A Cesàro-Volterra formula with little regularity

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Abstract

If a symmetric matrix field $\boldsymbol{e}$ of order three satisfies the Saint-Venant compatibility conditions in a simply-connected domain Ω in $\mathbb{R}^3$, there then exists a displacement field $\boldsymbol{u}$ of Ω with $\boldsymbol{e}$ as its associated linearized strain tensor, i.e., $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla u}^T + \boldsymbol{\nabla u})$ in Ω. A classical result, due to Cesàro and Volterra, asserts that, if the field $\boldsymbol{e}$ is sufficiently smooth, the displacement $\boldsymbol{u}(x)$ at any point $x \in \Omega$ can be explicitly computed as a function of the matrix fields $\boldsymbol{e}$ and $\mathbf{CURL}\,\boldsymbol{e}$, by means of a path integral inside Ω with endpoint x.

We assume here that the components of the field $\boldsymbol{e}$ are only in $L^2(\Omega)$ (as in the classical variational formulation of three-dimensional linearized elasticity), in which case the classical path integral formula of Cesàro and Volterra becomes meaningless. We then establish the existence of a "Cesàro-Volterra formula with little regularity", which again provides an explicit solution $\boldsymbol{u}$ to the equation $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla u}^T + \boldsymbol{\nabla u})$ in this case. We also show how the classical Cesàro-Volterra formula can be recovered from the formula with little regularity when the field $\boldsymbol{e}$ is smooth. Interestingly, our analysis also provides as a by-product a variational problem that satisfies all the assumptions of the Lax-Milgram lemma, and whose solution is precisely the unknown displacement field $\boldsymbol{u}$.

It is also shown how such results may be used in the mathematical analysis of "intrinsic" linearized elasticity, where the linearized strain tensor $\boldsymbol{e}$ (instead of the displacement vector $\boldsymbol{u}$ as is customary) is regarded as the primary unknown.

Résumé

Une formule de Cesàro-Volterra avec peu de régularité. Si un champ $\boldsymbol{e}$ de matrices symétriques d'ordre trois vérifie les conditions de compatibilité de Saint-Venant dans un ouvert Ω simplement connexe de $\mathbb{R}^3$, alors il existe un champ de déplacements $\boldsymbol{u}$ de Ω ayant $\boldsymbol{e}$ comme tenseur linéarisé des déformations associé, i.e., $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla u}^T + \boldsymbol{\nabla u})$ dans Ω. Un résultat classique de Cesàro et Volterra affirme que, si le champ $\boldsymbol{e}$ est suffisamment régulier, le déplacement $\boldsymbol{u}(x)$ en chaque point $x \in \Omega$ peut être calculé explicitement en fonction des champs de matrices $\boldsymbol{e}$ et $\mathbf{CURL}\,\boldsymbol{e}$, au moyen d'une intégrale curviligne dans Ω ayant x comme extrémité.

On suppose ici que les composantes du champ $\boldsymbol{e}$ sont seulement dans $L^2(\Omega)$ (comme dans la formulation variationnelle classique de l'élasticité linéarisée tri-dimensionnelle), auquel cas la formule classique de Cesàro-Volterra n'a plus de sens. On établit alors une "formule de Cesàro-Volterra avec peu de régularité", qui donne à nouveau une solution explicite $\boldsymbol{u}$ de l'équation $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla u}^T + \boldsymbol{\nabla u})$ dans ce cas. On montre aussi comment la formule classique de Cesàro-Volterra peut être retrouvée à partir de la formule “avec peu de régularité” lorsque le champ $\boldsymbol{e}$ est régulier. Il est intéressant de noter que l'un des corollaires de notre analyse est la formulation d'un problème variationnel qui vérifie toutes les hypothèses du lemme de Lax-Milgram, et dont la solution est précisément le champ $\boldsymbol{u}$.

On montre également comment de tels résultats peuvent être utilisés dans l'analyse mathématique de l'élasticité linéarisée “intrinsèque”, où le tenseur linéarisé des déformations $\boldsymbol{e}$ (au lieu du champ de déplacements comme il est usuel) est considéré comme étant l'inconnue principale.

Keywords : Saint-Venant compatibility equations ; Poincaré's lemma ; Cesàro-Volterra formula, three-dimensional linearized elasticity.

1. Introduction

For simplicity, we consider only the three-dimensional case in this introduction. But the results presented here can be extended to, and are subsequently established in, the n-dimensional case for any $n \geq 2$.

Latin indices range in the set $\{1, 2, 3\}$ and the summation convention with respect to repeated Latin indices is used in conjunction with this rule. The sets of all real matrices of order three and of all real symmetric matrices of order three are respectively denoted $\mathbb{M}^3$ and $\mathbb{S}^3$. Other notations used, but not defined, in this introduction are defined in the next section.

Let Ω be an open subset of $\mathbb{R}^3$. Given a vector field $\boldsymbol{u} = (u_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$, let the associated *linearized strain tensor field* $\boldsymbol{e} = (e_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3)$ be defined by

$$e_{ij} := \frac{1}{2}(\partial_j u_i + \partial_i u_j) \text{ in } \Omega. \tag{1}$$

It is then immediately verified that the components e_{ij} defined in (1.1) *necessarily* satisfy the following *compatibility conditions*, which were discovered and analyzed by Saint-Venant [17] in 1864, and since then bear his name:

$$\partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } \mathcal{C}^0(\Omega). \tag{2}$$

It is well known that, if Ω is *simply-connected*, these compatibility conditions become also *sufficient*. This means that, if a matrix field $\boldsymbol{e} = (e_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3)$ satisfies the *Saint-Venant compatibility conditons* (1.2) in such an open set Ω, then conversely, there exists a vector field $\boldsymbol{u} = (u_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ that satisfies the equations

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \text{ in } \Omega. \tag{3}$$

Besides, all other solutions $\widetilde{\boldsymbol{u}} = (\widetilde{u}_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ to the equations $\frac{1}{2}(\partial_j \widetilde{u}_i + \partial_i \widetilde{u}_j) = e_{ij}$ in Ω are of the form

$$\widetilde{\boldsymbol{u}}(x) = \boldsymbol{u}(x) + \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{ox}, x \in \Omega, \text{ for some } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3. \tag{4}$$

It is less known (Ref. [15] constitutes an exception) that an *explicit solution* $\boldsymbol{u} = (u_i)$ *to the equations* (1.3) can be given in the form of the following *Cesàro-Volterra path integral formula*, so named after Cesàro [5] and Volterra [18], who discovered it in 1906 and 1907: Let $\gamma(x)$ be any path of class $\mathcal{C}^1$ contained in Ω and joining a point $x_0 \in \Omega$ (considered as fixed) to any point $x \in \Omega$. Then

$$u_i(x) = \int_{\gamma(x)} \{e_{ij}(y) + (\partial_k e_{ij}(y) - \partial_i e_{kj}(y))(x_k - y_k)\} dy_j, \ x \in \Omega. \tag{5}$$

It can then be verified that each component $u_i(x)$ computed by formula (1.5) is independent of the path chosen for joining x_0 to x (as it should be), precisely because the functions e_{ij} satisfy the compatibility conditions (1.2).

The Cesàro-Volterra path integral formula (1.5) can be equivalently rewritten in *vector-matrix form*, as

$$\boldsymbol{u}(x) = \int_{\gamma(x)} \boldsymbol{e}(y) \boldsymbol{dy} + \int_{\gamma(x)} \boldsymbol{yx} \wedge ([\mathbf{CURL}\, \boldsymbol{e}(y)] \boldsymbol{dy}), \ x \in \Omega, \tag{6}$$

where $\wedge$ designates the vector product in $\mathbb{R}^3$, and the matrix curl operator $\mathbf{CURL}\ : \mathcal{D}'(\Omega; \mathbb{M}^3) \to \mathcal{D}'(\Omega; \mathbb{M}^3)$ is defined by

$$(\mathbf{CURL}\, \boldsymbol{e})_{ij} := \varepsilon_{ilk} \partial_l e_{jk} \text{ for any matrix field } \boldsymbol{e} = (e_{ij}) \in \mathcal{D}'(\Omega; \mathbb{M}^3), \tag{7}$$

where (ε_{ilk}) denotes the orientation tensor.

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The sufficiency of the Saint-Venant compatibility conditons (1.2) was recently shown to hold under substantially *weaker regularity assumptions on the given tensor field* $\boldsymbol{e} = (e_{ij})$, according to the following result, due to Ciarlet & Ciarlet, Jr. [6]: *Let* Ω *be a bounded and simply-connected open subset of* $\mathbb{R}^3$ *with a Lipschitz-continuous boundary, and let there be given functions* $e_{ij} = e_{ji} \in L^2(\Omega)$ *that satisfy*

$$\partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} = 0 \text{ in } H^{-2}(\Omega). \tag{8}$$

Then there exists a vector field $(u_i) \in H^1(\Omega;\mathbb{R}^3)$ *that satisfies*

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \text{ in } L^2(\Omega). \tag{9}$$

Besides, all the other solutions $\widetilde{\boldsymbol{u}} = (\widetilde{u}_i) \in H^1(\Omega;\mathbb{R}^3)$ to the equations $\frac{1}{2}(\partial_j\widetilde{u}_i + \partial_i\widetilde{u}_j) = e_{ij}$ are again of the form (1.4).

Clearly, the "classical" Cesàro-Volterra path integral formula (1.5) becomes meaningless when the functions e_{ij} satisfying (1.8) are only in the space $L^2(\Omega)$. The question then naturally arises as to whether there exists any "*Cesàro-Volterra formula with little regularity*", which (i) would again provide an explicit solution to the equations (1.9) when the functions e_{ij} are only in $L^2(\Omega)$ and (ii) would in some way resemble (1.5).

One of our objectives is to provide the following positive answer to this question (thus justifying the title of this paper). Let $< \cdot\, , \, \cdot >$ denote the duality pairing between a topological space and its dual, and let

$$\boldsymbol{T} = (T_i) : L^2_0(\Omega) := \{v \in L^2(\Omega); \int_\Omega v dx = 0\} \to H^1_0(\Omega;\mathbb{R}^3) \tag{10}$$

be a specific continuous linear operator that satisfies (the precise definition of $\boldsymbol{T}$ is given in Lemma 2.5)

$$-\text{div}(\boldsymbol{T}v) = v \text{ for all } v \in L^2_0(\Omega). \tag{11}$$

Note that *the operator* $\boldsymbol{T}$ *of* (1.10)−(1.11) *plays a key role throughout this paper.*

We then show (cf. Theorem 4.2) that *a vector field* $\boldsymbol{u} = (u_i) \in H^1(\Omega;\mathbb{R}^3)$ *satisfies equations* (1.9) *if and only if*

$$< u_i, \varphi_i > = < e_{ij}, T_i\varphi_j + \partial_k[T_i(T_j\varphi_k - T_k\varphi_j)] > \tag{12}$$

for all vector field fields $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^3)$ *that satify*

$$\int_\Omega \varphi_i dx = \int_\Omega (x_j\varphi_i - x_i\varphi_j) dx = 0. \tag{13}$$

In other words, we are able to compute all the "components"

$$< \boldsymbol{u}, \boldsymbol{\varphi} > := < u_i, \varphi_i >$$

of the solution $\boldsymbol{u} = (u_i)$ against all vector fields $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^3)$ that satisfy (1.13). Note in passing that it is no surprise that conditions (1.13) should be satisfied: They simply reflect (cf. Lemma 2.3) that the solution to the equations (1.9) is defined only up to *infinitesimal rigid displacements*, i.e., vector fields in $\mathcal{D}'(\Omega;\mathbb{R}^3)$ of the form (cf.(1.4))

$$x \in \Omega \to \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{ox}, \text{ for some vectors } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3. \tag{14}$$

As a consequence, *the knowledge of the duality pairings* $< \boldsymbol{u}, \boldsymbol{\varphi} >$ *for all fields* $\boldsymbol{\varphi} \in \mathcal{D}(\Omega;\mathbb{R}^3)$ *satisfying* (1.13) *uniquely defines a vector field* $\boldsymbol{u} = (u_i) \in \mathcal{D}'(\Omega;\mathbb{R}^3)$ *up to infinitesimal rigid displacements.*

Our claim that formula (1.12) is indeed a *bona fide* generalization of the "classical" Cesàro-Volterra formula (1.5) rests on two justifications.

First, we show that formula (1.12) can be rewritten in the following *vector-matrix form*:

$$< \boldsymbol{u}, \boldsymbol{\varphi} > = \ll \boldsymbol{e}, \boldsymbol{T} \otimes \boldsymbol{\varphi} \gg + \ll \textbf{CURL}\, \boldsymbol{e}, \boldsymbol{T} \otimes (\boldsymbol{T} \wedge \boldsymbol{\varphi}) \gg \tag{15}$$

(cf. Theorem 5.1; the notations used in (1.15) are explained at the beginning of Section 5), which clearly displays a strong, *albeit* formal, resemblance with the vector-matrix form (1.6) of the classical Cesàro-Volterra formula.

Second, and surely more convincingly, we show (cf. Theorem 5.2) that, if the functions e_{ij} happen to be in the space $\mathcal{C}^2(\Omega)$ (as in the "classical" Saint-Venant conditons (1.2)), the classical Cesàro-Volterra path integral formula (1.5) can be indeed recovered from formulas (1.12).

The proof of the equivalence between equations (1.9) and (1.12) given in Theorem 4.2 crucially relies on the following *Poincaré lemma with little regularity* (due to Ciarlet & Ciarlet, Jr. [6]; see also Remark 3.1 for various recent extensions): Let Ω be a bounded and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, and let $f_i \in H^{-1}(\Omega)$ be distributions that satisfy

$$\partial_i f_j - \partial_j f_i = 0 \text{ in } H^{-2}(\Omega). \tag{16}$$

Then there exists a function $u \in L^2(\Omega)$, unique up to an additive constant, that satisfies

$$\partial_i u = f_i \text{ in } H^{-1}(\Omega). \tag{17}$$

We then prove (cf. Theorem 3.2) the following complement to the above Poincaré lemma, which may be also of interest by itself: Given distributions $f_i \in H^{-1}(\Omega)$ that satisfy (1.16), a function $u \in L^2(\Omega)$ satisfies (1.17) if and only if

$$< u, \varphi > = < f_i, T_i\varphi > \text{ for all } \varphi \in \mathcal{D}(\Omega) \text{ that satisfy } \int_\Omega \varphi dx = 0, \tag{18}$$

where $\boldsymbol{T} = (T_i)$ is again the mapping of (1.10)$-$(1.11).

Formula (1.18) thus provides a means to compute a solution to equations (1.17) in the same manner that formula (1.12) provides a means to compute a solution to equations (1.9), in both cases when the data have too little regularity for a path integral formula to make sense.

Incidentally, a noticeable feature of our analysis is that it provides, as a by-product, a way to find either the solution of equations (1.9), or the solution of equations (1.17), in each case as the solution of a *variational problem*, which satisfies all the assumptions of the Lax-Milgram lemma (cf. Theorems 3.3 and 4.3).

One of our main motivations here is to provide another building stone for the mathematical analysis of *intrinsic linearized three-dimensional elasticity*, as begun in Ref. [6] (see Ref. [9] for a general survey of intrinsic methods in elasticity). It was shown there that the *pure traction problem* (to fix ideas) *of linearized three-dimensional elasticity* could be reformulated in a novel way, where *the linearized strain tensor* $\boldsymbol{e} \in L^2(\Omega;\mathbb{S}^3)$ *is regarded as the primary unknown*, instead of the displacement field $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3)$ as is customary.

More specifically, define the space

$$\boldsymbol{E}(\Omega) := \{\boldsymbol{e} = (e_{ij}) \in L^2(\Omega;\mathbb{S}^3);\ \partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} = 0 \text{ in } H^{-2}(\Omega)\},$$

and let

$$\boldsymbol{R}(\Omega) := \{\boldsymbol{r} \in H^1(\Omega;\mathbb{R}^3);\ \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{ox},\ x \in \Omega, \text{ for some } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$$

denote the space of all infinitesimal rigid displacements of the set Ω. Then (cf. Theorem 4.1 in *ibid.*) the mapping

$$\boldsymbol{\mathcal{F}} : \boldsymbol{e} = (e_{ij}) \in \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega), \tag{19}$$

where $\dot{\boldsymbol{v}}$ denotes the equivalence class of any vector field $\boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^3)$ that satisfies $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ in $L^2(\Omega)$, is an isomorphism between the Hilbert spaces $\boldsymbol{E}(\Omega)$ and $\boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega)$.

Thanks to the isomorphism $\boldsymbol{\mathcal{F}}$ of (1.19), the pure traction problem of linearized elasticity can then be equivalently posed in terms of the new unknown $\boldsymbol{e} \in L^2(\Omega, \mathbb{S}^3)$ as the following constrained minimization problem: Find a matrix field $\boldsymbol{\varepsilon} \in \boldsymbol{E}(\Omega)$ that satisfies

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \boldsymbol{E}(\Omega)} j(\boldsymbol{e}), \tag{20}$$

where the functional $j : \boldsymbol{E}(\Omega) \to \mathbb{R}$ is defined by

$$j(\boldsymbol{e}) := \frac{1}{2} \int_\Omega \{\lambda \, tr \, \boldsymbol{e} \, tr \, \boldsymbol{e} + 2\mu \boldsymbol{e} : \boldsymbol{e}\} dx - \Lambda(\boldsymbol{e}) \text{ for all } \boldsymbol{e} \in \boldsymbol{E}(\Omega). \tag{21}$$

In (1.21), λ and μ denote the Lamé constants of the constituting material (assumed for simplicity to be homogeneous and isotropic), the notation : denotes the matrix inner product, and the continuous linear form $\Lambda : \boldsymbol{E}(\Omega) \to \mathbb{R}$ is defined by

$$\Lambda(\boldsymbol{e}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\mathcal{F}} \boldsymbol{e} dx + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{\mathcal{F}} \boldsymbol{e} d\Gamma \tag{22}$$

where $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^3)$, resp. $\boldsymbol{g} \in L^2(\Gamma; \mathbb{R}^3)$ where $\Gamma := \partial\Omega$, denotes the density of the applied body, resp. surface, forces.

Our main results (cf. Theorem 4.2 and 4.3) thus provide a means to handle, via an explicit formula for computing the mapping $\boldsymbol{\mathcal{F}}$, the term (1.22) involving the applied forces in the functional (1.21). They similarly provide a means to handle boundary conditions involving the displacement field, e.g., $\boldsymbol{u} = \boldsymbol{0}$ on a portion of the boundary Γ. Besides its mathematical interest regarding the minimization problem (1.20), the Cesàro-Volterra formula with little regularity could be as well conveniently put to use in the *numerical implementation of intrinsic models*, as recently advocated and analyzed in Ciarlet & Ciarlet, Jr. [7].

The results of this paper were announced in Ref. [10].

2. Notations and preliminaries

Latin indices henceforth range in the set $\{1, 2, \dots, n\}$, where n is any integer ≥ 2, and the summation convention with respect to repeated indices is used in conjunction with this rule.

The notations $\mathbb{M}^n, \mathbb{S}^n$, and $\mathbb{A}^n$, respectively designate the sets of all real square, symmetric, and anti-symmetric, matrices of order n. The notation (a_{ij}) designates the matrix in $\mathbb{M}^n$ with a_{ij} as its elements, the first index being the row index. The notation $(\boldsymbol{A})_{ij}$ designates the element at the i-th row and j-th column of a matrix $\boldsymbol{A}$. When it is identified with a matrix, a vector in $\mathbb{R}^n$ is a column vector.

The coordinates of a point $x \in \mathbb{R}^n$ are denoted x_i. Partial derivative operators, in the usual sense or in the sense of distributions, of the first and second order are denoted $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$.

All the vector spaces considered in this paper are over $\mathbb{R}$. Given an open subset Ω of $\mathbb{R}^n$, the notations $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ respectively designate the space of all functions that are infinitely differentiable in Ω and have compact support in Ω and the space of distributions over Ω. The notation $< \cdot \, , \, \cdot >$ denotes the duality pairing between a topological space and its dual space, such as $L^2(\Omega)$ and itself, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, or $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$.

The notation $\mathcal{C}^m(\Omega)$, $m \geq 0$, designates the space of all continuous if $m = 0$, or m times continuously differentiable if $m \geq 1$, functions over Ω. The notations $H^m(\Omega), H_0^m(\Omega)$, and $H^{-m}(\Omega) = (H_0^m(\Omega))', m \geq 1$, designate the usual Sobolev spaces. If $\mathbb{X}$ is a finite-dimensional space such as $\mathbb{R}^n, \mathbb{S}^n$, etc., notations such as $\mathcal{D}(\Omega; \mathbb{X}), H_0^1(\Omega; \mathbb{X})$, etc., designate spaces of vector fields or matrix fields with values in $\mathbb{X}$ and components in $\mathcal{D}(\Omega), H_0^1(\Omega)$, etc.

Lemmas 2.1 to 2.4 list some properties of specific subspaces of $\mathcal{D}(\Omega)$ and $\mathcal{D}(\Omega; \mathbb{R}^n)$ (these subspaces naturally appear in the next two sections).

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^n$. Define the space*

$$\mathcal{D}_0(\Omega) := \{\varphi \in \mathcal{D}(\Omega); \int_\Omega \varphi dx = 0\}. \tag{23}$$

Then a distribution $u \in \mathcal{D}'(\Omega)$ satisfies

$$< u, \varphi >= 0 \quad \text{for all} \quad \varphi \in \mathcal{D}_0(\Omega) \tag{24}$$

if and only if u is a constant function.

Proof. If $u(x) = C$ for all $x \in \Omega$, then $< u, \varphi >= C \int_\Omega \varphi dx$ for all $\varphi \in \mathcal{D}(\Omega)$, and thus $< u, \varphi >= 0$ for all $\varphi \in \mathcal{D}_0(\Omega)$. To establish the converse, let $\theta \in \mathcal{D}(\Omega)$ be a function that satisfies

$$\int_\Omega \theta dx = 1. \tag{25}$$

Given any function $\psi \in \mathcal{D}(\Omega)$, the function

$$\varphi := \psi - \lambda\theta, \text{ where } \lambda := \int_\Omega \psi dx =< 1, \psi >,$$

belongs to the space $\mathcal{D}_0(\Omega)$. If a distribution $u \in \mathcal{D}'(\Omega)$ satisfies (2.2), we thus have

$$< u, \psi >= \lambda < u, \theta >=< C, \psi > \quad \text{for all } \psi \in \mathcal{D}(\Omega), \text{ where } C :=< u, \theta > .$$

Hence $u = C$. □

Remark 1. *The above proof shows that, given any function $\theta \in \mathcal{D}(\Omega)$ that satisfies (2.3), the space $\mathcal{D}(\Omega)$ can be written as the direct sum $\mathcal{D}_0(\Omega)\oplus$ Span θ. More precisely, any function $\psi \in \mathcal{D}(\Omega)$ can be written as*

$$\psi = \varphi + \lambda\theta, \text{ with } \varphi \in \mathcal{D}_0(\Omega) \quad \text{and} \quad \lambda = \int_\Omega \psi dx.$$ □

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$. The space $\mathcal{D}_0(\Omega)$ defined in* (2.1) *is dense in the space*

$$L^2_0(\Omega) := \{v \in L^2(\Omega); \int_\Omega v dx = 0\}, \tag{26}$$

with respect to the norm of the space $L^2(\Omega)$.

Proof. Let $\theta \in \mathcal{D}(\Omega)$ be a function that satisfies (2.3).

Let $\|\cdot\|_{L^2}$ designate the norm in the space $L^2(\Omega)$. Given any function $v \in L^2_0(\Omega)$, there exist functions $\psi^k \in \mathcal{D}(\Omega), k \geq 1$, such that $\|\psi^k - v\|_{L^2} \to 0$ as $k \to \infty$ (the space $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$). For each $k \geq 1$, let

$$\varphi^k := \psi^k - \Big(\int_\Omega \psi^k dx\Big)\theta,$$

so that $\varphi^k \in \mathcal{D}_0(\Omega)$. Besides,

$$\|\varphi^k - v\|_{L^2} \leq \|\psi^k - v\|_{L^2} + \Big|\int_\Omega \psi^k dx\Big| \|\theta\|_{L^2}.$$

Therefore, $\|\varphi^k - v\|_{L^2} \to 0$ as $k \to \infty$, since

$$\int_\Omega \psi^k dx \underset{k\to\infty}{\to} \int_\Omega v dx = 0.$$

□

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^n, n \geq 2$. Define the space*

$$\mathcal{D}_1(\Omega;\mathbb{R}^n) := \{\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^n); \int_\Omega \varphi_i dx = \int_\Omega (x_j\varphi_i - x_i\varphi_j)dx = 0\}. \tag{27}$$

Then a vector field $\boldsymbol{u} = (u_i) \in \mathcal{D}'(\Omega;\mathbb{R}^n)$ satisfies

$$< \boldsymbol{u}, \boldsymbol{\varphi} > := < u_i, \varphi_i > = 0 \quad \text{for all} \quad \boldsymbol{\varphi} \in \mathcal{D}_1(\Omega;\mathbb{R}^n) \tag{28}$$

if and only if

$$\boldsymbol{u}(x) = \boldsymbol{a} + \boldsymbol{A}\,\boldsymbol{ox} \text{ for all } x = (x_i) \in \Omega, \text{ for some } \boldsymbol{a} = (a_i) \in \mathbb{R}^n \text{ and } \boldsymbol{A} = (a_{ij}) \in \mathbb{A}^n. \tag{29}$$

Proof. A vector field $\boldsymbol{u} \in \mathcal{D}'(\Omega;\mathbb{R}^n)$ of the form (2.7) satisfies

$$< \boldsymbol{u}, \boldsymbol{\varphi} > = < a_i + a_{ij}x_j, \varphi_i > = a_i \int_\Omega \varphi_i dx + \sum_{i<j} a_{ij} \int_\Omega (x_j\varphi_i - x_i\varphi_j)dx$$

for all $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^n)$, thanks to the antisymmetry of the matrix $\boldsymbol{A}$. Hence such a a vector field satisfies $< \boldsymbol{u}, \boldsymbol{\varphi} > = 0$ for all $\boldsymbol{\varphi} \in \mathcal{D}_1(\Omega;\mathbb{R}^n)$.

To establish the converse, we first notice that there is no loss of generality in assuming that $0 \in \Omega$. Otherwise, let $x_0 = (x_i^0) \in \Omega$, let $\widetilde{\Omega} := \{(x - x_0) \in \mathbb{R}^n; x \in \Omega\}$, and, given any function $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^3)$, let the function $\widetilde{\boldsymbol{\varphi}} = (\widetilde{\varphi}_i) : \widetilde{\Omega} \to \mathbb{R}^3$ be defined by $\widetilde{\boldsymbol{\varphi}}(x - x_0) := \boldsymbol{\varphi}(x)$ for all $x \in \Omega$, so that $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}(\widetilde{\Omega};\mathbb{R}^3)$. Furthermore,

$$\int_{\widetilde{\Omega}} \widetilde{\varphi}_i d\widetilde{x} = \int_\Omega \varphi_i dx = 0,$$
$$\int_{\widetilde{\Omega}} (\widetilde{x}_j\widetilde{\varphi}_i - \widetilde{x}_i\widetilde{\varphi}_j)d\widetilde{x} = \int_\Omega (x_j\varphi_i - x_i\varphi_j)dx - x_j^0 \int_\Omega \varphi_i dx + x_i^0 \int_\Omega \varphi_j dx = 0,$$

which shows that $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}_1(\widetilde{\Omega};\mathbb{R}^3)$. Besides, if a vector field $\widetilde{\boldsymbol{u}} \in \mathcal{D}'(\widetilde{\Omega};\mathbb{R}^n)$ is of the form $\widetilde{\boldsymbol{u}}(\widetilde{x}) = \widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{A}}\boldsymbol{o}\widetilde{\boldsymbol{x}}$ for some $\widetilde{\boldsymbol{a}} \in \mathbb{R}^n$ and $\widetilde{\boldsymbol{A}} \in \mathbb{A}^n$, then the field $\boldsymbol{u} \in \mathcal{D}'(\Omega;\mathbb{R}^n)$ defined by $\boldsymbol{u}(x) = \widetilde{\boldsymbol{u}}(x - x_0)$ is indeed of the form (2.7), with $\boldsymbol{a} := \widetilde{\boldsymbol{a}} - \widetilde{\boldsymbol{A}}\boldsymbol{ox}_0$ and $\boldsymbol{A} = \widetilde{\boldsymbol{A}}$.

Next, let $\theta \in \mathcal{D}(\Omega)$ and $\theta_j \in \mathcal{D}(\Omega), 2 \leq j \leq n$, be functions that satisfy

$$\int_\Omega \theta dx = 1 \quad \text{and} \quad \int_\Omega x_i\theta dx = 0, \tag{30}$$

$$\int_\Omega \theta_j dx = 0 \quad \text{and} \quad \int_\Omega x_i\theta_j dx = \delta_{ij}. \tag{31}$$

For instance, let $\omega(x) := \exp(\|x\|^2 - 1)^{-1})$ if $\|x\| < 1$ and $\omega(x) := 0$ if $\|x\| \geq 1$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$, and let $r > 0$ be such that $\{x \in \mathbb{R}^n; \|x\| \leq r\} \subset \Omega$ (recall that we may assume that $0 \in \Omega$). Then the function θ defined by $\theta(x) := \left(\int_\Omega \omega\left(\frac{x}{r}\right)dx\right)^{-1}\omega\left(\frac{x}{r}\right)$ for $x \in \Omega$ belongs to the space $\mathcal{D}(\Omega)$ and satisfies (2.8). Likewise for instance, let a function $\chi \in \mathcal{D}(\Omega)$ be such that $\int_\Omega \chi dx = -1$; then the functions $\theta_j := \partial_j\chi$ belong to the space $\mathcal{D}(\Omega)$ and they satisfy (2.9), since

$$\int_\Omega \theta_j dx = \int_\Omega \partial_j\chi dx = 0,$$
$$\int_\Omega x_i\theta_j dx = \int_\Omega x_i\partial_j\chi dx = -\int_\Omega \delta_{ij}\chi dx = \delta_{ij}.$$

Given functions $\theta \in \mathcal{D}(\Omega)$ and $\theta_j \in \mathcal{D}(\Omega), 2 \leq j \leq n$, satisfying (2.8)−(2.9), we then define vector fields $\boldsymbol{\theta}_i \in \mathcal{D}(\Omega;\mathbb{R}^n)$ and $\boldsymbol{\theta}_{ij} \in \mathcal{D}(\Omega;\mathbb{R}^n), 2 \leq j \leq n$, by letting

$$\boldsymbol{\theta}_i := \theta \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{\theta}_{ij} = \theta_j \boldsymbol{e}_i, 1 \leq i < j \leq n, \tag{32}$$

where $\boldsymbol{e}_i$ denote the vectors of the canonical basis of $\mathbb{R}^n$.

Given any vector field $\boldsymbol{\psi} = (\psi_i) \in \mathcal{D}(\Omega;\mathbb{R}^n)$, let the vector field $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^n)$ be defined by

$$\varphi_i := \psi_i - \lambda_i \theta - \sum_{i<j} \lambda_{ij} \theta_j,$$

or equivalently, by

$$\boldsymbol{\varphi} = \boldsymbol{\psi} - \lambda_i \boldsymbol{\theta}_i - \sum_{i<j} \lambda_{ij} \boldsymbol{\theta}_{ij}, \tag{33}$$

where

$$\lambda_i := \int_\Omega \psi_i dx \quad \text{and} \quad \lambda_{ij} := \int_\Omega (x_j \psi_i - x_i \psi_j) dx. \tag{34}$$

We then observe that, thanks to relations (2.8)−(2.9), the vector field $\boldsymbol{\varphi}$ defined in (2.11) belongs to the space $\mathcal{D}_1(\Omega;\mathbb{R}^n)$: First,

$$\int_\Omega \varphi_i dx = \int_\Omega \psi_i dx - \lambda_i \int_\Omega \theta dx - \sum_{i<j} \lambda_{ij} \int_\Omega \theta_j dx = 0.$$

Second, for $i < j$ (the case $j < i$ is similar),

$$\int_\Omega (x_j \varphi_i - x_i \varphi_j) dx = \lambda_{ij} - \sum_{i<p} \lambda_{ip} \int_\Omega x_j \theta_p dx + \sum_{j<q} \lambda_{jq} \int_\Omega x_i \theta_q dx = 0,$$

since $\sum_{i<p} \lambda_{ip} \int_\Omega x_j \theta_p dx = \lambda_{ij}$ and $\sum_{j<q} \lambda_{jq} \int_\Omega x_i \theta_q dx = 0$.

If a vector field $\boldsymbol{u} \in \mathcal{D}'(\Omega;\mathbb{R}^n)$ satisfies (2.6), we thus have

$$< \boldsymbol{u}, \boldsymbol{\psi} >= \lambda_i < \boldsymbol{u}, \boldsymbol{\theta}_i > + \sum_{i<j} \lambda_{ij} < \boldsymbol{u}, \boldsymbol{\theta}_{ij} > \quad \text{for all } \boldsymbol{\psi} \in \mathcal{D}(\Omega;\mathbb{R}^n), \tag{35}$$

where the vector fields $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_{ij}$ and the coefficients λ_i and λ_{ij} are respectively defined as in (2.10) and (2.12). Letting

$$a_i :=< \boldsymbol{u}, \boldsymbol{\theta}_i >, \; a_{ii} = 0, \; \text{ and } \; a_{ij} = -a_{ji} :=< \boldsymbol{u}, \boldsymbol{\theta}_{ij} > \text{ if } i < j,$$

we can rewrite relations (2.13) as

$$\begin{aligned} < \boldsymbol{u}, \boldsymbol{\psi} > &= \int_\Omega a_i \psi_i dx + \sum_{i<j} \int_\Omega (a_{ij} x_j \psi_i - a_{ij} x_i \psi_j) dx \\ &= \int_\Omega (a_i + a_{ij} x_j) \psi_i dx. \end{aligned} \tag{36}$$

Since relations (2.14) hold for all $\boldsymbol{\psi} \in \mathcal{D}(\Omega;\mathbb{R}^n)$, the vector field $\boldsymbol{u} \in \mathcal{D}'(\Omega;\mathbb{R}^n)$ is indeed of the announced form (2.7), with $\boldsymbol{a} := (a_i) \in \mathbb{R}^n$ and $\boldsymbol{A} := (a_{ij}) \in \mathbb{A}^n$. □

Remark 2. *(1) The above proof shows that, given any functions $\theta \in \mathcal{D}(\Omega)$ and $\theta_j \in \mathcal{D}(\Omega), 2 \leq j \leq n$, that satisfy (2.8)−(2.9), the space $\mathcal{D}(\Omega;\mathbb{R}^n)$ can be written as the direct sum $\mathcal{D}_1(\Omega;\mathbb{R}^n) \oplus$ Span $(\boldsymbol{\theta}_i) \oplus$ Span*

$(\boldsymbol{\theta}_{ij})_{i<j}$, *where the vector fields* $\boldsymbol{\theta}_i$ *and* $\boldsymbol{\theta}_{ij}$ *are those defined in (2.10). More precisely, any vector field* $\boldsymbol{\psi} \in \mathcal{D}(\Omega; \mathbb{R}^n)$ *can be written as*

$$\boldsymbol{\psi} = \boldsymbol{\varphi} + \lambda_i \boldsymbol{\theta}_i + \sum_{i<j} \lambda_{ij} \boldsymbol{\theta}_{ij},$$

with

$$\boldsymbol{\varphi} \in \mathcal{D}_1(\Omega; \mathbb{R}^n), \ \lambda_i := \int_\Omega \psi_i dx, \ \lambda_{ij} := \int_\Omega (x_j \psi_i - x_i \psi_j) dx.$$

(2) If $n = 3$, *a vector field of the form (2.7) is nothing but an infinitesimal rigid displacement, i.e., of the form (1.14).* □

Lemma 2.4. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$. *The space* $\mathcal{D}_1(\Omega; \mathbb{R}^n)$ *defined in* (2.5) *in dense in the space*

$$L^2_1(\Omega; \mathbb{R}^n) := \{\boldsymbol{v} = (v_i) \in L^2(\Omega; \mathbb{R}^n); \int_\Omega v_i dx = \int_\Omega (x_j v_i - x_i v_j) dx = 0\}, \tag{37}$$

with respect to the norm of the space $L^2(\Omega; \mathbb{R}^n)$.

Proof. Let the vector fields $\boldsymbol{\theta}_i \in \mathcal{D}(\Omega; \mathbb{R}^n)$ and $\boldsymbol{\theta}_{ij} \in \mathcal{D}(\Omega; \mathbb{R}^n), 2 \leq j \leq n$, be defined as in (2.10), where the functions $\theta \in \mathcal{D}(\Omega)$ and $\theta_j \in \mathcal{D}(\Omega), 2 \leq j \leq n$, satisfy relations (2.8)−(2.9).

Let $\|\cdot\|_{\boldsymbol{L}^2}$ designate the norm in the space $L^2(\Omega; \mathbb{R}^n)$. Given any vector field $\boldsymbol{v} \in L^2_1(\Omega; \mathbb{R}^n)$, there exist vector fields $\boldsymbol{\psi}^k = (\psi^k_i) \in \mathcal{D}(\Omega; \mathbb{R}^n), k \geq 1$, such that $\|\boldsymbol{\psi}^k - \boldsymbol{v}\|_{\boldsymbol{L}^2} \to 0$ as $k \to \infty$. For each $k \geq 1$, let

$$\boldsymbol{\varphi}^k := \boldsymbol{\psi}^k - \Big(\int_\Omega \psi^k_i dx\Big)\boldsymbol{\theta}_i - \sum_{i<j}\Big(\int_\Omega (x_j \psi^k_i - x_i \psi^k_j) dx\Big)\boldsymbol{\theta}_{ij},$$

so that $\boldsymbol{\varphi}^k \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$ (to see this, argue as in the proof of Lemma 2.3). Besides,

$$\|\boldsymbol{\varphi}^k - \boldsymbol{v}\|_{\boldsymbol{L}^2} \leq \|\boldsymbol{\psi}^k - \boldsymbol{v}\|_{\boldsymbol{L}^2} + \left|\int_\Omega \psi^k_i dx\right| \|\boldsymbol{\theta}_i\|_{\boldsymbol{L}^2} + \sum_{i<j}\left|\int_\Omega (x_j \psi^k_i - x_i \psi^k_j) dx\right| \|\boldsymbol{\theta}_{ij}\|_{\boldsymbol{L}^2}.$$

Therefore, $\|\boldsymbol{\varphi}^k - \boldsymbol{v}\|_{\boldsymbol{L}^2} \to 0$ as $k \to \infty$, since

$$\int_\Omega \psi^k_i dx \underset{k\to\infty}{\to} \int_\Omega v_i dx = 0,$$
$$\int_\Omega (x_j \psi^k_i - x_i \psi^k_j) dx \underset{k\to\infty}{\to} \int_\Omega (x_j v_i - x_i v_j) dx = 0.$$

□

While Lemmas 2.1 and 2.3, resp. 2.2 and 2.4, hold in any open, resp. bounded open, subset of $\mathbb{R}^n$, some restrictions need to be imposed in the next lemma (which concludes our list of "preliminaries"), according to the following definition : A *domain* in $\mathbb{R}^n$ is an open, bounded, connected subset Ω of $\mathbb{R}^n$, with a Lipschitz-continuous boundary Γ, the set Ω being locally on one side of Γ.

The mapping $\boldsymbol{T} = (T_i)$ defined in the next lemma plays a key role in the rest of the paper.

Lemma 2.5. *Let* Ω *be a domain in* $\mathbb{R}^n$. *Then the Hilbert space* $H^1_0(\Omega; \mathbb{R}^n)$ *equipped with the norm* $(v_i) \mapsto (\int_\Omega \partial_j v_i \partial_j v_i dx)^{1/2}$ *can be written as the direct sum*

$$H^1_0(\Omega; \mathbb{R}^n) = \boldsymbol{V} \oplus \boldsymbol{V}^\perp, \tag{38}$$

where the subspace $\boldsymbol{V}$ *and its orthogonal complement* $\boldsymbol{V}^{\perp}$ *are defined by*

$$\boldsymbol{V} := \{\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^n);\ \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \ L^2(\Omega)\}, \tag{39}$$

$$\boldsymbol{V}^{\perp} := \{\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^n); -\Delta\boldsymbol{v} = \mathbf{grad}\ \ q \ \text{ for some } \ q \in L^2(\Omega)\}. \tag{40}$$

Let the space $L_0^2(\Omega)$ *be defined as in (2.4). Then there exists a bijection*

$$\boldsymbol{T} = (T_i) : v \in L_0^2(\Omega) \mapsto \boldsymbol{T}v = (T_i v) \in \boldsymbol{V}^{\perp} \subset H_0^1(\Omega;\mathbb{R}^n), \tag{41}$$

which is linear and continuous, hence an isomorphism, between the spaces $L_0^2(\Omega)$ *and* $\boldsymbol{V}^{\perp}$*, and that satisfies*

$$-\operatorname{div}(\boldsymbol{T}v) = v \ \text{ for all } v \in L_0^2(\Omega). \tag{42}$$

Proof. That the space $H_0^1(\Omega;\mathbb{R}^n)$ can be written as the direct sum (2.16), with the spaces $\boldsymbol{V}$ and $\boldsymbol{V}^{\perp}$ being defined as in (2.17)−(2.18), is proved in Corollary 2.3, Chapter 1, of Girault & Raviart [14]. It is also shown in Corollary 2.4, Chapter 1, of *ibid.*, that the operator div is an isomorphism of $\boldsymbol{V}^{\perp}$ onto $L_0^2(\Omega)$; hence the operator $\boldsymbol{T}$ of (2.19) is an isomorphism of $L_0^2(\Omega)$ onto $\boldsymbol{V}^{\perp}$ since, in view of (2.20), $\boldsymbol{T}$ is nothing but the inverse of the operator −div. □

Remark 3. *(1) That the domain of the operator* $\boldsymbol{T}$ *should be the subspace* $L_0^2(\Omega)$ *of* $L^2(\Omega)$ *is clear, since the range of* $\boldsymbol{T}$ *is a subspace of* $H_0^1(\Omega;\mathbb{R}^n)$*.*

(2) For a given function $v \in L_0^2(\Omega)$*, all the solutions* $\boldsymbol{u} \in H_0^1(\Omega;\mathbb{R}^n)$ *to the equation* $-\operatorname{div}\,\boldsymbol{u} = v$ *are thus of the form* $\boldsymbol{u} = \boldsymbol{T}v + \boldsymbol{w}$ *for some* $\boldsymbol{w} \in \boldsymbol{V}$*.*

(3) It is shown in Theorem 2' of Bourgain & Brezis [4] that, more generally for any $1 < p < \infty$*, there likewise exists a linear and continuous mapping*

$$\boldsymbol{T} : L_0^p(\Omega) := \Big\{v \in L^p(\Omega); \int_\Omega v dx = 0\Big\} \rightarrow W_0^{1,p}(\Omega;\mathbb{R}^n)$$

such that $-\operatorname{div}(\boldsymbol{T}v) = v$ *for all* $v \in L_0^p(\Omega)$*.* □

3. A Poincaré lemma with little regularity

A *classical lemma of Poincaré* asserts that, if functions $f_i \in \mathcal{C}^1(\Omega)$ satisfy $\partial_i f_j - \partial_j f_i = 0$ in a *simply-connected open subset* Ω *of* $\mathbb{R}^n$, then there exists a function $u \in \mathcal{C}^2(\Omega)$ such that $\partial_i u = f_i$ in Ω. It is easily verified that, in this case, an *explicit solution* to the equations $\partial_i u = f_i$ in Ω is given by the *path integral formula*

$$u(x) = \int_{\gamma(x)} f_i(y)dy_i \ \text{ for all } \ x \in \Omega, \tag{43}$$

where $\gamma(x)$ is any path of class $\mathcal{C}^1$ contained in Ω and joining a point $x_0 \in \Omega$ (considered as fixed) to the point $x \in \Omega$, the relations $\partial_i f_j - \partial_j f_i = 0$ in Ω insuring that the value $u(x)$ computed by (3.1) is independent of the path chosen for joining x_0 to x.

The above classical lemma of Poincaré was extended in Theorem 2.9, Chapter 1, of Girault & Raviart [14], as follows: If functions $f_i \in L^2(\Omega)$ satisfy $\partial_i f_j - \partial_j f_i = 0$ in $H^{-1}(\Omega)$, where Ω is a *simply-connected domain in* $\mathbb{R}^n$ (domains are defined before Lemma 2.5), then there exists a function $u \in H^1(\Omega)$ such that $\partial_i u = f_i$ in $L^2(\Omega)$. This extension was then carried out one step further in Theorem 3.1 of Ciarlet & Ciarlet, Jr. [6], according to the next theorem.

Theorem 3.1 (Poincaré lemma with little regularity)**.** *Let Ω be a simply-connected domain in $\mathbb{R}^n$, and let $f_i \in H^{-1}(\Omega)$ be distributions that satisfy*

$$\partial_i f_j - \partial_j f_i = 0 \ \text{ in } \ H^{-2}(\Omega).$$

Then there exists a function $u \in L^2(\Omega)$, unique up to an additive constant, such that

$$\partial_i u = f_i \ \text{ in } \ H^{-1}(\Omega).$$ □

Remark 4. Theorem 3.1 holds in the more general situation where $f_i \in H^{-m}(\Omega)$ for any integer $m \geq 2$ (in which case $u \in H^{-m+1}(\Omega)$); see Amrouche, Ciarlet & Ciarlet, Jr. [1,2] and Geymonat & Krasucki [12,13], where the extension to a non simply-connected domain is also treated. The last word in this direction is due to S. Mardare [16], who has shown that the Poincaré lemma holds in fact in the sense of distributions. □

We first show that, even under the weaker regularity assumptions of Theorem 3.1 (in which case formula (3.1) becomes meaningless), there is still a way to "compute" a solution $u \in L^2(\Omega)$ to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$. This objective is achieved by means of an *explicit expression* in terms of the data f_i of the duality pairings $< u, \varphi >$ for all functions $\varphi \in \mathcal{D}(\Omega)$ that satisfy $\int_\Omega \varphi dx = 0$; cf. (3.4) below. Note that, by Lemma 2.1, the knowledge of such duality pairings determines the distribution u only up to an additive constant (as expected).

Theorem 3.2. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $\mathcal{D}_0(\Omega)$ be defined as in (2.1), viz.,*

$$\mathcal{D}_0(\Omega) := \{\varphi \in \mathcal{D}(\Omega); \int_\Omega \varphi dx = 0\},$$

and let $f_i \in H^{-1}(\Omega)$ be distributions that satisfy

$$\partial_i f_j - \partial_j f_i = 0 \ \text{ in } \ H^{-2}(\Omega). \tag{44}$$

Then a function $u \in L^2(\Omega)$ satisfies

$$\partial_i u = f_i \ \text{ in } \ H^{-1}(\Omega) \tag{45}$$

if and only if

$$< u, \varphi >=< f_i, T_i\varphi > \quad \text{for all} \ \ \varphi \in \mathcal{D}_0(\Omega), \tag{46}$$

where $\boldsymbol{T} = (T_i) : L^2_0(\Omega) \to H^1_0(\Omega; \mathbb{R}^n)$ is the continuous linear operator defined in Lemma 2.5.

Proof. Note that, in (3.4), $< u, \varphi >= \int_\Omega u\varphi dx$, and $< f_i, T_i\varphi >$ is the duality pairing between the spaces $H^1_0(\Omega)$ and $H^{-1}(\Omega)$.

Assume first that a function $u \in L^2(\Omega)$ satisfies $\partial_i u = f_i$ in $H^{-1}(\Omega)$. Given any function $\varphi \in \mathcal{D}_0(\Omega) \subset L^2_0(\Omega)$, Lemma 2.5 shows that the vector field $\boldsymbol{T}\varphi = (T_i\varphi) \in H^1_0(\Omega; \mathbb{R}^n)$ satisfies $-\partial_i(T_i\varphi) = \varphi$ in the space $L^2_0(\Omega) = \{v \in L^2(\Omega); \int_\Omega v dx = 0\}$. Therefore,

$$< u, \varphi >=< u, -\partial_i(T_i\varphi) >=< \partial_i u, T_i\varphi >=< f_i, T_i\varphi > .$$

Assume next that a function $u \in L^2(\Omega)$ satisfies $< u, \varphi >=< f_i, T_i\varphi >$ for all $\varphi \in \mathcal{D}_0(\Omega)$. Since, given any vector field $(\psi_j) \in \mathcal{D}(\Omega; \mathbb{R}^n)$, the function $\partial_j\psi_j$ belongs to the space $\mathcal{D}_0(\Omega)$, it follows that

$$-\partial_i T_i(\partial_j\psi_j) = \partial_j\psi_j \ \text{ in } \ L^2_0(\Omega),$$

which in turn implies that

$$\begin{aligned} < \partial_j u, \psi_j > &=< u, -\partial_j\psi_j >= - < f_i, T_i(\partial_j\psi_j) > \\ &=< f_i, \psi_i > - < f_i, \psi_i + T_i(\partial_j\psi_j) > . \end{aligned}$$

But
$$\partial_i(\psi_i + T_i(\partial_j\psi_j)) = \partial_i\psi_i + \partial_i T_i(\partial_j\psi_j) = \partial_i\psi_i - \partial_j\psi_j = 0,$$
and since $\partial_i f_j - \partial_j f_i = 0$ in $H^{-2}(\Omega)$, there exists by Theorem 3.1 a function $p \in L^2(\Omega)$ such that $\partial_i p = f_i$ in $H^{-1}(\Omega)$. Therefore,
$$< f_i, \psi_i + T_i(\partial_j\psi_j) >=< \partial_i p, \psi_i + T_i(\partial_j\psi_j) >=< p, -\partial_i(\psi_i + T_i(\partial_j\psi_j)) >= 0.$$
We are thus left with $< \partial_j u, \psi_j >=< f_i, \psi_i >$ for all $(\psi_i) \in \mathcal{D}(\Omega; \mathbb{R}^n)$, which shows that $\partial_i u = f_i$ in $H^{-1}(\Omega)$. □

Remark 5. *The function p appearing in the above proof is of course of the form $p = u + C$ for some constant C, but this observation is not used in the above proof. The only reason for introducing p is to allow to rewrite the vector field (f_i) as a gradient, in this case the gradient of the function p.* □

We next show that the solution to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$ can also be found by solving a *variational problem* (cf. (3.5) below), which satisfies all the assumptions of the *Lax-Milgram lemma.* The operators $T_i : L^2_0(\Omega) \to H^1_0(\Omega)$ appearing in (3.5) are again those defined in Lemma 2.5.

Theorem 3.3. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $L^2_0(\Omega)$ be defined as in (2.4), viz.,*
$$L^2_0(\Omega) := \{v \in L^2(\Omega); \int_\Omega v dx = 0\},$$
and let there be given distributions $f_i \in H^{-1}(\Omega)$ that satisfy
$$\partial_i f_j - \partial_j f_i = 0 \text{ in } H^{-2}(\Omega).$$
Then the variational problem: Find a function $u \in L^2_0(\Omega)$ such that
$$< u, v >=< f_i, T_i v > \text{ for all } v \in L^2_0(\Omega), \tag{47}$$
has a unique solution, which is also a solution to the equations
$$\partial_i u = f_i \text{ in } H^{-1}(\Omega), \tag{48}$$
in effect the only solution to (3.6) *that satisfies* $\int_\Omega u dx = 0$.

Proof. Since $< u, v >= \int_\Omega uv dx$, the bilinear form appearing in the left-hand side of the variational equations (3.5) is clearly continuous and coercive over the space $L^2_0(\Omega)$. The linear form appearing in their right-hand side is clearly continuous, since $T_i \in \mathcal{L}(L^2_0(\Omega); H^1_0(\Omega))$ (Lemma 2.5). Hence the variational equations (3.5) have a unique solution u in the space $L^2_0(\Omega)$. Furthermore, u is a solution to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$, by Theorem 3.2. □

Remark 6. *Interestingly, the existence of a solution to the variational equations (3.5) can be obtained without a recourse to the Lax-Milgram lemma (its uniqueness is clear): Let $u \in L^2_0(\Omega)$ denote the unique solution to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$ that satisfies $\int_\Omega u dx = 0$ (the existence of such a solution follows from Theorem 3.1; its uniqueness is again clear). By Theorem 3.2, this solution satisfies*
$$< u, \varphi >=< f_i, T_i\varphi > \text{ for all } \varphi \in \mathcal{D}_0(\Omega).$$
But the space $\mathcal{D}_0(\Omega)$ is dense in the space $L^2_0(\Omega)$ (Lemma 2.2) and the operators $T_i : L^2_0(\Omega) \to H^1_0(\Omega)$ are continuous (Lemma 2.5); hence the above variational equations hold more generally for all $\varphi \in L^2_0(\Omega)$. □

4. A Cesàro-Volterra formula with little regularity

As shown in Ref. [6], the classical Saint-Venant compatibility conditions (1.2) remain sufficient when they take the weaker form of the equations (4.1) below, which we will call the *Saint-Venant compatibility conditions with little regularity* (although the proof in *ibid.* was given for $n = 3$, it readily extends to any integer $n \geq 2$):

Theorem 4.1 (Saint-Venant compatibility conditions with little regularity)**.** *Let Ω be a simply-connected domain in $\mathbb{R}^n$, and let $e_{ij} = e_{ji} \in L^2(\Omega)$ be functions that satisfy*

$$\partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} = 0 \text{ in } H^{-2}(\Omega). \tag{49}$$

Then there exists a vector field $\boldsymbol{u} = (u_i) \in H^1(\Omega;\mathbb{R}^n)$, unique up to the addition of a vector field of the form $x \in \Omega \to \boldsymbol{a} + \boldsymbol{A}\,\boldsymbol{ox}$ for some $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{A}^n$, such that

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \text{ in } L^2(\Omega). \tag{50}$$

Remark 7. Theorem 4.1 can be extended to non simply-connected domains; see Geymonat & Krasucki [11] and Ciarlet, Ciarlet, Jr., Geymonat & Krasucki [8]. Theorem 4.1 similarly holds in the more general situation where $e_{ij} = e_{ji} \in H^{-1}(\Omega)$ for some integer $m \geq 0$ (in which case $\boldsymbol{u} \in H^{-m+1}(\Omega;\mathbb{R}^n)$; see Amrouche, Ciarlet, Gratie & Kesavan [3]). □

Under the weak regularity assumptions of Theorem 4.1, the classical Cesàro-Volterra path integral formula (1.5) becomes meaningless. But we nevertheless show that there is still a way in this case to "compute" a solution $\boldsymbol{u} = (u_i) \in H^1(\Omega;\mathbb{R}^n)$ to the equations (4.2) in this case.

This objective is achieved by means of an *explicit expression* in terms of the data $e_{ij} \in L^2(\Omega)$ of the duality pairings $< \boldsymbol{u}, \boldsymbol{\varphi} > = < u_i, \varphi_i >$ for all vector fields $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^n)$ that satisfy $\int_\Omega \varphi_i dx = \int_\Omega (x_j\varphi_i - x_i\varphi_j)dx = 0$; cf. (4.3) below. Note that, by Lemma 2.3, the knowledge of such duality pairings determines the vector field $\boldsymbol{u}$ only up to a vector field of the form $\boldsymbol{a} + \boldsymbol{A}\,\boldsymbol{ox}$ for some $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{A}^n$ (as expected). By reference with the classical Cesàro-Volterra path integral formula (1.5), we will say that relations (4.3) contitute the *Cesàro-Volterra formula with little regularity* (this terminology will be further substantiated in Theorem 5.1 and, especially, in Theorem 5.2).

Theorem 4.2 (Cesàro-Volterra formula with little regularity)**.** *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $\mathcal{D}_1(\Omega;\mathbb{R}^n)$ be defined as in* (2.5)*, viz.,*

$$\mathcal{D}_1(\Omega;\mathbb{R}^n) := \{\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega;\mathbb{R}^n); \int_\Omega \varphi_i dx = \int_\Omega (x_j\varphi_i - x_i\varphi_j)dx = 0\},$$

and let there be given a matrix field $\mathbf{e} = (e_{ij}) \in L^2(\Omega;\mathbb{S}^3)$ whose components $e_{ij} = e_{ji} \in L^2(\Omega)$ satisfy the Saint-Venant compatibility conditions with little regularity (4.1)*, viz.,*

$$\partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} = 0 \text{ in } H^{-2}(\Omega).$$

Then a vector field $\mathbf{u} = (u_i) \in H^1(\Omega;\mathbb{R}^n)$ satisfies equations (4.2)*, viz.,*

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \text{ in } L^2(\Omega),$$

if and only if

$$< u_i, \varphi_i > = < e_{ij}, T_i\varphi_j + \partial_k[T_i(T_j\varphi_k - T_k\varphi_j)] > \text{ for all } \boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n), \tag{51}$$

where $\boldsymbol{T} = (T_i) : L^2_0(\Omega) \to H^1_0(\Omega;\mathbb{R}^n)$ is the continuous linear operator defined in Lemma 2.5.

Proof. Note that the duality pairings $< \cdot \,, \cdot >$ appearing in (4.3) are simply those of the space $L^2(\Omega)$.

(i) Assume first that a vector field $\boldsymbol{u} = (u_i) \in H^1(\Omega; \mathbb{R}^n)$ satisfies $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$, and let there be given a vector field $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$.

Define the functions

$$a_{ij} = -a_{ij} := \frac{1}{2}(\partial_j u_i - \partial_i u_j) \in L^2(\Omega),$$

so that $\partial_j u_i = e_{ij} + a_{ij}$. Since each component φ_i of the vector field $\boldsymbol{\varphi}$ belongs to the space $\mathcal{D}_0(\Omega) = \{\varphi \in \mathcal{D}(\Omega); \int_\Omega \varphi dx = 0\} \subset L^2_0(\Omega)$, Lemma 2.5 shows that, for each i, the vector field $\boldsymbol{T}\varphi_i = (T_j \varphi_i) \in H^1_0(\Omega; \mathbb{R}^n)$ satisfies $-\partial_j (T_j \varphi_i) = \varphi_i$ in $L^2(\Omega)$. Consequently,

$$\begin{aligned} < u_i, \varphi_i > &= - < u_i, \partial_j (T_j \varphi_i) > = < \partial_j u_i, T_j \varphi_i > = < e_{ij} + a_{ij}, T_j \varphi_i > \\ &= < e_{ij}, T_i \varphi_j > + \frac{1}{2} < a_{ij}, T_j \varphi_i - T_i \varphi_j >, \end{aligned} \tag{52}$$

since $e_{ij} = e_{ji}$ and $a_{ij} = -a_{ji}$.

We next note that each function $(T_j \varphi_i - T_i \varphi_j) \in H^1_0(\Omega)$ also belongs to the space $L^2_0(\Omega)$, since

$$\begin{aligned} 0 &= \int_\Omega (x_j \varphi_i - x_i \varphi_j) dx = \int_\Omega \{x_j[-\partial_k (T_k \varphi_i)] + x_i[\partial_k (T_k \varphi_j)]\} dx \\ &= \int_\Omega \{\delta_{jk} T_k \varphi_i - \delta_{ik} T_k \varphi_j\} dx = \int_\Omega (T_j \varphi_i - T_i \varphi_j) dx. \end{aligned}$$

Consequently,

$$T_j \varphi_i - T_i \varphi_j = -\partial_k T_k (T_j \varphi_i - T_i \varphi_j). \tag{53}$$

We also note that

$$\partial_k a_{ij} = \frac{1}{2}(\partial_{jk} u_i - \partial_{ik} u_j) = -\partial_i e_{kj} + \partial_j e_{ki} \ \text{ in } \ H^{-1}(\Omega). \tag{54}$$

Using (4.5) and (4.6), we then obtain

$$\begin{aligned} < a_{ij}, & T_j \varphi_i - T_i \varphi_j > = - < a_{ij}, \partial_k T_k (T_j \varphi_i - T_i \varphi_j) > \\ &= < \partial_k a_{ij}, T_k (T_j \varphi_i - T_i \varphi_j) > = < -\partial_i e_{kj} + \partial_j e_{ki}, T_k (T_j \varphi_i - T_i \varphi_j) > \\ &= < e_{kj}, \partial_i [T_k (T_j \varphi_i - T_i \varphi_j)] > - < e_{ki}, \partial_j [T_k (T_j \varphi_i - T_i \varphi_j)] > \\ &= < e_{ij}, \partial_k [T_i (T_j \varphi_k - T_k \varphi_j)] > - < e_{ji}, \partial_k [T_j (T_k \varphi_i - T_i \varphi_k)] > \\ &= 2 < e_{ij}, \partial_k [T_i (T_j \varphi_k - T_k \varphi_j)] > . \end{aligned} \tag{55}$$

Therefore, relations (4.3) follow from (4.4) and (4.7).

(ii) Assume next that a vector field $\boldsymbol{u} = (u_i) \in H^1(\Omega; \mathbb{R}^n)$ satisfies relations (4.3).

Let then a matrix field $\boldsymbol{\psi} = (\psi_{ij}) \in \mathcal{D}(\Omega; \mathbb{S}^n)$ be given. We first note that $(\partial_j \psi_{ij})_{i=1}^n \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$, since

$$\int_\Omega \partial_j \psi_{ij} = 0, \tag{56}$$

$$\int_\Omega (x_k \partial_j \psi_{lj} - x_l \partial_j \psi_{kj}) dx = - \int_\Omega (\delta_{jk} \psi_{lj} - \delta_{jl} \psi_{kj}) dx = - \int_\Omega (\psi_{lk} - \psi_{kl}) dx = 0. \tag{57}$$

We thus have, by (4.3),

$$\begin{aligned} \frac{1}{2} < \partial_j u_i + \partial_i u_j, \psi_{ij} > &= < \partial_j u_i, \psi_{ij} > = - < u_i, \partial_j \psi_{ij} > \\ &= - < e_{ij}, T_i(\partial_k \psi_{jk}) + \partial_k [T_i (T_j (\partial_l \psi_{kl}) - T_k (\partial_l \psi_{jl}))] > \\ &= - < e_{ij}, T_i(\partial_k \psi_{jk}) > + < \partial_k e_{ij}, T_i (T_j (\partial_l \psi_{kl}) - T_k (\partial_l \psi_{jl})) > \\ &= - < e_{ij}, T_i(\partial_k \psi_{jk}) > + < \partial_k e_{ij} - \partial_j e_{ik}, T_i (T_j (\partial_l \psi_{kl})) > . \end{aligned} \tag{58}$$

We next observe that the Saint-Venant compatibility conditions with little regularity (4.1) may be rewritten as

$$\partial_l h_{jki} = \partial_i h_{jkl} \text{ in } H^{-2}(\Omega), \text{ where } h_{jki} = -h_{kji} := \partial_k e_{ji} - \partial_j e_{ki} \in H^{-1}(\Omega).$$

The Poincaré lemma with little regularity (Theorem 3.1) then shows that there exist functions $p_{jk} \in L^2(\Omega)$, each one being unique up to an additive constant, such that

$$\partial_i p_{jk} = h_{jki} = \partial_k e_{ij} - \partial_j e_{ik} \text{ in } H^{-1}(\Omega). \tag{59}$$

Since $\partial_i(p_{jk} + p_{kj}) = h_{jki} + h_{kji} = 0$, these additive constants can be adjusted in such a way that

$$p_{jk} + p_{kj} = 0 \text{ in } L^2(\Omega). \tag{60}$$

Thanks to relations (4.11)−(4.12), we thus have

$$\begin{aligned} &< \partial_k e_{ij} - \partial_j e_{ik}, T_i(T_j(\partial_l \psi_{kl})) >=< \partial_i p_{jk}, T_i(T_j(\partial_l \psi_{kl})) > \\ &\quad = - < p_{jk}, \partial_i[T_i(T_j(\partial_l \psi_{kl}))] > \\ &\quad = -\frac{1}{2} < p_{jk}, \partial_i[T_i(T_j(\partial_l \psi_{kl}) - T_k(\partial_l \psi_{jl}))] > . \end{aligned} \tag{61}$$

As shown in (4.8), for each $k = 1, \dots, n$, the function $\partial_l \psi_{kl}$ belongs to the space $L^2_0(\Omega)$. Consequently, relations (4.9) combined with the definition of the operator $\boldsymbol{T} = (T_i) : L^2_0(\Omega) \to H^1_0(\Omega; \mathbb{R}^n)$ (cf. Lemma 2.5) give

$$\begin{aligned} 0 &= \int_\Omega (x_j \partial_l \psi_{kl} - x_k \partial_l \psi_{jl}) dx \\ &= \int_\Omega \big(-x_j \partial_p T_p(\partial_l \psi_{kl}) + x_k \partial_q T_q(\partial_l \psi_{jl})\big) dx \\ &= \int_\Omega \big(\delta_{jp} T_p(\partial_l \psi_{kl}) - \delta_{kq} T_q(\partial_l \psi_{jl})\big) dx = \int_\Omega \big(T_j(\partial_l \psi_{kl}) - T_k(\partial_l \psi_{jl}\big) dx, \end{aligned}$$

which means that, for each $j = 1, \dots, n$ and each $k = 1, \dots, n$, the function $(T_j(\partial_l \psi_{kl}) - T_k(\partial_l \psi_{jl}))$ also belongs to the space $L^2_0(\Omega)$. As a result, relations (4.13) become

$$\begin{aligned} < \partial_k e_{ij} - \partial_j e_{ik}, T_i(T_j(\partial_l \psi_{kl})) > &= \frac{1}{2} < p_{jk}, T_j(\partial_l \psi_{kl}) - T_k(\partial_l \psi_{jl}) > \\ &=< p_{jk}, T_j(\partial_l \psi_{kl}) >, \end{aligned} \tag{62}$$

thanks again to relations (4.12).

Using (4.14) in (4.10) then gives

$$\begin{aligned} \frac{1}{2} < \partial_j u_i + \partial_i u_j, \psi_{ij} > &= - < e_{ij}, T_i(\partial_k \psi_{jk}) > + < p_{jk}, T_j(\partial_l \psi_{kl}) > \\ &=< e_{ij}, \psi_{ij} > + < p_{jk} - e_{jk}, \psi_{jk} + T_j(\partial_l \psi_{kl}) >, \end{aligned} \tag{63}$$

since $< p_{jk}, \psi_{jk} >= 0$ (recall that $p_{jk} = -p_{kj}$ and $\psi_{jk} = \psi_{kj}$). Noting that, by (4.11), the functions

$$q_{jk} := p_{jk} - e_{jk} \in L^2(\Omega)$$

satisfy

$$\partial_l q_{jk} = \partial_j q_{lk} \text{ in } H^{-1}(\Omega),$$

we again resort to the Poincaré lemma with little regularity (Theorem 3.1) to conclude that there exist functions $v_k \in H^1(\Omega)$, each one being unique up to an additive constant, such that

$$q_{jk} = \partial_j v_k = p_{jk} - e_{jk} \text{ in } L^2(\Omega).$$

Consequently,

$$\begin{aligned} < p_{jk} - e_{jk}, \psi_{jk} + T_j(\partial_l \psi_{kl}) > &= < \partial_j v_k, \psi_{jk} + T_j(\partial_l \psi_{kl}) > \\ &= - < v_k, \partial_j \psi_{jk} + \partial_j T_j(\partial_l \psi_{kl}) >, \end{aligned} \tag{64}$$

since $(\psi_{jk} + T_j(\partial_l \psi_{kl})) \in H_0^1(\Omega)$. But the definition of the operators T_j (recall that $\partial_l \psi_{kl} \in \mathcal{D}_0(\Omega) \subset L_0^2(\Omega)$) and the symmetries $\psi_{kl} = \psi_{lk}$ together imply that

$$-\partial_j T_j(\partial_l \psi_{kl}) = \partial_l \psi_{kl} = \partial_j \psi_{jk}. \tag{65}$$

Combining (4.15), (4.16), and (4.17), we are thus left with

$$\frac{1}{2} < \partial_j u_i + \partial_i u_j, \psi_{ij} > = < e_{ij}, \psi_{ij} > .$$

Since this relation holds for any matrix field $\boldsymbol{\psi} = (\psi_{ij}) \in \mathcal{D}(\Omega; \mathbb{S}^n)$, it follows that $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$, as announced. □

We next show that the solution $\boldsymbol{u} = (u_i)$ to the equations $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$ can be found by solving a *variational problem* (cf (4.18) below), which satisfies all the assumptions of the *Lax-Milgram lemma*. The operators $T_i : L_0^2(\Omega) \to H_0^1(\Omega)$ are again those defined in Lemma 2.5.

Theorem 4.3. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $L_1^2(\Omega; \mathbb{R}^n)$ be defined as in* (2.15), *viz.,*

$$L_1^2(\Omega; \mathbb{R}^n) := \{\boldsymbol{v} = (v_i) \in L^2(\Omega; \mathbb{R}^n); \int_\Omega v_i dx = \int_\Omega (x_j v_i - x_i v_j) dx = 0\},$$

and let there be given functions $e_{ij} = e_{ji} \in L^2(\Omega)$ that satisfy the Saint-Venant compatibility conditions with little regularity, viz.,

$$\partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \ \text{ in } \ H^{-2}(\Omega).$$

Then the variational problem : Find a vector field $(u_i) \in L_1^2(\Omega; \mathbb{R}^n)$ such that

$$< u_i, v_i > = < e_{ij}, T_i v_j + \partial_k [T_i(T_j v_k - T_k v_j)] > \ \text{ for all } \ (v_i) \in L_1^2(\Omega; \mathbb{R}^n), \tag{66}$$

has a unique solution. Besides, (u_i) is in fact in the space $H^1(\Omega; \mathbb{R}^n)$ and is a particular solution to the equations

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \ \text{ in } \ L^2(\Omega), \tag{67}$$

in effect the only solutions to (4.19) *that satisfy $\int_\Omega u_i dx = \int_\Omega (x_j u_i - x_i u_j) dx = 0$.*

Proof. We first note that, given any vector field $(v_i) \in L_1^2(\Omega; \mathbb{R}^n)$, each function v_i belongs to the space $L_0^2(\Omega)$ (by definition of the space $L_1^2(\Omega; \mathbb{R}^n)$), and each function $(T_j v_k - T_k v_j)$ also belongs to $L_0^2(\Omega)$ (the proof is the same as that given for a vector field $(\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$ in part (i) of the proof of Theorem 4.2). Hence the right-hand side of the variational equations (4.18) makes sense; besides, it clearly defines a continuous linear form on the space $L_1^2(\Omega; \mathbb{R}^n)$ since the operators $T_i : L_0^2(\Omega) \to H_0^1(\Omega)$ are continuous. Since $< u_i, v_i > = \int_\Omega u_i v_i dx$, the bilinear form appearing in the left-hand side of equations (4.18) is clearly continuous and coercive over the space $L_1^2(\Omega; \mathbb{R}^n)$. Hence the variational equations (4.18) have a unique solution $\boldsymbol{u} = (u_i) \in L_1^2(\Omega; \mathbb{R}^n)$.

We then observe that there exists a unique vector field $\widetilde{\boldsymbol{u}} = (\widetilde{u}_i) \in H^1(\Omega; \mathbb{R}^n) \cap L_1^2(\Omega; \mathbb{R}^n)$ that satisfies $\frac{1}{2}(\partial_j \widetilde{u}_i + \partial_i \widetilde{u}_j) = e_{ij}$ in $L^2(\Omega)$ (the existence follows from Theorem 4.1; the uniqueness follows from Lemma 2.3). Therefore this vector field $\widetilde{\boldsymbol{u}} \in H^1(\Omega; \mathbb{R}^n)$ satisfies

$$< \widetilde{u}_i, \varphi_i > = < e_{ij}, T_i \varphi_j + \partial_k [T_i(T_j \varphi_k - T_k \varphi_j)] > \ \text{ for all } \ \boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n),$$

by Theorem 4.2. But, since the space $\mathcal{D}_1(\Omega;\mathbb{R}^n)$ is dense in the space $L^2_1(\Omega;\mathbb{R}^n)$ (cf. Lemma 2.4) and the operators $T_i : L^2_0(\Omega) \to H^1_0(\Omega)$ are continuous, the vector field $\boldsymbol{u}$ also satisfies

$$< \widetilde{u}_i, v_i > = < e_{ij}, T_i v_j + \partial_k[T_i(T_j v_k - T_k v_j)] > \quad \text{for all } \boldsymbol{v} \in L^2_1(\Omega;\mathbb{R}^n).$$

Hence $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, since the variational equations (4.18) have a unique solution in the space $L^2_1(\Omega;\mathbb{R}^n)$. Therefore, $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^n)$. □

5. The Cesàro-Volterra formula with little regularity is indeed a generalization of the classical formula

To begin with, we show how the Cesàro-Volterra formula with little regularity (4.3) in dimension $n = 3$ can be rewritten in a vector-matrix form (cf. (5.5) below) that is, at least *formally*, highly reminiscent of the vector-matrix form (1.6) of the classical Cesàro-Volterra path integral formula. To this end, we need some additional notation.

Given any vector fields $\boldsymbol{u} = (u_i)$ and $\boldsymbol{\varphi} = (\varphi_i)$ in $L^2(\Omega;\mathbb{R}^3)$, and given any matrix fields $\boldsymbol{e} = (e_{ij}) \in \mathcal{D}'(\Omega;\mathbb{M}^3)$ and $\boldsymbol{\psi} = (\psi_{ij}) \in \mathcal{D}'(\Omega;\mathbb{M}^3)$, we let

$$< \boldsymbol{u}, \boldsymbol{\varphi} > := \int_\Omega u_i \varphi_i dx \quad \text{and} \quad \ll \boldsymbol{e}, \boldsymbol{\varphi} \gg := < e_{ij}, \psi_{ij} > . \tag{68}$$

Given any matrix field $\boldsymbol{e} = (e_{ij}) \in \mathcal{D}'(\Omega;\mathbb{M}^3)$, we let the matrix field $\mathbf{CURL}\, \boldsymbol{e} \in \mathcal{D}'(\Omega;\mathbb{M}^3)$ be defined as in (1.7), viz.,

$$\mathbf{CURL}\ \boldsymbol{e} = \begin{pmatrix} \partial_2 e_{13} - \partial_3 e_{12} & \partial_3 e_{11} - \partial_1 e_{13} & \partial_1 e_{12} - \partial_2 e_{11} \\ \partial_2 e_{23} - \partial_3 e_{22} & \partial_3 e_{21} - \partial_1 e_{23} & \partial_1 e_{22} - \partial_2 e_{21} \\ \partial_2 e_{33} - \partial_3 e_{32} & \partial_3 e_{31} - \partial_1 e_{33} & \partial_1 e_{32} - \partial_2 e_{31} \end{pmatrix}. \tag{69}$$

Given any vector field $\boldsymbol{\varphi} = (\varphi_i) \in L^2_0(\Omega;\mathbb{R}^3)$, we define the vector field

$$\boldsymbol{T} \wedge \boldsymbol{\varphi} := \begin{pmatrix} T_2\varphi_3 - T_3\varphi_2 \\ T_3\varphi_1 - T_1\varphi_3 \\ T_1\varphi_2 - T_2\varphi_1 \end{pmatrix} \in H^1_0(\Omega;\mathbb{R}^3), \tag{70}$$

and the matrix field

$$\boldsymbol{T} \otimes \boldsymbol{\varphi} := \begin{pmatrix} T_1\varphi_1 & T_1\varphi_2 & T_1\varphi_3 \\ T_2\varphi_1 & T_2\varphi_2 & T_2\varphi_3 \\ T_3\varphi_1 & T_3\varphi_2 & T_3\varphi_3 \end{pmatrix} \in H^1_0(\Omega;\mathbb{M}^3), \tag{71}$$

where the operator $\boldsymbol{T} = (T_i) : L^2_0(\Omega) \to H^1_0(\Omega;\mathbb{R}^3)$ is that defined in Lemma 2.5.

Remark 8. The notations (5.3)–(5.4) are to be viewed as symbolic, like the notation $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ often used to denote the vector field $((\partial_j u_i)u_j)$ found in the Navier-Stokes equations. □

Theorem 5.1. *Let $n = 3$ and let the assumptions be those of* Theorem 4.2*. With the notations of* (5.1) – (5.4)*, the Cesàro-Volterra formula with little regularity* (4.3) *becomes*

$$< \boldsymbol{u}, \boldsymbol{\varphi} > = \ll \boldsymbol{e}, \boldsymbol{T} \otimes \boldsymbol{\varphi} \gg + \ll \mathbf{CURL}\ \boldsymbol{e}, \boldsymbol{T} \otimes (\boldsymbol{T} \wedge \boldsymbol{\varphi}) \gg \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_1(\Omega;\mathbb{R}^3). \tag{72}$$

Proof. Formula (4.3) may be equivalently rewritten as

$$< u_i, \varphi_i >=< e_{ij}, T_i\varphi_j > - < \partial_k e_{ij}, T_i(T_j\varphi_k - T_k\varphi_j) > . \tag{73}$$

It is then easily verified that formula (5.5) is simply the vector-matrix form of formula (5.6), rewritten with the notations defined in (5.1)−(5.4) (recall that each function φ_j and each function $(T_j\varphi_k - T_k\varphi_j)$ belongs to the domain $L^2_0(\Omega)$ of the operators T_i when $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^3)$; cf. the proof of Theorem 4.2). □

While the first justification above is admittedly not fully convincing, the second one (given in Theorem 5.2 below) is clearly so, since it establishes that the *Cesàro-Volterra formula with little regularity reduces to the classical Cesàro-Volterra formula* (1.5) (reproduced in (5.8) below) *when the data are smooth enough.*

Note that *relation* (5.7) *below, which only involves the functions* e_{ij}*, is established without using that its left-hand side is also given by* $< u_i, \varphi_i >$, by Theorem 4.2 (otherwise this information would immediately provide a "proof" of (5.7), through the expression of $u_i(x)$ given by the classical Cesàro-Volterra formula (1.5)). In the same vein, note that the following proof clearly associates each term in the classical formula with a corresponding one in the formula with little regularity.

Finally, note that, by contrast with Theorem 5.1, the next result holds in any dimension $n \geq 2$.

Theorem 5.2. *Let the assumptions be those of* Theorem 4.2, *the functions* $e_{ij} = e_{ji} \in L^2(\Omega)$ *being in addition assumed to be in the space* $\mathcal{C}^1(\Omega) \cap H^1(\Omega)$*, and let the operator* $(T_i) : L^2_0(\Omega) \to H^1_0(\Omega; \mathbb{R}^n)$ *be that defined in* Lemma 2.5.

Fix a point $x_0 \in \Omega$*, and, given any point* $x \in \Omega$*, let* $\gamma(x)$ *be any path of class* $\mathcal{C}^1$ *contained in* Ω *and joining* x_0 *to* x*. Then the right-hand side of the Cesàro-Volterra formula with little regularity* (4.3) *can be rewritten in this case as*

$$\begin{aligned} &< e_{ij}, T_i\varphi_j + \partial_k[T_i(T_j\varphi_k - T_k\varphi_j)] > \\ &\quad = \int_\Omega \left[\int_{\gamma(x)} \{e_{ij}(y) + (\partial_k e_{ij}(y) - \partial_i e_{kj}(y))(x_k - y_k)\} dy_j \right] \varphi_i(x) dx \end{aligned} \tag{74}$$

for all $(\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$.

Relations (5.7) *in turn imply that any vector field* $(u_i) \in H^1(\Omega; \mathbb{R}^n)$ *that satisfies the Cesàro-Volterra formula with little regularity* (4.3) *is also given by*

$$u_i(x) = \int_{\gamma(x)} \{e_{ij}(y) + (\partial_k e_{ij}(y) - \partial_i e_{kj}(y))(x_k - y_k)\} dy_j, \; x \in \Omega, \tag{75}$$

up to the addition of a vector field of the form $x \in \Omega \mapsto \boldsymbol{a} + \boldsymbol{Aox}$ *for some* $\boldsymbol{a} \in \mathbb{R}^n$ *and* $\boldsymbol{A} \in \mathbb{A}^n$*. Besides,* $(u_i) \in \mathcal{C}^2(\Omega; \mathbb{R}^n)$ *in this case.*

Proof. (i) *A preliminary result : Let* Ω *be a simply-connected domain in* $\mathbb{R}^n$*, and let* $f_i \in \mathcal{C}^0(\Omega) \cap L^2(\Omega)$ *be functions that satisfy*

$$\partial_i f_j - \partial_j f_i = 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{76}$$

Fix a point $x_0 \in \Omega$ *and, given any point* $x \in \Omega$*, let* $\gamma(x)$ *be any path of class* $\mathcal{C}^1$ *contained in* Ω *and joining* x_0 *to* x*. Then*

$$< f_i, T_i\varphi >= \int_\Omega \left[\int_{\gamma(x)} f_j(y) dy_j \right] \varphi(x) dx \;\text{ for all }\; \varphi \in L^2_0(\Omega). \tag{77}$$

Relations (5.9) imply that there exists a function $\widetilde{u} \in L^2(\Omega)$ such that

$$\partial_i \widetilde{u} = f_i \text{ in } \mathcal{C}^0(\Omega) \cap L^2(\Omega), \tag{78}$$

so that $\widetilde{u} \in \mathcal{C}^1(\Omega) \cap H^1(\Omega)$. Therefore, given any function $\varphi \in L^2_0(\Omega)$, Green's formula gives (recall that $T_i\varphi \in H^1_0(\Omega)$) :

$$< f_i, T_i\varphi >=< \partial_i\widetilde{u}, T_i\varphi >= - < \widetilde{u}, \partial_i T_i\varphi >=< \widetilde{u}, \varphi >,$$

by definition of the operator (T_i).

Since the function $\widetilde{u} \in \mathcal{C}^1(\Omega)$ satisfies equation (5.11), its value $\widetilde{u}(x)$ at any point $x \in \Omega$ is given by the path integral

$$\widetilde{u}(x) = \widetilde{u}(x_0) + \int_{\gamma(x)} f_j(y)dy_j.$$

Consequently,

$$< f_i, T_i\varphi >= \widetilde{u}(x_0)\int_\Omega \varphi dx + \int_\Omega \left[\int_{\gamma(x)} f_j(y)dy_j\right]\varphi dx.$$

Hence the conclusion follows, since $\int_\Omega \varphi dx = 0$.

(ii) *Let the assumption be those of Theorem* 5.2. First we observe that any vector field $(u_i) \in H^1(\Omega;\mathbb{R}^n)$ that satisfies (4.3) is in the space $\mathcal{C}^2(\Omega;\mathbb{R}^n) \cap H^2(\Omega;\mathbb{R}^n)$, since the relations $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ imply that

$$\partial_{jk}u_i = \partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk} \text{ in } \mathcal{C}^0(\Omega) \text{ and } L^2(\Omega),$$

for all indices i, j, k. Noting that $(\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n)$ implies $\varphi_i \in L^2_0(\Omega)$ for each index i, we next infer from the preliminary result of (i) that

$$< e_{ij}, T_i\varphi_j >= \int_\Omega \left[\int_{\gamma(x)} e_{ij}(y)dy_j\right]\varphi_i(x)dx \text{ for all } (\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n), \tag{79}$$

which takes care of the first term appearing in the left-hand side of (5.7).

(iii) It remains to take care of the remaining term $< e_{ij}, \partial_k[T_i(T_j\varphi_k - T_k\varphi_j)] >$ appearing in the left-hand side of (5.7). To this end, we first recall that $(\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n)$ implies that $(T_j\varphi_k - T_k\varphi_j) \in L^2_0(\Omega)$ (cf. the proof of Theorem 4.2). Noting that $e_{ij} = e_{ji} \in H^1(\Omega)$ and that $T_i(T_j\varphi_k - T_k\varphi_j) \in H^1_0(\Omega)$, we next obtain, by Green's formula:

$$\begin{aligned} < e_{ij}, \partial_k[T_i(T_j\varphi_k - T_k\varphi_j)] > &= - < \partial_k e_{ij}, T_i(T_j\varphi_k - T_k\varphi_j) > \\ &= -\frac{1}{2} < \partial_k e_{ij} - \partial_j e_{ik}, T_i(T_j\varphi_k - T_k\varphi_j) > . \end{aligned} \tag{80}$$

The functions $h_{jki} := \partial_k e_{ij} - \partial_j e_{ik} \in L^2(\Omega)$ satisfy $\partial_l h_{jki} = \partial_i h_{jkl}$ in $H^{-1}(\Omega)$. Therefore the Poincaré lemma with little regularity (Theorem 3.1) shows that there exist functions $\widetilde{p}_{jk} = -\widetilde{p}_{kj} \in H^1(\Omega)$ such that

$$\partial_i \widetilde{p}_{jk} = \partial_k e_{ij} - \partial_j e_{ik} \text{ in } L^2(\Omega).$$

Combining another application of Green's formula with the defining property of the operator (T_i), the preliminary result of (i), and the antisymmetries $\widetilde{p}_{jk} = -\widetilde{p}_{kj}$, we then obtain

$$\begin{aligned} -\frac{1}{2} < \partial_k e_{ij} - \partial_j e_{ik}, T_i(T_j\varphi_k - T_k\varphi_j) > &= -\frac{1}{2} < \partial_i\widetilde{p}_{jk}, T_i(T_j\varphi_k - T_k\varphi_j) > \\ &= \frac{1}{2} < \widetilde{p}_{jk}, \partial_i T_i(T_j\varphi_k - T_k\varphi_j) >= -\frac{1}{2} < \widetilde{p}_{jk}, T_j\varphi_k - T_k\varphi_j > \\ &= -\frac{1}{2}\int_\Omega \left[\int_{\gamma(x)} \widetilde{p}_{jk}(y)dy_j\right]\varphi_k(x)dx + \frac{1}{2}\int_\Omega \left[\int_{\gamma(x)} \widetilde{p}_{kj}(y)dy_j\right]\varphi_k(x)dx \\ &= \int_\Omega \left[\int_{\gamma(x)} \widetilde{p}_{ij}(y)dy_j\right]\varphi_i(x)dx. \end{aligned} \tag{81}$$

The path $\gamma(x)$ can be written as $\gamma(x) = \boldsymbol{f}([0,1])$, where the mapping $\boldsymbol{f} = (f_j) \in \mathcal{C}^1([0,1];\mathbb{R}^n)$ satisfies $\boldsymbol{f}(0) = x_0$ and $\boldsymbol{f}(1) = x$. Consequently,

$$\begin{aligned}
\int_{\gamma(x)} \widetilde{p}_{ij}(y)dy_j &= \int_0^1 \widetilde{p}_{ij}(\boldsymbol{f}(t))\frac{df_j}{dt}(t)dt \\
&= -\int_0^1 \left[\frac{d}{dt}\big(\widetilde{p}_{ij}(\boldsymbol{f}(t))\big]f_j(t)dt + \widetilde{p}_{ij}(\boldsymbol{f}(1))f_j(1) - p_{ij}(\boldsymbol{f}(0))f_j(0) \\
&= -\int_0^1 \partial_j\widetilde{p}_{ik}(\boldsymbol{f}(t))f_k(t)\frac{df_j}{dt}(t)dt + x_k\widetilde{p}_{ik}(x) - x_k^0\widetilde{p}_{ik}(x_0) \\
&= -\int_{\gamma(x)} \partial_j\widetilde{p}_{ik}(y)y_k dy_j + x_k(\widetilde{p}_{ik}(x) - \widetilde{p}_{ik}(x_0)) + (x_k - x_k^0)\widetilde{p}_{ik}(x_0),
\end{aligned}$$

where x_k^0 designates the k-th coordinate of x_0. Since

$$\begin{aligned}
\widetilde{p}_{ik}(x) - \widetilde{p}_{ik}(x_0) &= \int_0^1 \frac{d}{dt}(\widetilde{p}_{ik}(\boldsymbol{f}(t))dt = \int_0^1 \partial_j\widetilde{p}_{ik}(\boldsymbol{f}(t))\frac{df_j}{dt}(t)dt \\
&= \int_{\gamma(x)} \partial_j\widetilde{p}_{ik}(y)dy_j,
\end{aligned}$$

it follows that

$$\begin{aligned}
\int_{\gamma(x)} \widetilde{p}_{ij}(y)dy_j &= (x_k - x_k^0)\widetilde{p}_{ik}(x_0) + \int_{\gamma(x)} (x_k - y_k)\partial_j\widetilde{p}_{ik}(y)dy_j \qquad (82)\\
&= (x_k - x_k^0)\widetilde{p}_{ik}(x_0) + \int_{\gamma(x)} (x_k - y_k)(\partial_k e_{ij}(y) - \partial_i e_{jk}(y))dy_j.
\end{aligned}$$

Combining relations (5.12)$-$(5.15) then yields

$$\begin{aligned}
&< e_{ij}, T_i\varphi_j + \partial_k[T_i(T_j\varphi_k - T_k\varphi_j)] > \qquad (83)\\
&= \int_\Omega \left[\int_{\gamma(x)} \{e_{ij}(y) + (\partial_k e_{ij}(y) - \partial_i e_{jk}(y))(x_k - y_k)\}dy_j\right]\varphi_i(x)dx \\
&+ \int_\Omega \widetilde{p}_{ik}(x_0)(x_k - x_k^0)\varphi_i(x)dx \text{ for all } (\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n).
\end{aligned}$$

(iv) By Lemma 2.3,

$$\int_\Omega \widetilde{p}_{ik}(x_0)(x_k - x_k^0)\varphi_i(x)dx = 0 \text{ for all } (\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n), \qquad (84)$$

since the matrix $(\widetilde{p}_{ik}(x_0))$ is antisymmetric. We therefore conclude from (5.16)$-$(5.17) that, when the functions $e_{ij} = e_{ji}$ belong to the space $\mathcal{C}^1(\Omega) \cap H^1(\Omega)$, any vector field $(u_i) \in H^1(\Omega;\mathbb{R}^n)$ that satisfies equations (4.3) for all $(\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n)$ also satisfies

$$< u_i, \varphi_i >= \int_\Omega \left[\int_{\gamma(x)} \{e_{ij}(y) + (\partial_k e_{ij}(y) - \partial_i e_{jk}(y))(x_k - y_k)\}dy_j\right]\varphi_i(x)dx$$

for all $(\varphi_i) \in \mathcal{D}_1(\Omega;\mathbb{R}^n)$, and is in the space $\mathcal{C}^2(\Omega;\mathbb{R}^n) \cap H^2(\Omega;\mathbb{R}^n)$ by part (ii).

Lemma 2.3 then shows that there exist a vector $(a_i) \in \mathbb{R}^n$ and an antisymmetric matrix $(a_{ij}) \in \mathbb{A}^n$ such that

$$u_i(x) = \left[\int_{\gamma(x)} \{e_{ij}(y) + (\partial_k e_{ij}(y) - \partial_i e_{jk}(y))(x_k - y_k)\}dy_j\right] + a_i + a_{ij}x_j$$

for all $x = (x_i) \in \Omega$, which completes the proof. □

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