theta}}^n(x_0) - x_0\}\right) \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3),\ n \geq 0,$$

clearly satisfy

$$\begin{aligned}
(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n &= \boldsymbol{C}^n \text{ in } \Omega,\ n \geq 0,\\
\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{1,K} = \lim_{n\to\infty} |\boldsymbol{\nabla}\boldsymbol{\Theta}^n &- \boldsymbol{I}|_{0,K} = 0 \text{ for all } K \Subset \Omega,\\
\boldsymbol{\Theta}^n(x_0) &= x_0,\ n \geq 0.
\end{aligned}$$

Applying again the theorem about the differentiability of the limit of a sequence of mappings used in Part (iii), we conclude from the last two relations that the mappings $\boldsymbol{\Theta}^n$ uniformly converge on every compact subset of Ω toward a limit $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ that satisfies

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \lim_{n\to\infty} \boldsymbol{\nabla}\boldsymbol{\Theta}^n(x) = \boldsymbol{I} \text{ for all } x \in \Omega.$$

This shows that $(\boldsymbol{\Theta} - \boldsymbol{id})$ is a constant mapping since Ω is connected. Consequently, $\boldsymbol{\Theta} = \boldsymbol{id}$ since in particular $\boldsymbol{\Theta}(x_0) = \lim_{n\to\infty} \boldsymbol{\Theta}^n(x_0) = x_0$. We have thus established that

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{0,K} = 0 \text{ for all } K \Subset \Omega,$$

and the proof is complete. $\square$

Lemma 2 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying respectively $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Assume that there exists an injective mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ such that $\boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta} = \boldsymbol{C}$ in Ω (any mapping $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ that satisfies $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \boldsymbol{C}$ in Ω is thus also injective by Theorem 2). *Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T \boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. The assumptions made on the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \boldsymbol{E}^3$ imply that the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega) \subset \boldsymbol{E}^3$ is open, connected, and simply connected and that its inverse mapping $\widehat{\boldsymbol{\Theta}} : \widehat{\Omega} \subset \boldsymbol{E}^3 \to \mathbb{R}^3$ belongs to the space $\mathcal{C}^3(\widehat{\Omega}; \mathbb{R}^3)$. Define the matrix fields $(\widehat{g}^n_{ij}) \in \mathcal{C}^2(\widehat{\Omega}; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$(\widehat{g}^n_{ij}(\widehat{x})) := \boldsymbol{\nabla\Theta}(x)^{-T}(g^n_{ij}(x))\boldsymbol{\nabla\Theta}(x)^{-1} \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}.$$

Given any compact subset $\widehat{K}$ of $\widehat{\Omega}$, let $K := \widehat{\boldsymbol{\Theta}}(\widehat{K})$. Since $\lim_{n\to\infty} \|g^n_{ij} - g_{ij}\|_{2,K} = 0$ as K is a compact subset of Ω, the definition of the functions $\widehat{g}^n_{ij} : \widehat{\Omega} \to \mathbb{R}$ and the chain rule together imply that

$$\lim_{n\to\infty} \|\widehat{g}^n_{ij} - \delta_{ij}\|_{2,\widehat{K}} = 0.$$

Given $\widehat{x} = (\widehat{x}_i) \in \widehat{\Omega}$, let $\widehat{\partial}_i = \dfrac{\partial}{\partial \widehat{x}_i}$. Since it is easily verified that the fields $(\widehat{g}^n_{ij})$ satisfy $\widehat{R}^n_{qijk} = 0$ in $\widehat{\Omega}$ (with self–explanatory notations), Lemma 1 applied over the set $\widehat{\Omega}$ shows that there exist mappings $\widehat{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\widehat{\Omega}; \boldsymbol{E}^3)$ satisfying $\widehat{\partial}_i \widehat{\boldsymbol{\Theta}}^n \cdot \widehat{\partial}_j \widehat{\boldsymbol{\Theta}}^n = \widehat{g}^n_{ij}$ in $\widehat{\Omega}$, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0 \text{ for all } \widehat{K} \Subset \widehat{\Omega},$$

where $\widehat{\boldsymbol{id}}$ denotes the identity mapping of $\boldsymbol{E}^3$, identified here with $\mathbb{R}^3$. Define the mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$\boldsymbol{\Theta}^n(x) = \widehat{\boldsymbol{\Theta}}^n(\widehat{x}) \text{ for all } x = \widehat{\boldsymbol{\Theta}}(\widehat{x}) \in \Omega.$$

Given any compact subset K of Ω, let $\widehat{K} := \boldsymbol{\Theta}(K)$. Since $\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0$, the definition of the mappings $\boldsymbol{\Theta}^n$ and the chain rule together imply that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0,$$

on the one hand. Since, on the other, $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, the proof is complete. $\square$

Lemma 3 *The assumption that the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \boldsymbol{E}^3$ is injective is superfluous in* Lemma 2, *all its other assumptions holding* verbatim. *In other words,* Theorem 3 *holds.*

Proof. The proof is broken into four parts. In what follows, $\boldsymbol{C}$ and $\boldsymbol{C}^n$ designate matrix fields possessing the properties listed in the statement of Theorem 3.

(i) *Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{C}$ in Ω. Then there exists a countable number of open balls $B_r \subset \Omega$, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^{\infty} B_r$ and such that, for each $r \geq 1$, the set $\bigcup_{s=1}^{r} B_s$ is connected and the restriction of $\boldsymbol{\Theta}$ to B_r is injective.*

Given any $x \in \Omega$, there exists an open ball $V_x \subset \Omega$ such that the restriction of $\boldsymbol{\Theta}$ to V_x is injective. Since $\Omega = \bigcup_{x\in\Omega} V_x$ can be also written as a countable union of compact subsets of Ω, there already exist countably many such open balls, denoted V_r, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^{\infty} V_r$.

Let $r_1 := 1$, $B_1 := V_{r_1}$, and $r_2 := 2$. If the set $B_{r_1} \cup V_{r_2}$ is connected, let $B_2 := V_{r_2}$ and $r_3 := 3$. Otherwise, let γ_1 be a path in Ω joining the centers of V_{r_1} and V_{r_2}. Then there exists a finite set $I_1 = \{r_1(1), r_1(2), \dots, r_1(N_1)\}$ of integers, with $N_1 \geq 1$ and $2 < r_1(1) < r_1(2) < \dots < r_1(N_1)$, such that

$$\gamma_1 \subset V_{r_1} \cup V_{r_2} \cup \Big(\bigcup_{r\in I_1} V_r\Big).$$

Furthermore there exists a permutation σ_1 of $\{1, 2, \dots, N_1\}$ such that the sets $V_{r_1} \cup \bigcup_{s=1}^{r} V_{\sigma_1(s)})$ $1 \leq r \leq N_1$, and $V_{r_1} \cup (\bigcup_{s=1}^{N_1} V_{\sigma_1(N_1)}) \cup V_{r_2}$ are connected. Let then

$$B_r := V_{\sigma_1(r-1)},\ 2 \leq r \leq N_1 + 1, \quad B_{N_1+2} := V_{r_2},$$

and

$$r_3 := \min\big\{i \in \{\sigma_1(1), \dots, \sigma_1(N_1)\};\ i \geq 3\big\}.$$

If the set $(\bigcup_{r=1}^{N_1+2} B_r) \cup V_{r_3}$ is connected, let $B_{N_1+3} = V_{r_3}$. Otherwise, apply the same argument as above to a path γ_2 in Ω joining the centers of V_{r_2} and V_{r_3}, and so on.

The iterative procedure thus produces a countable number of open balls B_r, $r \geq 1$, that possess the announced properties. In particular, $\Omega = \bigcup_{r=1}^{\infty} B_r$ since, by construction, the integer r_i appearing at the i–th stage satisfies $r_i \geq i$.

(ii) By Lemma 2, *there exist mappings* $\boldsymbol{\Theta}_1^n \in \mathcal{C}^3(B_1; \boldsymbol{E}^3)$ *and* $\widetilde{\boldsymbol{\Theta}}_2^n \in \mathcal{C}^3(B_2; \boldsymbol{E}^3)$, $n \geq 0$, *that satisfy*

$$
\begin{aligned}
(\boldsymbol{\nabla}\boldsymbol{\Theta}_1^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_1^n &= \boldsymbol{C}^n \text{ in } B_1 \quad \text{and} \quad \lim_{n\to\infty} \|\boldsymbol{\Theta}_1^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1,\\
(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_2^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_2^n &= \boldsymbol{C}^n \text{ in } B_2 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_2,
\end{aligned}
$$

and by Theorem 2, *there exist vectors* $\boldsymbol{a}^n \in \boldsymbol{E}^3$ *and matrices* $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geq 0$, *such that*

$$
\widetilde{\boldsymbol{\Theta}}_2^n(x) = \boldsymbol{a}^n + \boldsymbol{Q}^n \boldsymbol{\Theta}_1^n(x) \text{ for all } x \in B_1 \cap B_2.
$$

Then

$$
\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0} \text{ and } \lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}.
$$

Let $(\boldsymbol{Q}^p)_{p\geq 0}$ be a subsequence of the sequence $(\boldsymbol{Q}^n)_{n\geq 0}$ that converges to an orthogonal matrix $\boldsymbol{Q}$ and let x_1 denote a point in the set $B_1 \cap B_2$. Since $\boldsymbol{c}^p = \widetilde{\boldsymbol{\Theta}}_2^p(x_1) - \boldsymbol{Q}^p \boldsymbol{\Theta}_1(x_1)$ and $\lim_{n\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x_1) = \lim_{n\to\infty} \boldsymbol{\Theta}_1^p(x_1) = \boldsymbol{\Theta}(x_1)$, the subsequence $(\boldsymbol{a}^p)_{p\geq 0}$ also converges. Let $\boldsymbol{a} := \lim_{p\to\infty} \boldsymbol{a}^p$. We thus have

$$
\begin{aligned}
\boldsymbol{\Theta}(x) &= \lim_{p\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x)\\
&= \lim_{p\to\infty} (\boldsymbol{a}^p + \boldsymbol{Q}^p \boldsymbol{\Theta}_1^p(x)) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \text{ for all } x \in B_1 \cap B_2,
\end{aligned}
$$

on the one hand. On the other, the differentiability of the mapping $\boldsymbol{\Theta}$ implies that

$$
\boldsymbol{\Theta}(x) = \boldsymbol{b} + \boldsymbol{A}(x - x_1) + o(|x - x_1|) \text{ for all } x \in B_1 \cap B_2,
$$

where $\boldsymbol{b} := \boldsymbol{\Theta}(x_1)$ and $\boldsymbol{A} := \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)$. Note that $\boldsymbol{A}$ is an invertible matrix, since $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1) = (g_{ij}(x_1))$.

Together, the last two relations imply that

$$
\boldsymbol{b} + \boldsymbol{A}(x - x_1) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{b} + \boldsymbol{Q}\boldsymbol{A}(x - x_1) + o(|x - x_1|),
$$

hence (letting $x = x_1$ shows that $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{b}$) that

$$\boldsymbol{A}(x - x_1) = \boldsymbol{Q}\boldsymbol{A}(x - x_1) + o(|x - x_1|) \text{ for all } x \in B_1 \cap B_2.$$

The invertibility of $\boldsymbol{A}$ thus implies that $\boldsymbol{Q} = \boldsymbol{I}$ and therefore that $\boldsymbol{a} = \boldsymbol{b} - \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{0}$. The uniqueness of these limits shows that the whole sequences $(\boldsymbol{Q}^n)_{n\geq 0}$ and $(\boldsymbol{c}^n)_{n\geq 0}$ converge.

(iii) *Let the mappings* $\boldsymbol{\Theta}_2^n \in \mathcal{C}^3(B_1 \cup B_2; \boldsymbol{E}^3)$, $n \geq 0$, *be defined by*

$$\begin{aligned} \boldsymbol{\Theta}_2^n(x) &:= \boldsymbol{\Theta}_1^n(x) \text{ for all } x \in B_1, \\ \boldsymbol{\Theta}_2^n(x) &:= (\boldsymbol{Q}^n)^T(\widetilde{\boldsymbol{\Theta}}_2^n(x) - \boldsymbol{a}^n) \text{ for all } x \in B_2. \end{aligned}$$

Then

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_2^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_2^n = \boldsymbol{C}^n \text{ in } B_1 \cup B_2$$

(as is clear), *and*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1 \cup B_2.$$

The plane containing the intersection of the boundaries of the open balls B_1 and B_2 is the common boundary of two closed half–spaces in $\mathbb{R}^3$, H_1 containing the center of B_1, and H_2 containing that of B_2 (by construction, the set $B_1 \cup B_2$ is connected; see Part (i)). Any compact subset K of $B_1 \cup B_2$ may thus be written as $K = K_1 \cup K_2$, where $K_1 := (K \cap H_1) \subset B_1$ and $K_2 := (K \cap H_2) \subset B_2$ (that the open sets found in Part (i) may be chosen as *balls* thus plays an essential rôle here). Hence

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_1} = 0 \text{ and } \lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0,$$

the second relation following from the definition of the mapping $\boldsymbol{\Theta}_2^n$ on $B_2 \supset K_2$ and on the relations $\lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$ (Part (ii)) and $\lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}$ and $\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0}$ (Part (iii)).

(iv) *It remains to iterate the procedure described in* Parts (ii) *and* (iii).

Assume that, for some $r \geq 2$, mappings $\boldsymbol{\Theta}_r^n \in \mathcal{C}^3(\bigcup_{s=1}^r B_s; \boldsymbol{E}^3)$, $n \geq 0$, have been found that satisfy

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_r^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_r^n = \boldsymbol{C}^n \text{ in } \bigcup_{s=1}^r B_s \text{and } \lim_{n\to\infty} \|\boldsymbol{\Theta}_r^n - \boldsymbol{\Theta}\|_{2,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^r B_s.$$

Since the restriction of $\mathbf{\Theta}$ to B_{r+1} is injective (Part (i)), Lemma 2 shows that there exist mappings $\widetilde{\mathbf{\Theta}}^n_{r+1} \in \mathcal{C}^3(B_{r+1}; \boldsymbol{E}^3)$, $n \geq 0$, that satisfy

$$(\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n_{r+1})^T \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n_{r+1} = \boldsymbol{C}^n \text{ in } B_{r+1}, \lim_{n\to\infty} \|\widetilde{\mathbf{\Theta}}^n_{r+1} - \mathbf{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_{r+1},$$

and since the set $\bigcup_{s=1}^{r+1} B_s$ is connected (Part (i)), Theorem 2 shows that there exist vectors $\boldsymbol{c}^n \in \boldsymbol{E}^3$ and matrices $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$\widetilde{\mathbf{\Theta}}^n_{r+1}(x) = \boldsymbol{a}^n + \boldsymbol{Q}^n \mathbf{\Theta}^n_r(x) \text{ for all } x \in \Big(\bigcup_{s=1}^{r} B_s\Big) \cap B_{r+1}.$$

Then an argument similar to that used in Part (ii) shows that $\lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}$ and $\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0}$ and an argument similar to that used in Part (iii) (note that the ball B_{r+1} may intersect more than one of the balls B_s, $1 \leq s \leq r$) shows that the mappings $\mathbf{\Theta}^n_{r+1} \in \mathcal{C}^3(\bigcup_{s=1}^{r} B_s; \boldsymbol{E}^3)$, $n \geq 0$, defined by

$$\begin{aligned}
\mathbf{\Theta}^n_{r+1}(x) &:= \mathbf{\Theta}^n_r(x) \text{ for all } x \in \bigcup_{s=1}^{r} B_s,\\
\mathbf{\Theta}^n_{r+1}(x) &:= (\boldsymbol{Q}^n)^T(\widetilde{\mathbf{\Theta}}^n_r(x) - \boldsymbol{a}^n) \text{ for all } x \in B_{r+1},
\end{aligned}$$

satisfy

$$\lim_{n\to\infty} \|\mathbf{\Theta}^n_{r+1} - \mathbf{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^{r} B_s.$$

Then the mappings $\mathbf{\Theta}^n : \Omega \to \boldsymbol{E}^3$, $n \geq 0$, defined by

$$\mathbf{\Theta}^n(x) = \mathbf{\Theta}^n_r(x) \text{ for all } x \in \bigcup_{s=1}^{r} B_s,\ r \geq 1,$$

possess all the required properties: They are unambiguously defined since for all $s > r$, $\mathbf{\Theta}^n_s(x) = \mathbf{\Theta}^n_r(x)$ for all $x \in \bigcup_{s=1}^{r} B_s$ by construction; they are of class $\mathcal{C}^3$ since the mappings $\mathbf{\Theta}^n_r : \bigcup_{s=1}^{r} B_s \to \boldsymbol{E}^3$ are themselves of class $\mathcal{C}^3$; they satisfy $(\boldsymbol{\nabla}\mathbf{\Theta}^n)^T \boldsymbol{\nabla}\mathbf{\Theta}^n = \boldsymbol{C}^n$ in Ω since the mappings $\mathbf{\Theta}^n_r$ satisfy the same relations in $\bigcup_{s=1}^{r} B_s$; and finally, they satisfy $\lim_{n\to\infty} \|\mathbf{\Theta}^n - \mathbf{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$ since any compact subset of Ω is contained in $\bigcup_{s=1}^{r} B_s$ for r large enough. $\square$

3 CONTINUITY IN METRIC SPACES

Let Ω be an open subset of $\mathbb{R}^3$. For any integers $\ell \geq 0$ and $d \geq 1$, the space $\mathcal{C}^\ell(\Omega; \mathbb{R}^d)$ becomes a *locally convex topological space* when its topology is defined by the family of semi–norms $\|\cdot\|_{\ell,K}$, $K \Subset \Omega$, and a sequence $(\boldsymbol{\Theta}^n)_{n\geq 0}$ converges to $\boldsymbol{\Theta}$ with respect to this topology if and only if

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega.$$

Furthermore, this topology is *metrizable*: Let $(K_i)_{i\geq 0}$ be any sequence of subsets of Ω that satisfy

$$K_i \Subset \Omega \text{ and } K_i \subset \operatorname{int} K_{i+1} \text{ for all } i \geq 0, \text{ and } \Omega = \bigcup_{i=0}^{\infty} K_i.$$

Then

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \Longleftrightarrow \lim_{n\to\infty} d_\ell(\boldsymbol{\Theta}^n, \boldsymbol{\Theta}) = 0,$$

where

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\Theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}.$$

For details, see, e.g., Yosida [1966, Chapter 1].

Let $\dot{\mathcal{C}}^3(\Omega; \boldsymbol{E}^3) := \mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$ denote the quotient set of $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ by the equivalence relation $\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$. Then it is easily verified that the set $\dot{\mathcal{C}}^3(\Omega; \boldsymbol{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\Theta}}, \dot{\boldsymbol{\psi}}) := \inf_{\begin{cases}\boldsymbol{\kappa}\in\dot{\boldsymbol{\Theta}}\\ \boldsymbol{\chi}\in\dot{\boldsymbol{\psi}}\end{cases}} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}) = \inf_{\begin{cases}\boldsymbol{a}\in\boldsymbol{E}^3\\ \boldsymbol{Q}\in\mathbb{O}^3\end{cases}} d_3(\boldsymbol{\Theta}, \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\Theta}}$ denotes the equivalence class of $\boldsymbol{\Theta}$ *modulo* $\mathcal{R}$.

The announced continuity of a deformation as a function of its Cauchy-Green tensor is then a corollary to Theorem 3. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 4 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let*

$$\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>) := \{(g_{ij}) \in \mathcal{C}^2(\Omega;\mathbb{S}^3_>);\ R_{qijk} = 0 \text{ in } \Omega\},$$

where the functions R_{qijk} are defined in terms of the functions g_{ij} as in Theorem 1*. Given any matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$, let $\mathcal{F}(\boldsymbol{C}) \in \dot{\mathcal{C}}^3(\Omega;\boldsymbol{E}^3)$ denote the equivalence class* modulo $\mathcal{R}$ *of any $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega;\boldsymbol{E}^3)$ that satisfies $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \boldsymbol{C}$ in Ω.*

Then the mapping

$$\mathcal{F} : \{\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>); d_2\} \longrightarrow \{\dot{\mathcal{C}}^3(\Omega;\boldsymbol{E}^3); \dot{d}_3\}$$

defined in this fashion is continuous.

Proof. Since $\{\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>); d_2\}$ and $\{\dot{\mathcal{C}}^3(\Omega;\boldsymbol{E}^3); \dot{d}_3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through $\mathcal{F}$ into convergent sequences.

Let then $\boldsymbol{C} \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$ and $\boldsymbol{C}^n \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$, $n \geq 0$, be such that

$$\lim_{n\to\infty} d_2(\boldsymbol{C}^n, \boldsymbol{C}) = 0,$$

i.e., such that $\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0$ for all $K \Subset \Omega$. Given any $\boldsymbol{\Theta} \in \mathcal{F}(\boldsymbol{C})$, Theorem 3 shows that there exist $\boldsymbol{\Theta}^n \in \mathcal{F}(\boldsymbol{C}^n)$, $n \geq 0$, such that $\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$, i.e., such that

$$\lim_{n\to\infty} d_3(\boldsymbol{\Theta}^n, \boldsymbol{\Theta}) = 0.$$

Consequently,

$$\lim_{n\to\infty} \dot{d}_3(\mathcal{F}(\boldsymbol{C}^n), \mathcal{F}(\boldsymbol{C})) = 0.$$

$\square$

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