A new approach to the fundamental theorem of surface theory, by means of the Darboux-Vallée-Fortuné compatibility relation

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Abstract

Let ω be a simply-connected open subset of $\mathbb{R}^2$. Given two smooth enough fields of positive definite symmetric, and symmetric, matrices defined over ω, the well-known fundamental theorem of surface theory asserts that, if these fields satisfy the Gauss and Codazzi-Mainardi relations in ω, then there exists an immersion from ω into $\mathbb{R}^3$ such that these fields are the first and second fundamental forms of the surface (ω)

We revisit here this classical result by establishing that a new compatibility relation, shown to be necessary by C. Vallée and D. Fortuné in 1996 through the introduction, following an idea of G. Darboux, of a rotation field on a surface, is also sufficient for the existence of such an immersion .

This approach also constitutes a first step toward the analysis of models for nonlinear elastic shells where the rotation field along the middle surface is considered as one of the primary unknowns.

Résumé

Soit ω un ouvert simplement connexe de $\mathbb{R}^2$. Etant donné deux champs suffisamment réguliers définis dans ω, l'un de matrices symétriques définies positives et l'autre de matrices symétriques, le théorème fondamental de la théorie des surfaces affirme que, si ces deux champs satisfont les relations de Gauss et Codazzi-Mainardi dans ω, alors il existe une immersion de ω dans $\mathbb{R}^3$ telle que ces champs soient les première et deuxième forme fondamentales de la surface (ω).

On donne ici une autre approche de ce résultat classique, en montrant qu'une nouvelle relation de compatibilité, dont C. Vallée et D. Fortuné ont montré en 1996 la nécessité en suivant une idée de G. Darboux, est également suffisante pour l'existence d'une telle immersion .

Cette approche constitue également un premier pas vers l'analyse de modèles de coques non linéairement élastiques où le champ de rotations le long de la surface moyenne est pris comme l'une des inconnues principales.

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1. Introduction

All the notions and notations used, but not defined, in this introduction are defined in the next section.

Latin and Greek indices range respectively in $\{1,2,3\}$ and $\{1,2\}$ and the summation convention with respect to repeated indices is used. The symbols $\mathbb{M}^n, \mathbb{M}^{m\times n}, \mathbb{S}^n, \mathbb{S}^n_>$, $\mathbb{O}^n$, and $\mathbb{O}^n_+$ designate the sets of all $n\times n, m\times n, n\times n$ symmetric, $n\times n$ positive definite symmetric, $n\times n$ orthogonal, and $n\times n$ proper orthogonal, real matrices. The notation $Df(a)$ designates the Fréchet derivative of a mapping f at a point a.

Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$ be an immersion. The *first* and *second fundamental forms* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2_>)$ of the *surface* $\boldsymbol{\theta}(\omega) \subset \mathbb{R}^3$ are then defined by means of their covariant components

$$a_{\alpha\beta} := \partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta} \qquad \text{and} \qquad b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}.$$

The matrix fields $(a_{\alpha\beta})$ and $(\beta_{\alpha\beta})$ cannot be arbitrary : Let

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \quad \text{and} \quad C^\sigma_{\alpha\beta} := a^{\sigma\tau}C_{\alpha\beta\tau}, \tag{1.1}$$

where $(a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}$. Then the *Gauss* and *Codazzi-Mainardi compatibility relations*, viz.

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\nu_{\alpha\beta}C_{\sigma\tau\nu} - C^\nu_{\alpha\sigma}C_{\beta\tau\nu} = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau}, \tag{1.2}$$

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\nu_{\alpha\sigma}b_{\beta\nu} - C^\nu_{\alpha\beta}b_{\sigma\nu} = 0, \tag{1.3}$$

necessarily hold in ω (they simply express in an appropriate way that $\partial_{\alpha\sigma\beta}\boldsymbol{\theta} = \partial_{\alpha\beta\sigma}\boldsymbol{\theta}$). The functions $C_{\alpha\beta\tau}$ and $C^\sigma_{\alpha\beta}$ are the *Christoffel symbols* of the first and second kinds.

Notice that the Gauss equations reduce in fact to a *single* equation, corresponding to $(\alpha,\beta,\sigma,\tau) = (1,2,1,2)$, and that the Codazzi-Mainardi equations reduce in fact to *two* equations, corresponding to $(\alpha,\beta,\sigma) = (1,2,1)$ and $(\alpha,\beta,\sigma) = (1,2,2)$ (other choices of indices are clearly possible).

When ω is *simply-connected*, the Gauss and Codazzi-Mainardi relations become also *sufficient* for the existence of such a mapping $\boldsymbol{\theta}$, according to the following classical *fundamental theorem of surface theory*: Let $\omega \subset \mathbb{R}^2$ be open and simply-connected and let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2)$ satisfy the *Gauss and Codazzi-Mainardi compatibility relations* in ω. *Then there exists an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$ *such that* $(a_{\alpha\beta})$ *and* $(b_{\alpha\beta})$ *are the first and second fundamental forms of the surface* $\boldsymbol{\theta}(\omega)$.

In 1996, a *different*, more "geometrical" and substantially simpler, *necessary compatibility relation* has been obtained in *vector form*, through the introduction of an appropriate *rotation field* $\mathbf{R}$ *on a surface*, by Vallée & Fortuné [29], an idea that in fact goes back to Darboux [12] (for convenience, we also provide here an "independent" proof of the necessity of these relations; cf. Theorem 5.1)

More specifically, let $\boldsymbol{\theta} = (\theta_i) \in \mathcal{C}^3(\omega;\mathbb{R}^3)$ be an immersion, let $\boldsymbol{\nabla\theta} := (\partial_\alpha\theta_i) \in \mathcal{C}^2(\omega;\mathbb{M}^{3\times 2})$, and let

$$\mathbf{R} := \boldsymbol{\nabla\theta}\ \mathbf{A}^{-1/2} \in \mathcal{C}^2(\omega;\mathbb{M}^{3\times 2}),$$

where $\mathbf{A} := \boldsymbol{\nabla\theta}^T\boldsymbol{\nabla\theta} = (a_{\alpha\beta})$ denotes the first fundamental form of the surface $\boldsymbol{\theta}(\omega)$. Let

$$\boldsymbol{a}_3 := \frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|} \in \mathcal{C}^2(\omega;\mathbb{R}^3),$$

and let $\mathbf{Q} \in \mathcal{C}^2(\omega; \mathbb{O}^3_+)$ denote the matrix field with the two columns of $\mathbf{R}$ as its first two columns and $\boldsymbol{a}_3$ as its third one. Then the orthogonality of the matrices $\mathbf{Q}(y)$ at all points, $y \in \omega$, implies the existence of a matrix field $\mathbf{L} \in \mathcal{C}^1(\omega; \mathbb{M}^{3\times 2})$ such that

$$(D\mathbf{Q}(y)\mathbf{h})\mathbf{k} = \mathbf{Q}(y)(\mathbf{L}(y)\mathbf{h})\mathbf{k} \quad \text{for all } y \in \omega, \mathbf{h} \in \mathbb{R}^2, \mathbf{k} \in \mathbb{R}^3,$$
$$\partial_2 \boldsymbol{\lambda}_1 - \partial_1 \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 \quad \text{in} \quad \omega,$$

where $\boldsymbol{\lambda}_1 = (\lambda_{i1})$ and $\boldsymbol{\lambda}_2 = (\lambda_{i2})$ denote the two columns of $\mathbf{L}$. That $\mathbf{R} = \boldsymbol{\nabla}\boldsymbol{\theta}\ \mathbf{A}^{-1/2}$ further implies that

$$\begin{pmatrix} \lambda_{31} \\ \lambda_{32} \end{pmatrix} = \mathbf{J}\ \mathbf{A}^{-1/2}\mathbf{J}\begin{pmatrix} \partial_2 u^0_{11} - \partial_1 u^0_{12} \\ \partial_2 u^0_{21} - \partial_1 u^0_{22} \end{pmatrix}, \text{ where } \mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } (u^0_{\alpha\beta}) := \mathbf{A}^{1/2},$$

and also that

$$\mathbf{B} = \mathbf{A}^{1/2}\mathbf{J}\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix},$$

where $\mathbf{B}$ is the second fundamental form $(b_{\alpha\beta})$ of the surface $\boldsymbol{\theta}(\omega)$.

An illuminating geometrical interpretation of the above matrix field $\mathbf{Q}$ is provided by the *canonical extension* of the immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ as the mapping $\boldsymbol{\Theta} : (y, x_3) \in \omega \times \mathbb{R} \to \mathbb{R}^3$, which is defined by

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \mathbf{a}_3(y) \text{ for all } (y, x_3) \in \omega \times \mathbb{R}.$$

For it is easily seen that, at each $y \in \omega$, *the proper orthogonal matrix* $\mathbf{Q}(y)$ *is nothing else than the restriction to* $x_3 = 0$ *of the proper orthogonal matrix found in the polar factorization of the gradient matrix* $\boldsymbol{\nabla\Theta}(y, x_3)$ (which is invertible for $|x_3|$ small enough).

As advocated notably by Simmonds & Danielson [26], Valid [27], Pietraszkiewicz [20], Basar [2], or Galka & Telega [13], *rotation fields* can be introduced as *bona fide* unknowns in nonlinear shell models. In particular, rotation fields are often introduced by way of *one-director Cosserat surfaces* (an excellent introduction to this approach is found in Chapter 14, Section 13, of Antman [1]).

References about the *existence theory* for models based on such principles with the rotation field as one of the unknowns are scarce. For linearized, or partially linearized, models, the contributions of Bielski & Telega [4], Bernadou, Ciarlet & Miara [3], or Grandmont, Maday & Métier [14] constitute noteworthy exceptions in what seems to be an essentially virgin territory.

Our *main objective* in this paper consists in showing that the above *necessary conditions* become also *sufficient* for the existence of the mapping $\boldsymbol{\theta}$ when ω is *simply-connected*, according to the following result (Theorem 4.1), which thus constitutes the *other approach to the fundamental theorem of surface theory* announced in the tile: Let ω be a *simply-connected* open subset of $\mathbb{R}^2$ and let $\mathbf{A} = (a_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $\mathbf{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ be two matrix fields that satisfy the *Darboux-Vallée-Fortuné compatibility relation*

$$\partial_2 \boldsymbol{\lambda}_1 - \partial_1 \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 \text{ in } \mathcal{D}'(\omega; \mathbb{R}^3), \tag{1.4}$$

where the components $\lambda_{\alpha\beta} \in \mathcal{C}^1(\omega)$ and $\lambda_{3\beta} \in \mathcal{C}^0(\omega)$ of the vector fields $\boldsymbol{\lambda}_1 := (\lambda_{i1}) : \omega \to \mathbb{R}^3$ and $\boldsymbol{\lambda}_2 := (\lambda_{i2}) : \omega \to \mathbb{R}^3$ are defined in terms of the matrix fields $\mathbf{A}$ and $\mathbf{B}$ by the matrix equations

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} := -\mathbf{J}\mathbf{A}^{-1/2}\mathbf{B} \text{ and } \begin{pmatrix} \lambda_{31} \\ \lambda_{32} \end{pmatrix} := \mathbf{J}\mathbf{A}^{-1/2}\mathbf{J}\begin{pmatrix} \partial_2 u^0_{11} - \partial_1 u^0_{12} \\ \partial_2 u^0_{21} - \partial_1 u^0_{22} \end{pmatrix}, \tag{1.5}$$

where $(u^0_{\alpha\beta}) := \mathbf{A}^{1/2}$. *Then there exists an immersion* $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbb{R}^3)$ *such that*

$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \theta|} = b_{\alpha\beta} \text{ in } \omega.$$

Notice in passing that, like the Gauss and Codazzi-Mainardi relations (1.2)-(1.3), the Darboux-Vallée-Fortuné (1.4) relation in vector form consists of *three* independent scalar equations.

Our strategy for proving the existence of the immersion $\boldsymbol{\theta}$ critically hinges on a new version of the fundamental theorem of Riemannian geometry recently proved in Ciarlet, Gratie, Iosifescu, C. Mardare & Vallée [9], which asserts the following (we state it in $\mathbb{R}^3$ for coherence, but it holds as well in $\mathbb{R}^n$ for any $n \geq 2$):

Let Ω be a simply-connected open subset of $\mathbb{R}^3$ and let $\mathbf{C} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies the following *Shield-Vallée compatibility relation* in matrix form (so named after Shield [24] and Vallée [28]):

$$\mathbf{CURL}\ \boldsymbol{\Lambda} + \mathbf{COF}\ \boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3),$$

where the matrix field $\boldsymbol{\Lambda} \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ is defined in terms of the matrix field $\mathbf{C}$ by

$$\boldsymbol{\Lambda} := \frac{1}{\det \widetilde{\mathbf{U}}} \widetilde{\mathbf{U}} \{(\mathbf{CURL}\ \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}} - \frac{1}{2}(\text{ tr } [(\mathbf{CURL}\ \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}}])\mathbf{I}\},$$

where

$$\widetilde{\mathbf{U}} := \mathbf{C}^{1/2} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>).$$

Then there exist an immersion $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ *that satisfies*

$$\boldsymbol{\nabla}\boldsymbol{\Theta}^\tau \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C} \text{ in } \mathcal{C}^1(\Omega; \mathbb{S}^3_>).$$

Then the proof relies on the following observation, which was also the basis of a new proof of the fundamental theorem of surface theory (in its "classical" version recalled at the beginning of this introduction), due to Ciarlet & Larsonneur [11]: Given a smooth immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ and given $\varepsilon > 0$, let $\Omega := \omega \times]-\varepsilon, \varepsilon[$, and let the canonical extension $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ of $\boldsymbol{\theta}$ be defined as before by $\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \mathbf{a}_3(y)$ for all $(y, x_3) \in \Omega$, where $\mathbf{a}_3 := \dfrac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}$, and let

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau} \text{ and } g_{i3} = \delta_{i3} \text{ in } \Omega,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $(a^{\sigma\tau}) = (a_{\alpha\beta})^{-1}$.

This observation is put to use as follows: Assume that the matrices (g_{ij}) constructed in this fashion from the given matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are *invertible*, hence positive definite, over the set Ω (if they are not invertible, the resulting difficulty is easily circumvented). Then the field $(g_{ij}) : \Omega \to \mathbb{S}^3$ becomes a natural candidate for applying the above "three-dimensional" existence result, provided of course that the "*three-dimensional" sufficient relation*

$$\mathbf{CURL}\ \boldsymbol{\Lambda} + \mathbf{COF}\ \boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3)$$

can be shown to be hold, as consequences of the "two-dimensional" relations:

$$\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3).$$

That this is indeed the case is the essence of our proof: By the "three-dimensional" theorem, there exists an immersion $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ that satisfies $g_{ij} = \partial_i\boldsymbol{\Theta} \cdot \partial_j\boldsymbol{\Theta}$ in Ω. It thus remains to check that the immersion $\boldsymbol{\theta} := \boldsymbol{\Theta}(\cdot, 0)$ indeed satisfies the announced conclusions.

These conclusions are drawn through computations that, by virtue of their more vector-like or matrix-like nature, are to a large extent more concise and substantially simpler than the lengthy computations in Ciarlet & Larsonneur [11], which relied on a massive usage of indices, combined with the consideration of infinite series.

It is to be emphasized that the most striking feature of the Darboux-Vallée-Fortuné compatibility relation is its *geometrical nature*, illustrated by its relation to a *surface rotation field*, as explained earlier. That, by contrast with the Gauss and Codazzi-Mainardi equations, the Christoffel symbols do *not* appear in the Darboux-Vallée-Fortuné relation is equally noteworthy.

Particularly relevant to the present work are the interesting analyses of Pietraszkiewicz & Vallée [23], Pietraskiewicz & Szwabowicz [21], and Pietraszkiewicz, Szwabowicz & Vallée [22], which show how the midsurface of a deformed thin shell can be reconstructed from the knowledge of the undeformed midsurface and of the surface strains and bendings. In particular, these authors also use in a crucial way the polar factorization of the deformation gradient of the midsurface.

Finally, we mention the related existence theorem of Ciarlet, Gratie & Mardare [8], where a different (and new to the authors' best knowledge) compatibility relation, expressed again in terms of the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$, have been proposed that are likewise related to rotation fields. This relation takes the form of the matrix equation

$$\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1 = \mathbf{0} \text{ in } \omega,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are antisymmetric matrix fields of order three that are functions of the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$, the field $(a_{\alpha\beta})$ appearing in particular through the square root $\mathbf{U}$ of the matrix field $\mathbf{C} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The main novelty in the proof of existence then lies in an explicit use of the rotation field $\mathbf{R}$ that appears in the polar factorization $\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{RU}$ of the restriction to the unknown surface of the gradient of the canonical three-dimensional extension $\boldsymbol{\Theta}$ of the unknown immersion $\boldsymbol{\theta}$. As in the recent extensions of the fundamental theorem of surface theory due to S. Mardare [17, 18], the unknown immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ is found in ibid. in function spaces "with little regularity", such as $W^{2,p}_{\text{loc}}(\omega;\mathbb{R}^3)$, $2 < p \leq \infty$.

The results of this paper have been announced in [10].

2. Notations

The rules governing Latin and Greek indices have already been set forth in Section 1. Specific sets of matrices, such as $\mathbb{M}^n, \mathbb{M}^{m\times n}$, etc., have also been defined there.

The same symbol $\mathbf{I}$ designates the identity matrix in $\mathbb{M}^n$ for any $n \geq 2$. The notation (a_{ij}) designates a matrix with a_{ij} as its elements, the first index being the row index. Given a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{M}^{m\times n}$, the notation $(\mathbf{A})_{ij}$ designates its element a_{ij}. When it is identified with a matrix, a vector in $\mathbb{R}^m$ will always be understood as a column vector, i.e, a matrix in $\mathbb{M}^{m\times 1}$. To avoid confusions, the notation

$(a_{11}; a_{12})$ (instead of $(a_{11}\ a_{12})$) will be occasionally introduced to designate a row vector in $\mathbb{M}^{1\times 2}$. The notation $(\mathbf{a})_i$ denotes the i-th component of a vector $\mathbf{a}$ and the notation $[\mathbf{A}]_j$ designates the j-th column of a matrix $\mathbf{A}$.

The Euclidean norm of $\mathbf{a} \in \mathbb{R}^m$ is denoted $|\mathbf{a}|$ and the Euclidean inner-product of $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$ is denoted $\mathbf{a} \cdot \mathbf{b}$. The vector product of $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$ is denoted $\mathbf{a} \wedge \mathbf{b}$. The *cofactor matrix* **COF A** associated with a matrix $\mathbf{A} \in \mathbb{M}^3$ is the matrix in $\mathbb{M}^3$ defined by

$$\mathbf{COF\ A} := \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{32}a_{13} - a_{33}a_{12} & a_{33}a_{11} - a_{31}a_{13} & a_{31}a_{12} - a_{32}a_{11} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

Given any matrix $\mathbf{C} \in \mathbb{S}^n_>$, there exists a unique matrix $\mathbf{U} \in \mathbb{S}^n_>$ such that $\mathbf{U}^2 = \mathbf{C}$ (for a proof, see, e.g., Ciarlet [5, Theorem 3.2-1). The matrix $\mathbf{U}$ is denoted $\mathbf{C}^{1/2}$ and is called the *square root* of $\mathbf{C}$. The mapping $\mathbf{C} \in \mathbb{S}^n_> \to \mathbf{C}^{1/2} \in \mathbb{S}^n_>$ defined in this fashion is of class C^∞ (for a proof, see, e.g., Gurtin [15, Section 3]).

Any invertible matrix $\mathbf{F} \in \mathbb{M}^n$ admits a unique *polar factorization* $\mathbf{F} = \mathbf{R}\,\mathbf{U}$. This means that $\mathbf{F}$ can be factored in a unique fashion as a product of a matrix $\mathbf{R} \in \mathbb{O}^n$ by a matrix $\mathbf{U} \in \mathbb{S}^n_>$ with $\mathbf{U} := (\mathbf{F}^T\mathbf{F})^{1/2}$ and $\mathbf{R} := \mathbf{F}\,\mathbf{U}^{-1}$ (the existence and uniqueness of such a factorization easily follow from the existence and uniqueness of the square root of a matrix $\mathbf{C} \in \mathbb{S}^n_>$).

The coordinates of a point $x \in \mathbb{R}^3$ are denoted x_i and partial derivatives operators, in the usual sense or in the sense of distributions, of the first order are devoted ∂_i. The coordinates of a point $y \in \mathbb{R}^2$ are denoted y_α and partial derivatives of the first and second order are denoted ∂_α and $\partial_{\alpha\beta}$.

All the vector spaces considered in this paper are over $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^n$. The notation $U \Subset \Omega$ means that $\overline{U}$ is a compact subset of Ω. The notations $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ respectively designate the space of all functions $\varphi \in \mathcal{C}^\infty(\Omega)$ whose support is compact and contained in Ω, and the space of distributions over Ω. The notations $\mathcal{C}^\ell(\Omega), \ell \geq 0$, and $W^{m,\infty}(\Omega), m \geq 0, 1 \leq p \leq \infty$, respectively designate the spaces of continuous functions over Ω for $\ell = 0$, or ℓ-times continuously differentiable functions over Ω for $\ell \geq 1$, and the usual Sobolev spaces, with $L^\infty(\Omega) = W^{0,\infty}(\Omega)$. Finally, $W^{m,\infty}_{\text{loc}}(\Omega)$ designates the space of equivalent classes $\dot{f}$ of measurable functions $f : \Omega \to \mathbb{R}$ such that $f|_U \in W^{m,\infty}(U)$ for all open sets $U \Subset \Omega$, where $f|_U$ denotes the restriction to f to U.

Let $\mathbb{X}$ be any finite-dimensional space, such as $\mathbb{R}^n, \mathbb{M}^{m\times n}, \mathbb{A}^n$, etc., or a subset thereof, such as $\mathbb{S}^n_>, \mathbb{O}^n$, etc. Then notations such as $\mathcal{D}'(\Omega; \mathbb{X}), \mathcal{C}^\ell(\Omega; \mathbb{X}), L^\infty_{\text{loc}}(\Omega; \mathbb{X})$, etc., designate spaces or sets of vector fields or matrix fields with values in $\mathbb{X}$ and whose components belong to $\mathcal{D}'(\Omega), \mathcal{C}^\ell(\Omega), L^\infty_{\text{loc}}(\Omega)$, etc.

Given a mapping $\boldsymbol{\Theta} = (\Theta_i) \in \mathcal{D}'(\Omega; \mathbb{R}^3)$, the matrix field $\boldsymbol{\nabla\Theta} \in \mathcal{D}'(\Omega; \mathbb{M}^3)$ is defined by $(\boldsymbol{\nabla\Theta})_{ij} = \partial_j \Theta_i$. Given a matrix field $\mathbf{A} = (a_{ij}) \in \mathcal{D}'(\Omega; \mathbb{M}^3)$, the notation **CURL A** designates the matrix field

$$\mathbf{CURL\ A} := \begin{pmatrix} \partial_2 a_{13} - \partial_3 a_{12} & \partial_3 a_{11} - \partial_1 a_{13} & \partial_1 a_{12} - \partial_2 a_{11} \\ \partial_2 a_{23} - \partial_3 a_{22} & \partial_3 a_{21} - \partial_1 a_{23} & \partial_1 a_{22} - \partial_2 a_{21} \\ \partial_2 a_{33} - \partial_3 a_{32} & \partial_3 a_{31} - \partial_1 a_{33} & \partial_1 a_{32} - \partial_2 a_{31} \end{pmatrix} \in \mathcal{D}'(\Omega; \mathbb{M}^3).$$

3. The Fundamental theorem of Riemannian geometry in $\mathbb{R}^3$

The *fundamental theorem of Riemannian geometry in* $\mathbb{R}^3$ classically asserts that, if the Riemann curvature tensor associated with a field $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ vanishes in a simply-connected open subset of $\mathbb{R}^3$, then $\mathbf{C}$ is the *metric tensor field* of a manifold isometrically imbedded in $\mathbb{R}^3$, i.e., there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ such that $\mathbf{C} = \boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta}$ in Ω

The above regularity assumption on the field $\mathbf{C}$ can be weakened in various ways. For instance, C. Mardare [16] has shown that the following existence theorem holds if $\mathbf{C} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$.

Theorem 3.1. *Let Ω be a simply-connected open subset of $\mathbb{R}^3$ and let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$ be a matrix field whose components satisfy the compatibility relations*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \mathcal{D}'(\Omega) \tag{3.1}$$

for all $i, j, k, q \in \{1, 2, 3\}$, where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}), \quad \Gamma^p_{ij} := g^{pq} \Gamma_{ijq}, \text{ and } (g^{pq}) = g_{ij})^{-1}. \tag{3.2}$$

Then there exist an immersion $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ that satisfies

$$\boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta} = \mathbf{C} \text{ in } \mathcal{C}^1(\Omega; \mathbb{S}^3_>). \tag{3.3}$$

Such an immersion $\boldsymbol{\Theta}$ becomes uniquely defined if Ω is connected and conditions such as

$$\boldsymbol{\Theta}(x_0) = \mathbf{a}_0 \text{ and } \boldsymbol{\nabla\Theta}(x_0) = \mathbf{F}_0 \tag{3.4}$$

are imposed, where $x_0 \in \Omega, \mathbf{a}_0 \in \mathbb{R}^3$, and $\mathbf{F}_0 \in \mathbb{M}^3$ is any matrix that satisfies $\mathbf{F}_0^T \mathbf{F}_0 = \mathbf{C}(x_0)$ (for instance, $\mathbf{F}_0 = (\mathbf{C}(x_0))^{1/2}$). □

The functions R_{qijk} defined in (3.1) are the (covariant) *components of the Riemann curvature tensor* associated with the field $\mathbf{C} = (g_{ij})$, and the functions Γ_{ijq} and Γ^p_{ij} defined in (3.2) are the *Christoffel symbols* of the first and second kinds. It is easily seen that the relations (3.1) reduce in fact to *six* independent relations, such as

$$R_{1212} = R_{1213} = R_{1223} = R_{1313} = R_{1323} = R_{2323} = 0$$

(other such six relations are clearly possible).

Ciarlet, Gratie, Iosifescu, C. Mardare & Vallée [9] have recently shown that an existence theorem similar to Theorem 3.1 holds, but under a *different* compatibility relation, this time involving the *square root of the matrix field* $\mathbf{C}$. More specifically, the following *new formulation of the fundamental theorem of Riemannian geometry in* $\mathbb{R}^3$ has been established in Theorem 6.2 in ibid.

Theorem 3.2. *Let Ω be a simply-connected open subset of $\mathbb{R}^3$ and let $\mathbf{C} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies the Shield-Vallée compatibility relation* (in matrix form)

$$\mathbf{CURL}\, \boldsymbol{\Lambda} + \mathbf{COF}\, \boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3), \tag{3.5}$$

where the matrix field $\boldsymbol{\Lambda} \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ is defined in terms of the matrix field $\mathbf{C}$ by

$$\boldsymbol{\Lambda} := \frac{1}{\det \widetilde{\mathbf{U}}} \widetilde{\mathbf{U}}\{(\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}} - \frac{1}{2}(\text{tr } [(\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}}])\mathbf{I}\}, \tag{3.6}$$

where

(3.7) $$\widetilde{\mathbf{U}} := \mathbf{C}^{1/2} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>).$$

Then there exists an immersion $\mathbf{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ *that satisfies*

(3.8) $$\boldsymbol{\nabla}\mathbf{\Theta}^T \boldsymbol{\nabla}\mathbf{\Theta} = \mathbf{C} \text{ in } \mathcal{C}^1(\Omega; \mathbb{S}^3_>).$$

Such an immersion $\mathbf{\Theta}$ *becomes uniquely defined if* Ω *is connected and conditions such as* (3.4) *are imposed.* □

The specific form of the relation (3.5), with the fields $\mathbf{\Lambda}$ and $\widetilde{\mathrm{U}}$ defined as in (3.6) and (3.7), is due to Vallée [28], who showed that it is *necessarily* satisfied by the metric tensor field $\mathbf{C} := \boldsymbol{\nabla}\mathbf{\Theta}^T \boldsymbol{\nabla}\mathbf{\Theta}$ associated with a smooth enough immersion $\mathbf{\Theta} : \Omega \to \mathbb{R}^3$, where Ω is any open subset of $\mathbb{R}^3$ (simply-connected or not). It is easily verified that, like relations (3.1), the matrix equation (3.5) reduce again to only six independent scalar equations.

If the set Ω is connected, but no condition such as (3.4) are imposed, then the immersions found in either Theorem 3.1 or Theorem 3.2 are *uniquely defined up to isometries in* $\mathbb{R}^3$. This means that, given an immersion $\mathbf{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ that satisfies (3.3) or (3.8), any immersion $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ that satisfies

$$\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^T \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} = \mathbf{C} \text{ in } \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$$

is necessarily of the form

$$\widetilde{\mathbf{\Theta}}(x) = \mathbf{a} + \mathbf{Q}\mathbf{\Theta}(x) \text{ for all } x \in \Omega, \text{ for some vector } \mathbf{a} \in \mathbb{R}^3 \text{ and some matrix } \mathbf{Q} \in \mathbb{O}^3.$$

Theorem 3.2 is the point of departure of our subsequent analysis.

4. A new formulation of the fundamental theorem of surface theory

We now establish the main result of this paper, viz., that the *Darboux-Vallée-Fortuné compatibility relation* (cf. (4.1) below), which is *necessarily* satisfied by the matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$ associated with a *given* smooth immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ (see Section 1), are also *sufficient* for the existence of such an immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ if the open set $\omega \subset \mathbb{R}^2$ is simply-connected.

Theorem 4.1. *Let* ω *be simply-connected open subset of* $\mathbb{R}^2$ *and let* $\mathbf{A} = (a_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ *and* $\mathbf{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ *be two matrix fields that satisfy the Darboux-Vallée-Fortuné compatibility relation*

(4.1) $$\partial_2 \boldsymbol{\lambda}_1 - \partial_1 \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 \text{ in } \mathcal{D}'(\omega; \mathbb{R}^3),$$

where the components $\lambda_{\alpha\beta} \in \mathcal{C}^1(\omega)$ *and* $\lambda_{3\beta} \in \mathcal{C}^0(\omega)$ *of the two vector fields* $\boldsymbol{\lambda}_\beta = (\lambda_{i\beta}) : \omega \to \mathbb{R}^3$ *are defined in terms of the matrix fields* $\mathbf{A}$ *and* $\mathbf{B}$ *by the matrix equations*

(4.2) $$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} := -\mathbf{J}\mathbf{A}^{-1/2}\mathbf{B},$$

(4.3) $$(\lambda_{31}; \quad \lambda_{32}) := (\partial_2 u^0_{11} - \partial_1 u^0_{12}; \quad \partial_2 u^0_{21} - \partial_1 u^0_{22})\mathbf{J}\mathbf{A}^{-1/2}\mathbf{J},$$

where

(4.4) $$\mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } (u^0_{\alpha\beta}) := \mathbf{A}^{1/2} \in \mathcal{C}^1(\omega; \mathbb{S}^2_>).$$

Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbb{R}^3)$ such that

(4.5) $$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ in } \mathcal{C}^1(\omega) \text{ and } \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \mathcal{C}^0(\omega).$$

Such an immersion $\boldsymbol{\theta}$ becomes uniquely defined if ω is connected and conditions such as

(4.6) $$\boldsymbol{\theta}(y_0) = \mathbf{a}_0 \text{ and } \partial_\alpha \boldsymbol{\theta}(y_0) = \mathbf{a}^0_\alpha,$$

are imposed, where $y_0 \in \omega, \mathbf{a}_0 \in \mathbb{R}^3$, and $\mathbf{a}^0_\alpha \in \mathbb{R}^3$ are two linearly independent vectors that satisfy $\mathbf{a}^0_\alpha \cdot \mathbf{a}^0_\beta = a_{\alpha\beta}(y_0)$.

Proof. For clarity, the proof is broken into parts, numbered (i) to (xi). Note that parts (i) to (iii) hold *verbation for any* matrix fields $\mathbf{A} \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $\mathbf{B} \in \mathcal{C}^1(\omega; \mathbb{S}^2)$, i.e., irrespective of whether these fields satisfy the compatibility relation (4.1).

(i) *Let ω_0 be an open subset of $\mathbb{R}^2$ such that $\overline{\omega}_0$ is a compact subset of ω. Then there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that*

(4.7) $$\{\mathbf{A}(y) - 2x_3\mathbf{B}(y) + x_3^2\mathbf{B}(y)\mathbf{A}^{-1}(y)\mathbf{B}(y)\} \in \mathbb{S}^2_> \text{ for all } (y, x_3) \in \overline{\Omega}_0,$$

and

(4.8) $$\text{tr } (\mathbf{A}(y)^{1/2} - x_3 \text{ tr } (\mathbf{B}(y)\mathbf{A}(y)^{-1/2}) > 0 \text{ for all } (y, x_3) \in \overline{\Omega}_0,$$

where

(4.9) $$\Omega_0 := \omega_0 \times] - \varepsilon_0, \varepsilon_0[.$$

To see this, it suffices to combine a straightforward compactness-continuity argument with the assumptions that $\mathbf{A} \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $\mathbf{B} \in \mathcal{C}^1(\omega; \mathbb{S}^2)$.

In what follows, various functions, vector fields, or matrix fields, will be defined over the set $\overline{\Omega}_0 = \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$. However, in order to avoid lengthy and cumbersome formulas, their dependence on the variable $y \in \overline{\omega}_0$ will be often omitted, while their dependence on the variable $x_3 \in [-\varepsilon_0, \varepsilon_0]$ will be occasionally omitted. For instance, the relation $\mathbf{U} := (\mathbf{A} - 2x_3\mathbf{B} + x_3^2\mathbf{B}^{-1}\mathbf{A}\mathbf{B})^{1/2}$ that appears in eq. (4.10) means that $\mathbf{U}(y, x_3) := (\mathbf{A}(y) - 2x_3\mathbf{B}(y) + x_3^2\mathbf{B}^{-1}(y)\mathbf{A}(y)\mathbf{B}(y))^{1/2}$ for all $(y, x_3) \in \overline{\Omega}_0 = \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$, so that the matrix field $\mathbf{U}$ is effectively a function of both $y \in \overline{\omega}_0$ and $x_3 \in [-\varepsilon_0, \varepsilon_0]$, even though neither y nor x_3 appear on the left-hand side of (4.9) (by contrast, it is essential that x_3 appear in the right-hand side); likewise, it should be clear that the matrix field $\mathbf{Q}$ and the function φ appearing in eqs. (4.11) and (4.12) are also both functions of $y \in \overline{\omega}_0$ and $x_3 \in [-\varepsilon_0, \varepsilon_0]$; etc.

(ii) *Define the matrix field*

(4.10) $$\mathbf{U} := (\mathbf{A} - 2x_3\mathbf{B} + x_3^2\mathbf{B}^{-1}\mathbf{A}\mathbf{B})^{1/2} \in \mathcal{C}^1(\overline{\Omega}_0; \mathbb{S}^2_>).$$

Then the field $\mathbf{U}$ is also given by

$$\mathbf{U} = \mathbf{Q}^T(\mathbf{A}^{1/2} - x_3\mathbf{A}^{-1/2}\mathbf{B}), \text{ where } \mathbf{Q} := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \tag{4.11}$$

the function $\varphi \in \mathcal{C}^1(\overline{\Omega}_0)$ being defined by

$$\varphi := \arctan\left(\frac{x_3 \text{ tr } (\mathbf{BJA}^{-1/2})}{\text{tr } \mathbf{A}^{1/2} - x_3 \text{ tr } (\mathbf{BA}^{-1/2})}\right) \text{ with } \mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.12}$$

The matrix field $\mathbf{U} \in \mathcal{C}^1(\overline{\Omega}_0; \mathbb{S}^2_>)$ being defined as in (4.10) *or* (4.11), *define the matrix fields*

$$\widetilde{\mathbf{U}} := \begin{pmatrix} \boxed{\mathbf{U}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \in \mathcal{C}^1(\overline{\Omega}_0; \mathbb{S}^3_>) \text{ and } (g_{ij}) := \widetilde{\mathbf{U}}^2 \in \mathcal{C}^1(\overline{\Omega}_0; \mathbb{S}^3_>). \tag{4.13}$$

Then

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau} \text{ and } g_{i3} = \delta_{i3}, \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}. \tag{4.14}$$

The matrix fields $\mathbf{U}$ and $\mathbf{Q}$ being defined as in (4.10) and (4.11), elementary matrix algebra shows that the matrix field $\mathbf{U}$ is symmetric if and only if

$$(\mathbf{MQ})_{12} = (\mathbf{MQ})_{21}, \text{ where } \mathbf{M} = (m_{\alpha\beta}) := \mathbf{A}^{1/2} - x_3\mathbf{BA}^{-1/2},$$

a relation itself satisfied if and only if

$$(m_{11} + m_{22}) \sin\varphi = (m_{12} - m_{21}) \cos\varphi.$$

Noting that

$$m_{11} + m_{22} = \text{ tr } \mathbf{A}^{1/2} - x_3 \text{ tr } (\mathbf{BA}^{-1/2}) \text{ and } (m_{12} - m_{21}) = x_3 \text{ tr } (\mathbf{BJA}^{-1/2}),$$

we thus infer from relation (4.8) that the matrix $\mathbf{U}$ field is symmetric if the function φ is defined as in (4.12).

The relations $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{QQ}^T = \mathbf{I}$ imply that

$$\begin{aligned} \mathbf{U}^2 = \mathbf{U}^T\mathbf{U} &= (\mathbf{A}^{1/2} - x_3\mathbf{BA}^{-1/2})\mathbf{QQ}^T(\mathbf{A}^{1/2} - x_3\mathbf{A}^{-1/2}\mathbf{B}) \\ &= \mathbf{A} - 2x_3\mathbf{B} + x_3^2\mathbf{BA}^{-1}\mathbf{B}. \end{aligned}$$

Hence the matrix field $\mathbf{U}$ defined in (4.11) is indeed the unique square root of the matrix field $(\mathbf{A} - 2x_3\mathbf{B} + x_3^2\mathbf{B}^{-1}\mathbf{AB}) \in \mathcal{C}^1(\overline{\Omega}_0; \mathbb{S}^2_>)$.

Relations (4.14) immediately follow from the definitions (4.10) and (4.13) of the matrix fields $\mathbf{U}$ and $\widetilde{\mathbf{U}}$.

(iii) In what follows, the same symbol $\mathbf{I}$ denotes the 2×2 and the 3×3 identity matrices (for instance $\mathbf{I} \in \mathbb{M}^3$ in (4.15); $\mathbf{I} \in \mathbb{M}^2$ in (4.20); etc.). *The matrix field $\widetilde{\mathbf{U}} \in \mathcal{C}^1(\Omega_0; \mathbb{S}^3_>)$ being defined as in* (4.13), *define the matrix field*

$$\mathbf{\Lambda} := \frac{1}{\det \widetilde{\mathbf{U}}} \widetilde{\mathbf{U}}\{(\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}} - \frac{1}{2}(\mathrm{tr}\, [(\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}}])\mathbf{I}\} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3). \tag{4.15}$$

Then the field $\mathbf{\Lambda}$ *is also given by*

$$\mathbf{\Lambda} = \begin{pmatrix} \boxed{\mathbf{J}^T \mathbf{Q}^T \mathbf{A}^{-1/2} \mathbf{B}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \Lambda_{31} \quad \Lambda_{32} & \partial_3 \varphi \end{pmatrix}, \tag{4.16}$$

where the function φ *is defined in equation* (4.12), *and*

$$(\Lambda_{31};\quad \Lambda_{32}) := (\partial_1 u_{12} - \partial_2 u_{11};\ \partial_1 u_{22} - \partial_2 u_{21}) \mathbf{J}^T \mathbf{U}^{-1} \mathbf{J} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^{1\times 2}),\ \text{with } (u_{\alpha\beta}) := \mathbf{U}. \tag{4.17}$$

By definition of the matrix **CURL** operator (Section 2),

$$\mathbf{CURL}\, \widetilde{\mathbf{U}} = \begin{pmatrix} \boxed{(\partial_3 \mathbf{U}) \mathbf{J}^T} & \begin{matrix} \partial_1 u_{12} - \partial_2 u_{11} \\ \partial_1 u_{22} - \partial_2 u_{21} \end{matrix} \\ 0 \quad\ 0 & 0 \end{pmatrix}.$$

Consequently,

$$(\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}} = \begin{pmatrix} \boxed{\mathbf{J}(\partial_3 \mathbf{U}) \mathbf{U}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ (\partial_1 u_{12} - \partial_2 u_{11};\ \partial_1 u_{22} - \partial_2 u_{21}) \mathbf{U} & 0 \end{pmatrix}. \tag{4.18}$$

Noting that, by (4.11),

$$(\partial_3 \mathbf{U}) \mathbf{U} = (\mathbf{A}^{1/2} \partial_3 \mathbf{Q} - \mathbf{B} \mathbf{A}^{-1/2} \mathbf{Q} - x_3 \mathbf{B} \mathbf{A}^{-1/2} \partial_3 \mathbf{Q}) \mathbf{Q}^T (\mathbf{A}^{1/2} - x_3 \mathbf{A}^{-1/2} \mathbf{B})$$

and that, again by (4.11),

$$\partial_3 \mathbf{Q} = (\partial_3 \varphi) \mathbf{J} \mathbf{Q} = (\partial_3 \varphi) \mathbf{Q} \mathbf{J},$$

since **Q** and **J** commute, we obtain, after some straightforward computations,

$$(\partial_3 \mathbf{U}) \mathbf{U} = -\mathbf{B} + (\partial_3 \varphi) \mathbf{U} \mathbf{J} \mathbf{U} + x_3 \mathbf{B} \mathbf{A}^{-1} \mathbf{B}.$$

Consequently,

$$\mathbf{J}(\partial_3 \mathbf{U}) \mathbf{U} = -\mathbf{J} \mathbf{B} - (\partial_3 \varphi)(\det \mathbf{U}) \mathbf{I} + x_3 \mathbf{J} \mathbf{B} \mathbf{A}^{-1} \mathbf{B}, \tag{4.19}$$

since

$$\mathbf{U} \mathbf{J} \mathbf{U} = (\det \mathbf{U}) \mathbf{J} \text{ for any } \mathbf{U} \in \mathbb{S}^2 \text{ and } \mathbf{J}^2 = -\mathbf{I}. \tag{4.20}$$

This takes care of the first term appearing in the right-hand side of definition (4.15).

Let us now examine the other term. Using (4.18) and (4.19), we get

$$\text{(4.21)}\qquad \text{tr}\,[(\mathbf{CURL}\,\widetilde{\mathbf{U}})^T\widetilde{\mathbf{U}}] = \text{tr}\,[\mathbf{J}(\partial_3\mathbf{U})\mathbf{U}] = -2(\partial_3\varphi)\det\mathbf{U},$$

since $\text{tr}[\mathbf{JB}] = \text{tr}\,[\mathbf{JBA}^{-1}\mathbf{B}] = 0$ (both matrix fields $\mathbf{B}$ and $\mathbf{BA}^{-1}\mathbf{B}$ are symmetric). Using (4.18), (4.19) and (4.21) in definition (4.15), we thus obtain

$$\mathbf{\Lambda} = \begin{pmatrix} \boxed{(\det\mathbf{U})^{-1}\mathbf{U}} & \begin{matrix}0\\0\end{matrix} \\ 0 \quad 0 & (\det\mathbf{U})^{-1}\end{pmatrix}\begin{pmatrix} \boxed{-\mathbf{JB} + x_3\mathbf{JBA}^{-1}\mathbf{B}} & \begin{matrix}0\\0\end{matrix} \\ (\partial_1 u_{12} - \partial_2 u_{11}; \partial_1 u_{22} - \partial_2 u_{21})\mathbf{U} & (\partial_3\varphi)(\det\mathbf{U})\end{pmatrix}.$$

In order to further transform the right-hand side of the above matrix equation, we first observe that, by (4.20),

$$(\det\mathbf{U})^{-1}\mathbf{U} = \mathbf{J}^T\mathbf{U}^{-1}\mathbf{J},$$

so that

$$\mathbf{\Lambda} = \begin{pmatrix} \boxed{\mathbf{J}^T\mathbf{U}^{-1}(\mathbf{B} - x_3\mathbf{B}^{-1}\mathbf{AB})} & \begin{matrix}0\\0\end{matrix} \\ \Lambda_{31} \qquad \Lambda_{32} & \partial_3\varphi\end{pmatrix},$$

where the functions Λ_{31} and Λ_{32} are defined as in (4.17) (the relation $\mathbf{JJ}^T = \mathbf{I}$ is also used here). We next note that relation (4.11) implies that

$$\mathbf{U} = \mathbf{U}^T = (\mathbf{A}^{1/2} - x_3\mathbf{BA}^{-1/2})\mathbf{Q}.$$

After multiplying this relation by $\mathbf{Q}^T\mathbf{A}^{-1/2}$ on the right and by $\mathbf{U}^{-1}$ on the left, we find that

$$\mathbf{U}^{-1}(\mathbf{B} - x_3\mathbf{BA}^{-1}\mathbf{B}) = \mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}.$$

Therefore the matrix field $\mathbf{\Lambda}$ is indeed of the form announced in (4.16).

(iv) *The row vector field* $(\Lambda_{31}; \Lambda_{32}) \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^{1\times 2})$ *as defined in* (4.17) *is also given by*

$$\text{(4.22)}\qquad (\Lambda_{31};\quad \Lambda_{32}) = (\lambda_{31};\quad \lambda_{32}) + (\partial_1\varphi;\quad \partial_2\varphi),$$

where

$$\text{(4.23)}\qquad (\lambda_{31};\quad \lambda_{32}) := (\partial_2 u^0_{11} - \partial_1 u^0_{12};\quad \partial_2 u^0_{21} - \partial_1 u^0_{22})\mathbf{JA}^{-1/2}\mathbf{J} \text{ and } (u^0_{\alpha\beta}) := \mathbf{A}^{1/2}.$$

The compatibility relation (4.1) (which has not yet been used so far) plays an indispensable role here for establishing relations (4.22) and (4.23). Recall that the notation $[\mathbf{A}]_\alpha$ denotes the α-th column of a matrix $\mathbf{A}$.

On the one hand, the *first two components of the compatibility relation* (4.1), coupled with the definition (4.2) of the components $\lambda_{\alpha\beta}$, give

$$\text{(4.24)}\qquad \partial_2[\mathbf{A}^{-1/2}\mathbf{B}]_1 - \partial_1[\mathbf{A}^{-1/2}\mathbf{B}]_2 = -\lambda_{32}[\mathbf{JA}^{-1/2}\mathbf{B}]_1 + \lambda_{31}[\mathbf{JA}^{-1/2}\mathbf{B}]_2,$$

the definition (4.3) of the components $\lambda_{3\beta}$ gives

$$\partial_2[\mathbf{A}^{1/2}]_1 - \partial_1[\mathbf{A}^{1/2}]_2 = -\lambda_{32}[\mathbf{J}\mathbf{A}^{1/2}]_1 + \lambda_{31}[\mathbf{J}\mathbf{A}^{1/2}]_2, \tag{4.25}$$

and the definition (4.17) of the functions $\Lambda_{3\beta}$ gives

$$\partial_2[\mathbf{U}]_1 - \partial_1[\mathbf{U}]_2 = -\Lambda_{32}[\mathbf{J}\mathbf{U}]_1 + \Lambda_{31}[\mathbf{J}\mathbf{U}]_2. \tag{4.26}$$

On the other hand, the definition (4.11) of the matrix field $\mathbf{U}$, combined with the relations

$$\partial_\alpha \mathbf{Q}^T = (\partial_\alpha \varphi)\mathbf{Q}^T\mathbf{J}^T$$

and with relations (4.24) and (4.25), gives

$$\begin{aligned}\partial_2[\mathbf{U}]_1 - \partial_1[\mathbf{U}]_2 =& -(\partial_2\varphi)[\mathbf{J}\mathbf{U}]_1 + (\partial_1\varphi)[\mathbf{J}\mathbf{U}]_2 + \mathbf{Q}^T(-\lambda_{32}[\mathbf{J}\mathbf{A}^{1/2}]_1 + \lambda_{31}[\mathbf{J}A^{1/2}]_2)\\ &- x_3\mathbf{Q}^T(-\lambda_{32}[\mathbf{J}\mathbf{A}^{-1/2}\mathbf{B}]_1 + \lambda_{31}[\mathbf{J}\mathbf{A}^{-1/2}\mathbf{B}]_2)\\ =& -(\lambda_{32} + \partial_2\varphi)[\mathbf{J}\mathbf{U}]_1 + (\lambda_{31} + \partial_1\varphi)[\mathbf{J}\mathbf{U}]_2.\end{aligned}$$

Together with (4.26), this last expression shows that the functions $\Lambda_{3\beta}$ are indeed of the form announced in (4.22) (the vector fields $[\mathbf{J}\mathbf{U}]_1$ and $[\mathbf{J}\mathbf{U}]_2$ are linearly independent).

The definition (4.12) of the function φ shows that $\varphi(y,0) = 0$ for all $y \in \overline{\omega}_0$; hence $\partial_\alpha\varphi(y,0) = 0$ for all $y \in \overline{\omega}_0$. Therefore, relations (4.17) and (4.22) combined imply that

$$(\lambda_{31};\quad \lambda_{32})(y) = (\Lambda_{31};\quad \Lambda_{32})(y,0) \text{ for all } y \in \overline{\omega}_0.$$

Hence relations (4.23) are established.

(v) By parts (iii) and (iv), *the matrix field* $\mathbf{\Lambda} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3)$ *of* (4.15) *is of the form*

$$\mathbf{\Lambda} = (\Lambda_{ij}) = \begin{pmatrix} \boxed{\mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}} & \begin{matrix}0\\0\end{matrix} \\ \Lambda_{31} \quad \Lambda_{32} & \Lambda_{33}\end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}_1 \,\Big|\, \mathbf{\Lambda}_2 \,\Big|\, \mathbf{\Lambda}_3 \end{pmatrix}, \text{ with } \mathbf{\Lambda}_3 := \begin{pmatrix}0\\0\\ \partial_3\varphi\end{pmatrix}, \tag{4.27}$$

where the row-vector field $(\Lambda_{31}; \Lambda_{32}) \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^{1\times 2})$ *is defined by* (4.22)-(4.23). *Then*

$$\mathbf{COF}\ \mathbf{\Lambda} = \begin{pmatrix} \boxed{-(\partial_3\varphi)\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J}} & \Big| & \mathbf{\Lambda}_1 \wedge \mathbf{\Lambda}_2 \\ 0 \quad 0 & \Big| & \end{pmatrix} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3). \tag{4.28}$$

By definition of the cofactor matrix (Section 2),

$$\mathbf{COF}\ \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_2 \wedge \mathbf{\Lambda}_3 \,\Big|\, \mathbf{\Lambda}_3 \wedge \mathbf{\Lambda}_1 \,\Big|\, \mathbf{\Lambda}_1 \wedge \mathbf{\Lambda}_2 \end{pmatrix},$$

which gives, in view of (4.27),

$$\mathbf{COF}\,\boldsymbol{\Lambda} = \left(\begin{array}{c|c} \boxed{\partial_3\varphi \begin{pmatrix} \Lambda_{22} & -\Lambda_{21} \\ -\Lambda_{12} & \Lambda_{11} \end{pmatrix}} & \boldsymbol{\Lambda}_1 \wedge \boldsymbol{\Lambda}_2 \\ 0 \quad\quad 0 & \end{array}\right), \text{ with } (\Lambda_{\alpha\beta}) := \mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}.$$

Then the matrix field $\mathbf{COF}\,\boldsymbol{\Lambda}$ is indeed of the form (4.28), since

$$\begin{pmatrix} \Lambda_{22} & -\Lambda_{21} \\ -\Lambda_{12} & \Lambda_{11} \end{pmatrix} = \mathbf{J}^T \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \mathbf{J} = -\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J}.$$

(vi) *Let* $\boldsymbol{\Lambda} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3)$ *be the matrix field of* (4.27). *Then*

$$\text{(4.29)} \qquad \mathbf{CURL}\,\boldsymbol{\Lambda} = \left(\begin{array}{c|c} \boxed{-\partial_3(\mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J})} & \partial_1\boldsymbol{\Lambda}_2 - \partial_2\boldsymbol{\Lambda}_1 \\ 0 \quad\quad 0 & \end{array}\right) \in \mathcal{D}'(\Omega_0; \mathbb{M}^3).$$

The relations $\Lambda_{\alpha 3} = 0$ imply that

$$(\mathbf{CURL}\,\boldsymbol{\Lambda})_{\alpha 1} = -\partial_3\Lambda_{\alpha 2} \quad \text{and} \quad (\mathbf{CURL}\,\boldsymbol{\Lambda})_{\alpha 2} = \partial_3\Lambda_{\alpha 1}.$$

The announced expression (4.29) thus follows by noting that

$$\partial_3 \begin{pmatrix} \Lambda_{12} & -\Lambda_{11} \\ \Lambda_{22} & -\Lambda_{21} \end{pmatrix} = \partial_3 \left\{ \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \mathbf{J} \right\}$$

and that

$$(\mathbf{CURL}\,\boldsymbol{\Lambda})_{31} = \partial_2\Lambda_{33} - \partial_3\Lambda_{32} = 0,$$

$$(\mathbf{CURL}\,\boldsymbol{\Lambda})_{32} = \partial_3\Lambda_{31} - \partial_1\Lambda_{33} = 0,$$

since, by (4.22) and (4.23),

$$\partial_3\Lambda_{3\beta} = \partial_3(\lambda_{3\beta} + \partial_\beta\varphi) = \partial_{3\beta}\varphi = \partial_\beta\Lambda_{33}$$

(the third column vector in the matrix $\mathbf{CURL}\,\boldsymbol{\Lambda}$ is simply that given by the definition of the matrix **CURL** operator).

(vii) *Let the matrix fields* $\mathbf{COF}\,\boldsymbol{\Lambda} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3)$ *and* $\mathbf{CURL}\,\boldsymbol{\Lambda} \in \mathcal{D}'(\Omega_0; \mathbb{M}^3)$ *be given by* (4.28) *and* (4.29). *Then*

$$\text{(4.30)} \qquad \mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega_0; \mathbb{M}^3)$$

Like in part (iv), the compatibility relation (4.1) plays an indispensable role here. To prove (4.30), we first note that

$$\partial_3(\mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J}) = (\partial_3\mathbf{Q}^T)\mathbf{J}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J} = -(\partial_3\varphi)\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J},$$

since $\mathbf{J}^T\mathbf{Q}^T = \mathbf{Q}^T\mathbf{J}^T$, the matrix field $\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J}$ is independent of the variable x_3, $\partial_3\mathbf{Q}^T = (\partial_3\varphi)\mathbf{Q}^T\mathbf{J}^T$, and $\mathbf{J}^T\mathbf{J}^T = -\mathbf{I}$. It thus remains to show that

$$\boldsymbol{\Lambda}_1 \wedge \boldsymbol{\Lambda}_2 = \partial_2\boldsymbol{\Lambda}_1 - \partial_1\boldsymbol{\Lambda}_2. \tag{4.31}$$

Together, definition (4.2) and equations (4.16) and (4.22) show that

$$\left(\begin{array}{c|c} \boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2 \end{array}\right) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \\ \Lambda_{31} & \Lambda_{32} \end{pmatrix} = \begin{pmatrix} \boxed{\mathbf{Q}^T} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \\ \lambda_{31} + \partial_1\varphi & \lambda_{32} + \partial_2\varphi \end{pmatrix}. \tag{4.32}$$

This relation, combined with the relations $\partial_\alpha\mathbf{Q}^T = (\partial_\alpha\varphi)\mathbf{Q}^T\mathbf{J}^T$, in turn yields

$$\partial_2\boldsymbol{\Lambda}_1 - \partial_1\boldsymbol{\Lambda}_2 = \begin{pmatrix} \boxed{\mathbf{Q}^T \begin{pmatrix} \partial_2\lambda_{11} - \partial_1\lambda_{12} \\ \partial_2\lambda_{21} - \partial_1\lambda_{22} \end{pmatrix} + \mathbf{J}\mathbf{Q}^T \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \mathbf{J} \begin{pmatrix} \partial_1\varphi \\ \partial_2\varphi \end{pmatrix}} \\ \partial_2\lambda_{31} - \partial_1\lambda_{32} \end{pmatrix}, \tag{4.33}$$

on the one hand. Since the vector product $\boldsymbol{\Lambda}_1 \wedge \boldsymbol{\Lambda}_2$ can be also written as

$$\boldsymbol{\Lambda}_1 \wedge \boldsymbol{\Lambda}_2 = \begin{pmatrix} \boxed{\mathbf{J} \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \mathbf{J} \begin{pmatrix} \Lambda_{31} \\ \Lambda_{32} \end{pmatrix}} \\ \det\,(\Lambda_{\alpha\beta}) \end{pmatrix},$$

relation (4.32) implies that

$$\boldsymbol{\Lambda}_1 \wedge \boldsymbol{\Lambda}_2 = \begin{pmatrix} \boxed{\mathbf{J}\mathbf{Q}^T \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \mathbf{J} \begin{pmatrix} \lambda_{31} + \partial_1\varphi \\ \lambda_{32} + \partial_2\varphi \end{pmatrix}} \\ \det\,(\lambda_{\alpha\beta}) \end{pmatrix}, \tag{4.34}$$

on the other hand.

The *first two components of the compatibility relation* (4.1) can be also written as

$$\mathbf{J} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \mathbf{J} \begin{pmatrix} \lambda_{31} \\ \lambda_{32} \end{pmatrix} = \begin{pmatrix} \partial_2\lambda_{11} - \partial_1\lambda_{12} \\ \partial_2\lambda_{21} - \partial_1\lambda_{22} \end{pmatrix}. \tag{4.35}$$

Hence equations (4.34) and (4.35), combined with the relation $\mathbf{J}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{J}$, show that

$$\mathbf{J}\mathbf{Q}^T\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}\mathbf{J}\begin{pmatrix}\lambda_{31}+\partial_1\varphi\\ \lambda_{32}+\partial_2\varphi\end{pmatrix}=\mathbf{Q}^T\begin{pmatrix}\partial_2\lambda_{11}-\partial_1\lambda_{12}\\ \partial_2\lambda_{21}-\partial_1\lambda_{22}\end{pmatrix}+\mathbf{J}\mathbf{Q}^T\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}\mathbf{J}\begin{pmatrix}\partial_1\varphi\\ \partial_2\varphi\end{pmatrix}.$$

Consequently, by equations (4.33) and (4.34)),

$$(\mathbf{\Lambda}_1\wedge\mathbf{\Lambda}_2)_\alpha=(\partial_2\mathbf{\Lambda}_1-\partial_1\mathbf{\Lambda}_2)_\alpha.$$

The *third component of the compatibility condition* (4.1) can be also written as

$$\det(\lambda_{\alpha\beta})=\partial_2\lambda_{31}-\partial_1\lambda_{32}.$$

Hence, again by equations (4.33) and (4.34),

$$(\mathbf{\Lambda}_1\wedge\mathbf{\Lambda}_2)_3=(\partial_2\mathbf{\Lambda}_1-\partial_1\mathbf{\Lambda}_2)_3.$$

Relation (4.31), and consequently relation (4.30), thus hold.

(viii) *Let ω be simply-connected open subset of $\mathbb{R}^2$. Then there exist open subsets $\omega_n, n\geqslant 0$,* of $\mathbb{R}^2$ *such that ω_n is a compact subset of ω for each $n\geqslant 0$ and*

$$\omega=\bigcup_{n\geq 0}\omega_n. \tag{4.36}$$

Furthermore, for each $n\geqslant 0$, there exists $\varepsilon_n=\varepsilon_n(\omega_n)>0$ such that relations (4.7)-(4.8) *hold with the set $\overline{\Omega}_0$ replaced by $\overline{\Omega}_n$, where*

$$\Omega_n:=\omega_n\times]-\varepsilon_n,\varepsilon_n[, \tag{4.37}$$

Finally, the open set

$$\Omega:=\bigcup_{n\geq 0}\Omega_n \tag{4.38}$$

is connected and simply-connected.

Let $\omega_n, n\geq 0$, be open subsets with compact closures contained in ω such that relation (4.36) holds. The existence of $\varepsilon_n=\varepsilon_n(\omega_n)>0$ with the required properties is established as in part (i), with the set $\overline{\omega}_0$ replaced by $\overline{\omega}_n$.

It is clear that the set Ω defined in (4.38) is connected. It is easily seen that Ω is simply-connected by considering a loop in Ω, projecting it onto ω, and using the assumed simple-connectedness of ω.

(ix) *Let the matrix field $\mathbf{U}\in\mathcal{C}^1(\Omega;\mathbb{S}^2_>)$ be defined by*

$$\mathbf{U}(y,x_3):=(\mathbf{A}(y)-2x_3\mathbf{B}(y)+x_3^2\mathbf{B}^{-1}(y)\mathbf{A}(y)\mathbf{B}(y))^{1/2}\in\mathbb{S}^2_>\text{ for }(y,x_3)\in\Omega_n, n\geq 0. \tag{4.39}$$

and let the matrix field $\mathbf{\Lambda}\in\mathcal{C}^0(\Omega;\mathbb{M}^3)$ be defined in terms of the matrix field

$$\widetilde{\mathbf{U}}:=\begin{pmatrix}\boxed{\mathbf{U}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 1\end{pmatrix}\in\mathcal{C}^1(\Omega;\mathbb{S}^3_>) \tag{4.40}$$

by

$$\mathbf{\Lambda} := \frac{1}{\det \widetilde{\mathbf{U}}} \widetilde{\mathbf{U}} \{ (\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}} - \frac{1}{2} (\mathrm{tr}[(\mathbf{CURL}\, \widetilde{\mathbf{U}})^T \widetilde{\mathbf{U}}]) \mathbf{I} \}. \tag{4.41}$$

Then

$$\mathbf{CURL}\, \mathbf{\Lambda} + \mathbf{COF}\, \mathbf{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3) \tag{4.42}$$

By construction, the restriction to each set $\Omega_n, n \geq 0$, of the matrix field $\mathbf{U}$ defined in (4.39) is continuously differentiable (it is even continuously differentiable on each $\overline{\Omega}_n, n \geq 0$). Hence $\mathbf{U} \in \mathcal{C}^1(\Omega; \mathbb{S}^2_>)$ and thus $\widetilde{\mathbf{U}} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$, where the field $\widetilde{\mathbf{U}}$ is defined in (4.40).

The same argument as that used in part (vii) shows that the restriction of the field $\mathbf{\Lambda} \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ defined in (4.41) to each set $\Omega_n, n \geq 0$, satisfies

$$\mathbf{CURL}\, \mathbf{\Lambda} + \mathbf{COF}\, \mathbf{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega_n; \mathbb{M}^3).$$

Since $\Omega = \bigcup_{n \geq 0} \Omega_n$ (part (viii)), the *principle of localization of distributions* (cf. Chapter 1 in Schwartz [25]) shows that the field $\mathbf{\Lambda}$ satisfies in fact the same relation in $\mathcal{D}'(\Omega; \mathbb{M}^3)$, i.e., relation (4.42) holds.

(x) *Given any two linearly independent vectors $\mathbf{a}^0_\alpha \in \mathbb{R}^3$ that satisfy $\mathbf{a}^0_\alpha \cdot \mathbf{a}^0_\beta = a_{\alpha\beta}(y_0)$, define the matrix*

$$\mathbf{F}_0 := \left(\mathbf{a}^0_1 \,\middle|\, \mathbf{a}^0_2 \,\middle|\, \frac{\mathbf{a}^0_1 \wedge \mathbf{a}^0_2}{|\mathbf{a}^0_1 \wedge \mathbf{a}^0_2|} \right), \tag{4.43}$$

which satisfies

$$\mathbf{F}_0^T \mathbf{F}_0 = \begin{pmatrix} \boxed{\mathbf{A}(y_0)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} = \widetilde{\mathbf{U}}^2(y_0, 0). \tag{4.44}$$

Since the compatibility relation (4.42) is satisfied, Theorem 3.2 shows that *there exists a unique immersion $\mathbf{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ that satisfies*

$$\mathbf{\nabla\Theta}^T \mathbf{\nabla\Theta} = \mathbf{U}^2 \text{ in } \mathcal{C}^1(\Omega; \mathbb{S}^3_>), \tag{4.45}$$

and

$$\mathbf{\Theta}(y_0, 0) = \mathbf{a}_0 \text{ and } \mathbf{\nabla\Theta}(y_0, 0) = \mathbf{F}_0. \tag{4.46}$$

Let the mapping $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbb{R}^3)$ be defined by

$$\boldsymbol{\theta}(y) := \mathbf{\Theta}(y, 0) \text{ for all } y \in \omega. \tag{4.47}$$

Then the mapping $\boldsymbol{\theta}$ is an immersion and it satisfies

(4.48)
$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ in } \mathcal{C}^1(\omega),$$

(4.49)
$$\boldsymbol{\theta}(y_0) = \mathbf{a}_0 \text{ and } \partial_\alpha \boldsymbol{\theta}(y_0) = \mathbf{a}_\alpha^0.$$

Let the matrix field $\mathbf{F} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ be defined by

$$\mathbf{F}(y) = \left(\begin{array}{c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \right)(y) := \boldsymbol{\nabla\Theta}(y,0) \text{ for all } y \in \omega,$$

so that the column vectors $\mathbf{a}_i(y)$ are linearly independent at all $y \in \omega$. Furthermore, definition (4.47) implies that

$$\mathbf{a}_\alpha(y) = \partial_\alpha \boldsymbol{\Theta}(y,0) = \partial_\alpha \boldsymbol{\theta}(y) \text{ for all } y \in \omega,$$

and relations (4.10), (4.13), and (4.45) together imply that

(4.50)
$$(\mathbf{F}^T\mathbf{F})(y) = (\boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta})(y,0) = \widetilde{\mathbf{U}}^2(y,0) = \begin{pmatrix} \boxed{\mathbf{A}(y)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \text{ for all } y \in \omega.$$

Relations (4.48)-(4.49) then immediately follow from the three relations above. That $(a_{\alpha\beta}(y)) \in \mathbb{S}^2_>$ for all $y \in \omega$ shows that the mapping $\boldsymbol{\theta}$ is an immersion.

(xi) *The immersion* $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbb{R}^3)$ *defined in* (4.47) *satisfies*

(4.51)
$$\partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \mathcal{C}^0(\omega).$$

Relation (4.50) also show that $\mathbf{a}_3(y) \cdot \mathbf{a}_i(y) = \delta_{3i}$ for all $y \in \omega$. Consequently,

$$\text{either } \mathbf{a}_3 = \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|} \text{ in } \omega \text{ or } \mathbf{a}_3 = -\frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|} \text{ in } \omega,$$

since $\mathbf{a}_3 \in \mathcal{C}^1(\omega; \mathbb{R}^3)$. But the second alternative is excluded in view of the condition $\mathbf{F}(y_0) = \mathbf{F}_0$, again because $\mathbf{a}_3 \in \mathcal{C}^1(\omega; \mathbb{R}^3)$. We thus have

(4.52)
$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|} \text{ in } \mathcal{C}^1(\omega; \mathbb{R}^3).$$

Let

$$\boldsymbol{\nabla\Theta}(x) = \left(\begin{array}{c|c|c} \mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \end{array} \right)(x) \text{ for all } x \in \Omega,$$

so that

(4.53) $$\mathbf{g}_i(x) = \partial_i \boldsymbol{\Theta}(x) \text{ and } \mathbf{g}_i(x)\cdot\mathbf{g}_j(x) = g_{ij}(x) \text{ for all } x\in\Omega,$$

where the components $g_{ij}\in\mathcal{C}^1(\Omega)$ of the matrix field $\mathbf{U}^2\in\mathcal{C}^1(\Omega;\mathbb{S}^3_>)$ are given by (4.14).

It is well known that

$$\partial_i\mathbf{g}_j = \Gamma^p_{ij}\mathbf{g}_p, \text{ where } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}, (g^{pq}) = (g_{ij})^{-1}, \text{ and } \Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}),$$

as a consequence of relations (4.53). But, in addition, $g_{i3}=\delta_{i3}$ by (4.14); hence $\Gamma^p_{33} = g^{pq}\Gamma_{33q} = 0$ in the present case. Consequently,

$$\partial_{33}\boldsymbol{\Theta} = \partial_3\mathbf{g}_3 = \Gamma^p_{33}\mathbf{g}_p = \mathbf{0} \text{ in } \Omega.$$

There thus exists a vector field $\boldsymbol{\theta}^1\in\mathcal{C}^2(\omega;\mathbb{R}^3)$ such that

$$\boldsymbol{\Theta}(y,x_3) = \boldsymbol{\theta}(y) + x_3\boldsymbol{\theta}^1(y) \text{ for all } (y,x_3)\in\Omega.$$

Since $\mathbf{a}_3(y) = \partial_3\boldsymbol{\Theta}(y,0)$ by definition of the vector field $\mathbf{a}_3$, it follows that the vector field $\boldsymbol{\Theta}\in\mathcal{C}^2(\Omega;\mathbb{R}^3)$ is of the form

(4.54) $$\boldsymbol{\Theta} = \boldsymbol{\theta} + x_3\mathbf{a}_3 \text{ with } \mathbf{a}_3 = \frac{\mathbf{a}_1\wedge\mathbf{a}_2}{|\mathbf{a}_1\wedge\mathbf{a}_2|}.$$

Nothing that $\partial_\alpha\boldsymbol{\theta}\cdot\mathbf{a}_3 = 0$ implies $\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\mathbf{a}_3 = -\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\mathbf{a}_3$, we deduce from (4.54) that

$$\partial_\alpha\boldsymbol{\Theta}\cdot\partial_\beta\boldsymbol{\Theta} = \partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta} - 2x_3\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\mathbf{a}_3 + x_3^2\partial_\alpha\mathbf{a}_3\cdot\partial_\beta\mathbf{a}_3 \text{ in } \Omega,$$

on the one hand. On the other hand, we know that by (4.14) and (4.45),

$$\partial_\alpha\boldsymbol{\Theta}\cdot\partial_\beta\boldsymbol{\Theta} = g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 a^{\sigma\tau}b_{\alpha\sigma}b_{\beta\tau} \text{ in } \Omega.$$

Hence

$$b_{\alpha\beta} = (\partial_{\alpha\beta}\boldsymbol{\theta})\cdot\mathbf{a}_3 = \partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\mathbf{a}_1\wedge\mathbf{a}_2}{|\mathbf{a}_1\wedge\mathbf{a}_2|} \text{ in } \omega,$$

as announced in (4.51). □

Remark 4.1. As vectors $\mathbf{a}^0_\alpha$ in condition (4.6), one may choose the first and second column vectors of the square root of the matrix $\begin{pmatrix} \boxed{\mathbf{A}(y_0)} & \begin{matrix}0\\0\end{matrix} \\ 0 \quad 0 & 1\end{pmatrix}\in\mathbb{S}^3_>$. □

Remark 4.2. Naturally, if no condition such as (4.6) are imposed, the immersion $\boldsymbol{\theta}$ found in Theorem 4.1 is unique only up to proper rigid displacements. This means that any other solution $\widetilde{\boldsymbol{\theta}}\in\mathcal{C}^2(\omega;\mathbb{R}^3)$ of equations (4.5) is necessarily of the form

$$\widetilde{\boldsymbol{\theta}}(y) = \mathbf{a} + \mathbf{Q}\boldsymbol{\theta}(y) \text{ for all } y\in\omega, \text{ for some vector } \mathbf{a}\in\mathbb{R}^3 \text{ and matrix } \mathbf{Q}\in\mathbb{O}^3_+.$$

For a proof, see, e.g., [6,Theorem 2.9-1]. □

Remark 4.3. Links between the "three-dimensional" Shield-Vallée compatibility relation (3.5) and the "two-dimensional" Darboux-Vallée-Fortuné compatibility relation (4.1) have played crucial role in the proof of Theorem 4.1. Other links between these relations have been already discussed by Vallée & Fortuné [30], albeit in a different context. □

Thanks to deep global existence theorems for Pfaff systems with little regularity recently obtained by S. Mardare [18], the existence result of Theorem 3.2 can be extended to cover the situation where the given field $\mathbf{C}$ is only in the space $W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^3_>)$, in which case the resulting immersion $\mathbf{\Theta}$ is only in the space $W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^3)$.

Using this extension of Theorem 3.2, we can likewise prove the following extension of Theorem 4.1, the proof of which is essentially the same; only some additional case must be taken to justify all the computations involved (suffice it to say here that a key use is made of the facts that the point values $f(x)$ of an equivalence class f in $L^\infty_{\rm loc}(\Omega)$ can be unambiguously defined at each point $x \in \Omega$ and that an equivalence class in the space $W^{1,\infty}_{\rm loc}(\Omega)$ can be identified with a function in the space $\mathcal{C}^0(\Omega)$).

Theorem 4.2. *Let ω be simply-connected open subset of $\mathbb{R}^2$ and let $\mathbf{A} = (a_{\alpha\beta}) \in W^{1,\infty}_{\rm loc}(\omega;\mathbb{S}^2_>)$ and $\mathbf{B} = (b_{\alpha\beta}) \in W^{1,\infty}_{\rm loc}(\omega;\mathbb{S}^2)$ be two matrix fields that satisfy the compatibility relation*

$$\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3),$$

where the components $\lambda_{\alpha\beta} \in W^{1,\infty}_{\rm loc}(\omega)$ and $\lambda_{3\beta} \in L^\infty_{\rm loc}(\omega)$ of the vector fields $\boldsymbol{\lambda}_1 := (\lambda_{i1}) : \omega \to \mathbb{R}^3$ and $\boldsymbol{\lambda}_2 := (\lambda_{i2}) : \omega \to \mathbb{R}^3$ are defined in terms of the matrix fields $\mathbf{A}$ and $\mathbf{B}$ by the matrix equations

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} := -\mathbf{J}\mathbf{A}^{-1/2}\mathbf{B},$$

$$\begin{pmatrix} \lambda_{31}; & \lambda_{32} \end{pmatrix} := \begin{pmatrix} \partial_2 u^0_{11} - \partial_1 u^0_{12}; & \partial_2 u^0_{21} - \partial_1 u^0_{22} \end{pmatrix} \mathbf{J}\mathbf{A}^{-1/2}\mathbf{B},$$

where

$$\mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } (u^0_{\alpha\beta}) := \mathbf{A}^{-1/2} \in W^{1,\infty}_{\rm loc}(\omega;\mathbb{S}^2_>).$$

Then there exists an immersion $\boldsymbol{\theta} \in W^{2,\infty}_{\rm loc}(\omega;\mathbb{R}^3)$ such that

$$\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta} = a_{\alpha\beta} \text{ in } W^{1,\infty}_{\rm loc}(\omega) \text{ and } \partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|} = (b_{\alpha\beta}) \text{ in } L^\infty_{\rm loc}(\omega).$$

□

Remark 4.4. More recently, S. Mardare [18] has further extended the fundamental theorem of surface theory (in its "classical" formulation) so as to cover the case where the given fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are only in the spaces $W^{1,p}_{\rm loc}(\omega)$ and $L^p_{\rm loc}(\omega)$ for some $p > 2$, with a resulting immersion $\boldsymbol{\theta}$ in the space $W^{2,p}_{\rm loc}(\omega)$. It is thus likely that Theorem 4.2 can be likewise extended, this time using another weakening of the regularity assumptions for Pfaff systems, again due to S. Mardare [19]. □

5. Necessity of the Darboux-Vallée-Fortuné compatibility relation

As recalled in the introduction, it is by exploiting an idea of Darboux [12] that Vallée & Fortuné [29] have shown that the two fundamental forms of a surface $\boldsymbol{\theta}(\omega)$ associated with a given immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ necessarily satisfy the compatibility relation $\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2$ in ω, where the two vector fields $\boldsymbol{\lambda}_\beta : \omega \to \mathbb{R}^3$ are defined as in (4.2)−(4.4) in terms of the two fundamental forms.

By contrast with the *Gauss* and *Codazzi-Mainardi* relations (whose necessity is easy to establish from the knowledge of an immersion), establishing the necessity of the *Darboux-Vallée-Fortuné relation* $\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2$ through a direct computation turns out to be substantially less easy, however.

We propose here a new proof of the necessity of the Darboux-Vallée-Fortuné relation, based on computations similar to those used in the proof of Theorem 4.1. For this reason, the proof is only sketched.

Theorem 5.1. *Let ω be an open subset of $\mathbb{R}^2$ and let there be given an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$. Let the two vector fields $\boldsymbol{\lambda}_\beta = (\lambda_{i\beta}) \in \mathcal{C}^1(\omega;\mathbb{R}^3)$ be defined in terms of the immersion $\boldsymbol{\theta}$ by*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} := -\mathbf{J}\ \mathbf{A}^{-1/2}\mathbf{B}, \text{ where } \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.1}$$

$$(\lambda_{31};\quad \lambda_{32}) := (\partial_2 u^0_{11} - \partial_1 u^0_{12};\quad \partial_2 u^0_{21} - \partial_1 u^0_{22})\mathbf{J}\ \mathbf{A}^{-1/2}\mathbf{J}, \quad \text{where}\,(u^0_{\alpha\beta}) := \mathbf{A}^{1/2}, \tag{5.2}$$

where

$$\mathbf{A} = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>) \text{ with } a_{\alpha\beta} := \partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta}, \tag{5.3}$$

$$\mathbf{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2) \text{ with } b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}. \tag{5.4}$$

Then the two vector fields $\boldsymbol{\lambda}_\beta$ necessarily satisfy

$$\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1\wedge\boldsymbol{\lambda}_2 \text{ in } \mathcal{C}^0(\omega;\mathbb{R}^3). \tag{5.5}$$

Proof. As already noted, parts (i) to (iii) of the proof of Theorem 4.1 hold *verbation* for *any* matrix fields $\mathbf{A} \in \mathcal{C}^1(\omega;\mathbb{S}^2_>)$ and $\mathbf{B} \in \mathcal{C}^1(\omega;\mathbb{S}^2)$. Using the particular matrix fields $\mathbf{A}$ and $\mathbf{B}$ of (5.3)$-$(5.4) and defining the connected set Ω as in (4.38), we may thus define a matrix field $\widetilde{\mathbf{U}} \in \mathcal{C}^1(\Omega;\mathbb{S}^3_>)$ as in (4.40). Then, by construction,

$$\widetilde{\mathbf{U}}^2 = (g_{ij}) \text{ with } g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau} \text{ and } g_{i3} = \delta_{i3},$$

i.e., the matrix field $\widetilde{\mathbf{U}}^2 \in \mathcal{C}^1(\Omega;\mathbb{S}^3_>)$ is nothing but the *metric tensor associated with the canonical extension* $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega;\mathbb{R}^3)$ *of the immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$, defined as usual by

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3\mathbf{a}_3(y) \text{ for all } (y, x_3) \in \Omega.$$

By Theorem 6.4 of Ciarlet, Gratie, Iosifescu, Mardare & Vallée [9], the field $\boldsymbol{\Lambda} = (\Lambda_{ij}) \in \mathcal{C}^0(\Omega;\mathbb{M}^3)$ defined as in (4.41) therefore *necessarily* satisfies the Shield-Vallée compatibility relation

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^3).$$

By part (ii) of the proof of Theorem 4.1, the matrix field $\mathbf{U} = (u_{\alpha\beta}) \in \mathcal{C}^1(\Omega;\mathbb{S}^2_>)$ used in the definition (4.40) of the matrix field $\widetilde{\mathrm{U}}$ can be also written as

$$\mathbf{U} = \mathbf{Q}^T(\mathbf{A}^{1/2} - x_3\mathbf{A}^{-1/2}\mathbf{B}),$$

where

$$\mathbf{Q} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad\text{and}\quad \varphi = \arctan\Big(\frac{x_3\ \mathrm{tr}\ (\mathbf{B}\ \mathbf{J}\ \mathbf{A}^{-1/2})}{\mathrm{tr}\ \mathbf{A}^{1/2} - x_3\ \mathrm{tr}\ (\mathbf{B}\ \mathbf{A}^{-1/2})}\Big).$$

Besides, by part (iii) of the same proof,

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} = \mathbf{J}^T \mathbf{Q}^T \mathbf{A}^{-1/2} \mathbf{B},$$
$$(\Lambda_{31}; \quad \Lambda_{32}) = (\partial_1 u_{12} - \partial_2 u_{11}; \partial_1 u_{22} - \partial_2 u_{21}) \mathbf{J}^T \mathbf{U}^{-1} \mathbf{J}.$$

Computations similar to those used in part (iv) of the same proof then show that the above row vector field is also given by

$$(\Lambda_{31}; \quad \Lambda_{32}) = (\lambda_{31} + \partial_1 \varphi; \quad \lambda_{32} + \partial_2 \varphi),$$

where *the vector field* $(\lambda_{31}; \quad \lambda_{32})$ *is precisely of the form* (5.2). Computations similar to those used in parts (v) to (vii) of the same proof further show that the equality of the third column vector fields in the relation $\mathbf{CURL}\ \mathbf{\Lambda} + \mathbf{COF}\ \mathbf{\Lambda} = \mathbf{0}$ reduces to

$$\partial_2 \mathbf{\Lambda}_1 - \partial_1 \mathbf{\Lambda}_2 = \mathbf{\Lambda}_1 \wedge \mathbf{\Lambda}_2,$$

where $\partial_2 \mathbf{\Lambda}_1 - \partial_1 \mathbf{\Lambda}_2$ and $\mathbf{\Lambda}_1 \wedge \mathbf{\Lambda}_2$ are respectively defined as in (4.33) and (4.34) and *the matrix field* $\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$ *is precisely of the form* (5.1). It then suffices to observe that for $x_3 = 0$, the relation $\partial_2 \mathbf{\Lambda}_1 - \partial_1 \mathbf{\Lambda}_2 = \mathbf{\Lambda}_1 \wedge \mathbf{\Lambda}_2$ reduces to $\partial_2 \boldsymbol{\lambda}_1 - \partial_1 \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2$. □

Theorem 5.1 in turn provides a simple way to prove the *equivalence between the Gauss and Codazzi-Mainardi relations and the Darboux-Vallée-Fortuné relation.*

Theorem 5.2. *Let* ω *be an open subset of* $\mathbb{R}^3$. *Then two matrix fields* $\mathbf{A} = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $\mathbf{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ *satisfy the Darboux-Vallée-Fortuné compatibility relation* (5.5), *where the vector fields* $\boldsymbol{\lambda}_\alpha \in \mathcal{C}^1(\omega; \mathbb{R}^3)$ *are defined in terms of the matrix fields* $\mathbf{A}$ *and* $\mathbf{B}$ *as in* (5.1)−(5.2) *if and only if they satisfy the Gauss and Codazzi-Mainardi equation* (1.2)−(1.3) *in* $\mathcal{D}'(\omega)$, *where the functions* $C_{\alpha\beta\tau}$ *and* $C^\sigma_{\alpha\beta}$ *are defined in terms of the functions* $a_{\alpha\beta}$ *and* $b_{\alpha\beta}$ *as in* (1.1).

Proof. Since the equivalence between the two sets of compatibility relations is a "local" property, the principle of localization of distributions (cf. Schwartz [25]) implies that the set ω may be assumed to be simply-connected without loss of generality.

This being the case, assume that two matrix fields $\mathbf{A} \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $\mathbf{B} \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ satisfy the Darboux-Vallée-Fortuné relations (5.5), where the two vector fields $\boldsymbol{\lambda}_\beta \in \mathcal{C}^1(\omega; \mathbb{R}^3)$ are defined as in (5.1)−(5.2). Then, by Theorem 4.1, there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ that satisfies

$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \omega. \tag{5.6}$$

and thus the functions $C_{\alpha\beta\tau}$ and $C^\sigma_{\alpha\beta}$ defined as in (1.1) necessarily satisfy the Gauss and Codazzi-Mainardi equations (1.2)−(1.3).

Assume conversely that two matrix fields $\mathbf{A} \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $\mathbf{B} \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ satisfy the Gauss and Codazzi-Mainardi relations (1.2)−(1.3) with the functions $C_{\alpha\beta\tau}$ and $C^\sigma_{\alpha\beta}$ defined as in (1.1). Then, by the fundamental theorem of surface theory, there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ that satisfies (5.6), and thus the vector fields $\boldsymbol{\lambda}_\beta$ defined by (5.1)−(5.4) satisfy (5.5) by Theorem 5.1. □

Naturally, yet another way to establish the necessity of the Darboux-Vallée-Fortuné relation consists in *directly* showing that they are equivalent to the Gauss and Codazzi-Mainardi equations, but this approach requires somewhat lengthy and delicate computations; cf. Ciarlet, Fortuné, Gratie & Vallée [7].

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References

[1] S. S. Antman [1995], Nonlinear Problems of Elasticity, Springer-Verlag, Berlin.

[2] Y. Basar, A consistent theory of geometrically non-linear shells with an independent rotation vector, Internat. J. Solids Structures 23 (1987), 1401-1445.

[3] M. Bernadou, P. G. Ciarlet, B. Miara, Existence theorems for two-dimensional linear shell theories, J. Elasticity 34 (1994), 111-138.

[4] W. Bielski, J. J. Telega, On existence of solutions for geometrically nonlinear shells and plates, Z. angew. Math. Mech. 68 (2008), 155-157.

[5] P. G. Ciarlet, Mathematical Elasticity, Volume I : Three-Dimensional Elasticity, North-Holland, Amsterdam 1988.

[6] P. G. Ciarlet, An Introduction to Differential Geometry with Applications to Elasticity, Springer, Heidelberg, 2005.

[7] P. G. Ciarlet, D. Fortuné, L. Gratie, C. Vallée, Equivalence between the vanishing of the Riemann curvature tensor in $\mathbb{R}^3$ and the Gauss and Codazzi-Mainardi equations, in preparation.

[8] P. G. Ciarlet, L. Gratie, C. Mardare: A new approach to the fundamental theorem of surface theory, Arch. Rational Mech. Anal. 188 (2008), 457-474.

[9] P. G. Ciarlet, L. Gratie, O. Iosifescu, C. Mardare, C. Vallée, Another approach to the fundamental theorem of Riemannian geometry in $\mathbb{R}^3$, by way of rotation fields, J. Math. Pures Appl. 87 (2007), 237-252.

[10] P. G. Ciarlet, O. Iosifescu, Justification of the Darboux-Vallée-Fortuné compatibility relation in the theory of surfaces, C. R. Acad. Sci. Paris (to appear).

[11] P. G. Ciarlet, F. Larsonneur, On the recovery of a surface with prescribed first and second fundamental forms, J. Math. Pures Appl. 81 (2002), 167-185.

[12] G. Darboux, Leçons sur la Théorie Générale des Surfaces et les Applications Géométriques du Calcul Infinitésimal, Volumes 1-4, Gauthier-Villars, Paris, 1894-1915 (re-published in 2000 by the American Mathematical Society, Providence).

[13] A. Galka, J. J. Telega, The complementary energy principle as a dual problem for a specific model of geometrically non-linear elastic shells with an independent rotation vector: general results, Eur. J. Mech. A-Solids 11 (1992), 245-270.

[14] C. Grandmont, Y. Maday, P. Métier, Modeling and analysis of an elastic problem with large displacements and small strains, J. Elasticity 87 (2007), 29-72.

[15] M. E. Gurtin, An Introduction to Continuum Mechanics, Academic Press, New York, 1981.

[16] C. Mardare, On the recovery of a manifold with prescribed metric tensor, Analysis and Applications 1 (2003), 433-453.

[17] S. Mardare, The fundamental theorem of surface theory for surfaces with little regularity, J. Elasticity 73 (2003), 251-290.

[18] S. Mardare, On Pfaff systems with L^p coefficients and their applications in differential geometry, J. Math. Pures Appl. 84 (2005), 1659-1692.

[19] S. Mardare, On systems of first order linear partial differential equations with L^p coefficients, Adv. Differential Equation 73 (2007), 301-360.

[20] W. Pietraszkiewicz, Finite Rotations and Lagrangean Description in the Non-Linear Theory of Shells, Polish Scientific Publishers (1979), Warszawa.

[21] W. Pietraszkiewicz, M. L. Szwabowicz, Determination of the midsurface of a deformed shell from prescribed fields of surface strains and bendings, Internat. J. Solids Structures 44 (2007), 6163-6172.

[22] W. Pietraszkiewicz, M. L. Szwabowicz, C. Vallée, Determination of the midsurface of a deformed shell from prescribed surface strains and bendings via the polar decomposition, Internat. J. Non-Linear Mech. (2008), to appear.

[23] W. Pietraszkiewicz, C. Vallée, A method of shell theory in determination of the surface from components of its two fundamental forms, Z. Angew. Math. Mech. 87 (2007), 603-615.

[24] R. T. Shield, The rotation associated with large strains, SIAM J. Appl. Math. 25 (1973), 483-491.

[25] L. Schwartz, Théorie des Distributions, Hermann, Paris, 1966.

[26] J. G. Simmonds, D. A. Danielson, Nonlinear shell theory with finite rotation and stress function vectors, J. Appl. Mech. (1972), 1085-1090.

[27] R. Valid, The principle of complementary energy in nonlinear shell theory, C. R. Acad. Sci. Paris B289 (1979), 293-296.

[28] C. Vallée, Compatibility equations for large deformations, Internat. J. Engrg. Sci. 30 (1992), 1753-1757.

[29] C. Vallée, D. Fortuné, Compatibility equations in shell theory, Internat. J. Engrg. Sci. 34 (1996), 495-499.

[30] C. Vallée, D. Fortuné, Link between compatibility conditions in nonlinear shell theory and 3d compatibility conditions, Arch. Civil Engrg XLV (1999), 369-374.