Existence theorems in intrinsic nonlinear elasticity

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Abstract

We first show how the displacement-traction problem of nonlinear three-dimensional elasticity can be recast either as a boundary value problem or as a minimization problem over a Banach manifold, where the unknown is the Cauchy-Green strain tensor instead of the deformation as is customary. We then consider the pure displacement problem, and we show that, under appropriate smoothness assumptions on the data, either problem recast in this fashion possesses at least a solution if the applied forces are sufficiently small and the stored energy function satisfies specific hypotheses. In particular, the minimization problem provides an example where the functional is not coercive.

Résumé

On montre d'abord comment le problème en déplacement-traction de l'élasticité non linéaire tri-dimensionnelle peut être ré-écrit soit comme un problème aux limites, soit comme un problème de minimisation sur une variété de Banach, où l'inconnue est le tenseur des déformations de Cauchy-Green au lieu de la déformation comme il est usuel. On considère ensuite le problème en déplacement pur et nous montrons que, sous des hypothèses appropriées de régularité sur les données, chacun de ces problèmes ainsi ré-écrits possède au moins une solution si les forces appliquées sont suffisamment petites et si la densité d'énergie satisfait des hypothèses spécifiques. En particulier, le problème de minimisation constitue un exemple où la fonctionnelle n'est pas coercive.

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1. Introduction

In what follows, Ω denotes a bounded open subset of $\mathbb{R}^3$ with a smooth enough boundary, and Γ_0 denotes a relatively open subset of $\Gamma = \partial\Omega$.

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The principal aim of elasticity theory is to predict the *stress field* and the *deformation field* arising in an elastic body in response to given forces. Such a prediction is made either by *solving a system of partial differential equations*, or by *minimizing a functional* representing the *total energy* of the elastic body. At each point x of the reference configuration $\overline{\Omega} \subset \mathbb{R}^3$ of an elastic body, the *second Piola-Kirchhoff stress tensor* $\boldsymbol{\Sigma}(x)$ is a function of the gradient $\boldsymbol{\nabla\varphi}(x)$ of the deformation field $\boldsymbol{\varphi} : \overline{\Omega} \to \mathbb{R}^3$ by means of a *constitutive equation* of the form

$$\boldsymbol{\Sigma}(x) = \hat{\boldsymbol{\Sigma}}(x, \boldsymbol{\nabla\varphi}(x)),$$

where $\hat{\boldsymbol{\Sigma}}$ is a given *response function* that characterizes the elastic material.

The relation above shows that the deformation $\boldsymbol{\varphi}$ can be considered as the single *primary* unknown, the stress field $\boldsymbol{\Sigma}$ being then recovered by means of the above constitutive equation. This observation, which is the basis of the *classical approach*, led to the *only two existence theorems* known as of now in nonlinear elasticity, one based on the *implicit function theorem*, and one, due to John Ball, based on the *minimisation of the total energy* (these results are briefly recalled in Section 5).

Another approach, called the *intrinsic approach*, is slowly coming out of age, however, after it was (apparently for the first time) suggested by Antman [2]. This new approach is based on the following two observations. Because of the *principle of material frame-indifference*, the stress tensor $\boldsymbol{\Sigma}(x)$ depends on the deformation $\boldsymbol{\varphi}$ in fact only via its associated *Cauchy-Green tensor* field $\boldsymbol{C} = \boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi}$. In other words, there exists another *response function* $\tilde{\boldsymbol{\Sigma}}$ such that the following *constitutive equation* holds:

$$\boldsymbol{\Sigma}(x) = \tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{C}(x)) \text{ at each point } x \in \overline{\Omega}.$$

The second observation is provided by a well-known theorem asserting that the deformation $\boldsymbol{\varphi}$ can be recovered (up to a rigid body motion) from the tensor field $\boldsymbol{C}$ provided the latter field satisfies specific *compatibility conditions*. *Therefore the tensor field $\boldsymbol{C}$ can also be considered as the primary unknown in elasticity theory*, since both the stress field $\boldsymbol{\Sigma}$ and the deformation $\boldsymbol{\varphi}$ are functions of $\boldsymbol{C}$. This is the basis of the intrinsic approach.

Note that an intrinsic approach also directly provides, by means of the constitutive equation, the *stress tensor field* $\boldsymbol{\Sigma}$, which is often the unknown of primary interest from the mechanical and computational viewpoints.

The main objective of this paper is to provide new *existence, uniqueness*, and *regularity theorems* for the equations of intrinsic nonlinear three-dimensional elasticity, i.e., when the primary unknown is the field $\boldsymbol{C}$.

In order to recast the *displacement-traction problem of nonlinear elasticity*, i.e., with a boundary condition $\boldsymbol{\varphi} = \boldsymbol{id}$ on Γ_0 imposed on the admissible deformations, in terms of the Cauchy-Green tensor $\boldsymbol{C}$ as the primary unknown instead of the deformation $\boldsymbol{\varphi}$ as in the classical approach, we need to characterize those matrix fields that can be considered as Cauchy-Green tensors constructed from deformations that are admissible for the displacement-traction problem.

More specifically, we show (Theorem 2.2) that the *set of admissible Cauchy-Green strain tensors* that is appropriate for our purposes take the following form (all the relevant definitions and notations not defined here are defined in Section 2):

$$\begin{aligned}\mathbb{T}(\Omega) = \{\boldsymbol{C} \in W^{2,s}(\Omega; \mathbb{S}^3);\ & \boldsymbol{C}(x) \in \mathbb{S}^3_> \text{ for all } x \in \overline{\Omega},\ R^p_{\cdot ijk}(\boldsymbol{C}) = 0 \text{ in } L^s(\Omega), \\ & A_x(\boldsymbol{C}) = A_x(\boldsymbol{I}) \text{ and } B_x(\boldsymbol{C}) = B_x(\boldsymbol{I}) \text{ on } T_x\Gamma_0 \times T_x\Gamma_0\}\end{aligned}$$

where s is any real number that satisfies $s > 3/2$, the functions $R^p_{\cdot ijk}(\boldsymbol{C})$ are the components of the Riemann tensor, and the boundary condition along Γ_0 express that the two fundamental forms of the surface Γ_0 and of the corresponding deformed surface are the same (this is the way the boundary condition $\boldsymbol{\varphi} = \boldsymbol{id}$ on Γ_0 is expressed in the intrinsic approach).

We then continue our analysis by showing that, when $\Gamma_0 = \Gamma$, the set $\mathbb{T}(\Omega)$ is a *Banach manifold of class* C^∞ *in the space* $W^{2,s}(\Omega; \mathbb{S}^3)$ (Theorem 4.1). To this end, we proceed along the same lines as in C. Mardare [12].

The *pure displacement problem* of three-dimensional nonlinear elasticity classically takes the form

$$-\mathbf{div}\,\{\boldsymbol{\nabla\varphi}\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi})\} = \boldsymbol{f} \text{ in } \Omega \text{ and } \boldsymbol{\varphi} = \boldsymbol{id} \text{ on } \Gamma.$$

We then show that the intrinsic formulation of the same problem consists in seeking a tensor field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ that satisfies

$$-\mathbf{div}\,\{\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{C})\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\} = \boldsymbol{f} \text{ in } \Omega,$$

where $\mathcal{G}$ is an ad hoc C^∞-diffeomorphism from the set $\mathbb{T}(\Omega)$ onto the set

$$\{\boldsymbol{\varphi} \in W^{3,s}(\Omega; \mathbb{R}^3);\ \inf_{x\in\Omega} \det \boldsymbol{\nabla\varphi}(x) > 0,\ \boldsymbol{\varphi} = \boldsymbol{id} \text{ on } \Gamma\}$$

(cf. Theorem 3.1, where the displacement-traction problem is also considered). We next show (Theorem 6.1) that this pure displacement problem has a unique solution in a neighborhood of the identity in the Banach manifold $\mathbb{T}(\Omega)$ if the applied force density is sufficiently small in the space $W^{1,s}(\Omega; \mathbb{R}^3)$. The proof makes an essential use of the *implicit function theorem in a Banach manifold*, in the form given in Abraham, Marsden & Ratiu [1].

We assume next that the elastic material is *hyperelastic*, with a stored energy function $\tilde{W}$ of the form proposed by Ciarlet & Geymonat [8]. We then show that the intrinsic formulation of the associated minimization problem, which consists in seeking a tensor field $\boldsymbol{C}_0 \in \mathbb{T}(\Omega)$ that satisfies

$$\mathcal{I}(\boldsymbol{C}_0) = \inf_{\boldsymbol{C}\in\mathbb{T}(\Omega)} \mathcal{I}(\boldsymbol{C}), \text{ where } \mathcal{I}(\boldsymbol{C}) = \int_\Omega \tilde{W}(\cdot, \boldsymbol{C})dx - \int_\Omega \boldsymbol{f}\cdot\mathcal{G}(\boldsymbol{C})dx,$$

has a unique solution, again in a neighborhood of the identity in the Banach manifold $\mathbb{T}(\Omega)$ if the applied force density is small enough in the space $W^{1,s}(\Omega; \mathbb{R}^3)$ (Theorem 7.3). The proof relies in particular on the comparison, due to Zhang [16], between the minimizers found in the fundamental existence theorem of Ball [3] for the classical approach and the solution found by the implicit function theorem, also applied to the classical approach.

It is worth noticing that the minimization problem solved here in the intrinsic approach provides an example where the functional, which is defined on a Banach manifold, is not coercive.

2. The set of admissible Cauchy-Green tensor fields

Throughout this paper, Ω denotes a *bounded*, *simply-connected*, *open* subset of $\mathbb{R}^3$, with a boundary $\Gamma := \partial\Omega$ of class C^4, Γ_0 denotes a *non-empty, connected*, *relatively open* subset of Γ, $\Gamma_1 := \Gamma \setminus \Gamma_0$, and $s > 3/2$ is a real number.

The notation $\mathbb{M}^3$, $\mathbb{M}^3_+$, $\mathbb{S}^3$, $\mathbb{S}^3_>$, $\mathbb{O}^3$, and $\mathbb{O}^3_+$ respectively designate the space of all square matrices of order three, the set of all matrices $\boldsymbol{F} \in \mathbb{M}^3$ such that $\det \boldsymbol{F} > 0$, the space of all symmetric matrices of order three, the set of all positive-definite symmetric matrices of order three, the set of all orthogonal matrices of order three, and the set of all proper orthogonal matrices of order three. Latin indices and exponents take their values in the set $\{1, 2, 3\}$ and Greek indices and exponents take their values in the set $\{1,2\}$, and the summation convention for repeated indices and exponents is used in conjunction with these rules.

A *deformation* of an elastic body with $\overline{\Omega}$ as its reference configuration is a smooth enough mapping $\boldsymbol{\varphi} : \overline{\Omega} \to \mathbb{R}^3$ that is *orientation preserving* (i.e., $\det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) > 0$ for all $x \in \overline{\Omega}$) and *injective* on the open set Ω (i.e., no interpenetration of matter occurs). For the displacement-traction problem, a deformation $\boldsymbol{\varphi}$ is called *admissible* if $\boldsymbol{\varphi}(x) = x$ for all $x \in \Gamma_0$, which means that the body is kept fixed on a portion Γ_0 of its boundary Γ (a more general boundary condition of the type $\boldsymbol{\varphi}(x) = \boldsymbol{\varphi}_0(x)$ for all $x \in \Gamma_0$, where $\boldsymbol{\varphi}_0 : \Gamma_0 \to \mathbb{R}^3$ is the trace on Γ_0 of a given function in $W^{3,s}(\Omega;\mathbb{R}^3)$ could be as well considered).

The *set of admissible deformations* that is best suited for our subsequent purposes turns out to be

$$\boldsymbol{D}(\Omega) := \{\boldsymbol{\varphi} \in W^{3,s}(\Omega;\mathbb{R}^3);\ \inf_{x\in\Omega} \det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) > 0,\ \boldsymbol{\varphi} = \boldsymbol{id} \text{ on } \Gamma_0\}, \tag{1}$$

for some $s > 3/2$. Note that $W^{3,s}(\Omega;\mathbb{R}^3) \subset W^{2,2s}(\Omega;\mathbb{R}^3) \subset C^1(\overline{\Omega};\mathbb{R}^3)$ and that the space $W^{2,s}(\Omega)$ is in fact an algebra since Ω is a three-dimensional domain and $2s > 3$. Note also that the requirement that $\boldsymbol{\varphi}$ be injective in Ω has been dropped from the definition of $\boldsymbol{D}(\Omega)$, as the injectivity is an issue that needs to be treated separately (see Remarks 6.2 and 7.4).

Remark 2.1. The condition $\inf_{x\in\Omega} \det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) > 0$ appearing in (1) simply means that any admissible deformation is orientation preserving in $\overline{\Omega}$ (naturally, $\inf_{x\in\Omega} \det \boldsymbol{\nabla}\boldsymbol{\varphi}(x)$ depends on $\boldsymbol{\varphi}$). □

With any deformation $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$, we associate the *Cauchy-Green tensor* $\boldsymbol{C}$, the *Christoffel symbols* Γ^k_{ij}, and the mixed components $R^p_{\cdot ijk}(\boldsymbol{C})$ of the *Riemann tensor field*, by letting

$$\begin{aligned}
&\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi},\ g_{ij} = (\boldsymbol{C})_{ij},\ g^{k\ell} := (\boldsymbol{C}^{-1})_{kl},\\
&\Gamma^k_{ij} := \frac{1}{2} g^{k\ell}\Big(\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{\ell i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell}\Big),\\
&R^p_{\cdot ijk}(\boldsymbol{C}) := \frac{\partial \Gamma^p_{ik}}{\partial x_j} - \frac{\partial \Gamma^p_{ij}}{\partial x_k} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell}.
\end{aligned}$$

Note that $\boldsymbol{C} \in W^{2,s}(\Omega;\mathbb{S}^3)$, $\Gamma^k_{ij} \in W^{1,s}(\Omega)$, and $R^p_{\cdot ijk} \in L^s(\Omega)$.

The corresponding *set of admissible Cauchy-Green tensors* is then naturally defined as the image

$$\mathbb{T}(\Omega) := \mathcal{F}(\boldsymbol{D}(\Omega))$$

through the mapping

$$\mathcal{F} : \boldsymbol{\varphi} \in W^{3,s}(\Omega;\mathbb{R}^3) \to \mathcal{F}(\boldsymbol{\varphi}) := \boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi} \in W^{2,s}(\Omega;\mathbb{S}^3).$$

Our first goal is to characterize the set $\mathbb{T}(\Omega)$ *without resorting to the mapping* $\mathcal{F}$.

First, since $W^{2,s}(\Omega;\mathbb{R}^3) \subset C^0(\overline{\Omega};\mathbb{R}^3)$, every matrix field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ is continuous over $\overline{\Omega}$; this means that each equivalence class $\boldsymbol{C}$ contains one and only one matrix field that is continuous over $\overline{\Omega}$. Hence the matrix $\boldsymbol{C}(x)$ is positive definite at all $x \in \overline{\Omega}$. Next, it is well known that the matrix field $\boldsymbol{C}$ necessarily satisfies the equations

$$R^p_{\cdot ijk}(\boldsymbol{C}) = 0 \text{ in } L^s(\Omega).$$

It thus remains to recast the boundary condition $\boldsymbol{\varphi} = \boldsymbol{id}$ on Γ_0 in terms of the matrix field $\boldsymbol{C}$. To this end, we will use the *fundamental theorem of surface theory*, which asserts that a sufficiently regular surface is uniquely determined up to a rigid motion of $\mathbb{R}^3$ by its two fundamental forms. More specifically, we will use the "optimal" version of this theorem due to S. Mardare [14, Theorem 9], where it is shown that the minimal regularity of the immersion that defines the surface is $W^{2,p}_{loc}$, $p > 2$.

Since, when viewed as surface tensors, the fundamental forms are intrinsic, i.e., they are independent of the choice of the immersion defining the surface, the condition $\boldsymbol{\varphi}(x) = x$ for all $x \in \Gamma_0$ is equivalent, up to a rigid motion of $\mathbb{R}^3$, to the condition that the two fundamental forms defined by the immersion $\boldsymbol{\varphi}|_{\Gamma_0}$ coincide with the two fundamental forms defined by the immersion $\boldsymbol{id}|_{\Gamma_0}$. Note that these immersions satisfy the hypotheses of [14, Theorem 9] since they belong to the space $W^{3-1/s,s}(\Gamma_0; \mathbb{R}^3)$, which, by virtue of the assumption $s > 3/2$, is contained in the space $W^{2,p}(\Gamma_0; \mathbb{R}^3)$ for some $p > 2$, by the Sobolev embedding theorem. Thus, to achieve our goal, it remains to express the fundamental forms of $\boldsymbol{\varphi}|_{\Gamma_0}$ and $\boldsymbol{id}|_{\Gamma_0}$ in terms of the matrix fields $\boldsymbol{C}$ and $\boldsymbol{I}$, respectively.

It is well known (see, e.g., [5]) that the first and second fundamental forms induced by the immersion $\boldsymbol{\varphi}|_{\Gamma_0}$ at a point $x \in \Gamma_0$ of the surface Γ_0 are the restrictions to the space $T_x\Gamma_0 \times T_x\Gamma_0$ of the bilinear forms

$$\boldsymbol{A}_x(\boldsymbol{C}) : (\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \boldsymbol{a}^T \boldsymbol{C}(x)\boldsymbol{b} \in \mathbb{R}$$

and

$$\boldsymbol{B}_x(\boldsymbol{C}) : (\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto -\frac{1}{2}\boldsymbol{a}^T(\mathcal{L}_{\boldsymbol{n}(\boldsymbol{C})}\boldsymbol{C})(x)\boldsymbol{b} \in \mathbb{R},$$

respectively, where $T_x\Gamma_0$ denotes the tangent space of Γ_0 at $x \in \Gamma_0$, $\mathcal{L}$ denotes the Lie derivative, and $\boldsymbol{n}(\boldsymbol{C})$ is a C^1-extension in a neighborhood of Γ_0 of a vector field that is unit and normal to the surface Γ_0 with respect to the metric in $\mathbb{R}^3$ induced by the field $\boldsymbol{C}$. In other words, the vector field $\boldsymbol{n}(\boldsymbol{C})$ is defined at $x \in \Gamma_0$ by the relations

$$\boldsymbol{a}^T \boldsymbol{C}(x)\boldsymbol{n}(\boldsymbol{C})(x) = 0 \text{ for all } \boldsymbol{a} \in T_xS \text{ and } \boldsymbol{n}(\boldsymbol{C})(x)^T \boldsymbol{C}(x)\boldsymbol{n}(\boldsymbol{C})(x) = 1.$$

To fix the sign of the second fundamental form, we choose $\boldsymbol{n}(\boldsymbol{C})$ pointing towards the inside of Ω. Note that the above expression of the second fundamental form does not depend on the choice of the extension $\boldsymbol{n}(\boldsymbol{C})$.

Consequently, up to a rigid motion of $\mathbb{R}^3$, the boundary condition $\boldsymbol{\varphi} = \boldsymbol{id}$ on Γ_0 is equivalent to the relation

$$\boldsymbol{A}_x(\boldsymbol{C}) = \boldsymbol{A}_x(\boldsymbol{I}) \text{ and } \boldsymbol{B}_x(\boldsymbol{C}) = \boldsymbol{B}_x(\boldsymbol{I}) \text{ on } T_x\Gamma_0 \times T_x\Gamma_0 \text{ for all } x \in \Gamma_0.$$

As we will show elsewhere [10], using *local curvilinear systems* for defining the surface Γ_0 allows to re-write the boundary conditions as explicit expressions in terms of the components of the tensor field $\boldsymbol{C}$ (the assumption that Γ is of class C^4 is needed here).

We are now in a position to characterize those matrix fields that are Cauchy-Green tensors induced by those deformations that are admissible for the displacement-traction problem of nonlinear elasticity.

Theorem 2.2. *The set of admissible Cauchy-Green tensors*

$$\mathbb{T}(\Omega) := \mathcal{F}(\boldsymbol{D}(\Omega))$$

is also given by

$$\begin{aligned}\mathbb{T}(\Omega) = \{&\boldsymbol{C} \in W^{2,s}(\Omega;\mathbb{S}^3);\ \boldsymbol{C}(x) \in \mathbb{S}^3_> \text{ for all } x \in \overline{\Omega},\ R^p_{\cdot ijk}(\boldsymbol{C}) = 0 \text{ in } L^s(\Omega),\\ &\boldsymbol{A}_x(\boldsymbol{C}) = \boldsymbol{A}_x(\boldsymbol{I}) \text{ and } \boldsymbol{B}_x(\boldsymbol{C}) = \boldsymbol{B}_x(\boldsymbol{I}) \text{ on } T_x\Gamma_0 \times T_x\Gamma_0 \text{ for all } x \in \Gamma_0\}.\end{aligned} \tag{2}$$

Besides, the mapping $\mathcal{F}$ is a homeomorphism from $\boldsymbol{D}(\Omega)$ onto its image $\mathbb{T}(\Omega)$.

Proof. (i) That the set $\mathcal{F}(\boldsymbol{D}(\Omega))$ is contained in the set appearing in the right-hand side of the relation (2) is a consequence of the above considerations.

To prove the other inclusion, let a matrix field $\boldsymbol{C}$ belong to the set defined by the right-hand side of (2). Since Ω is simply-connected and $W^{2,s}(\Omega;\mathbb{S}^3) \subset W^{1,p}(\Omega;\mathbb{S}^3)$ for some $p > 3$, a generalization due to S. Mardare [13, 15] of the fundamental theorem of Riemannian geometry for an open subset in $\mathbb{R}^n$ shows that there exists a vector field $\boldsymbol{\varphi} \in W^{2,p}_{loc}(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi} = \boldsymbol{C}$ in Ω.

Moreover, such a field $\boldsymbol{\varphi}$ is unique up to rigid body motions: a mapping $\tilde{\boldsymbol{\varphi}} \in W^{2,p}_{loc}(\Omega;\mathbb{R}^3)$ satisfies $\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}} = \boldsymbol{C}$ in Ω if and only if there exist a vector $\boldsymbol{a} \in \mathbb{R}^3$ and an orthogonal matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\tilde{\boldsymbol{\varphi}}(x) = \boldsymbol{a} + \boldsymbol{Q\varphi}(x)$ at all $x \in \Omega$.

The vector field $\boldsymbol{\varphi}$ belongs in fact to the space $W^{3,s}(\Omega;\mathbb{R}^3)$. To see this, note that the Sobolev embedding theorem implies that $\boldsymbol{C} \in C^0(\overline{\Omega};\mathbb{S}^3)$ since $s > 3/2$. Since

$$|\boldsymbol{\nabla\varphi}|^2 = \operatorname{tr}(\boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi}) = \operatorname{tr}\boldsymbol{C} \text{ and } \operatorname{tr}\boldsymbol{C} \in L^\infty(\Omega),$$

it follows that $\boldsymbol{\nabla\varphi} \in L^\infty(\Omega;\mathbb{M}^3)$. Combined with the equations

$$\frac{\partial^2\boldsymbol{\varphi}}{\partial x_i\partial x_j} = \Gamma^k_{ij}\frac{\partial\boldsymbol{\varphi}}{\partial x_k}$$

(which are consequences of the equation $\boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi} = \boldsymbol{C}$), and with the relations $\Gamma^k_{ij} \in W^{1,s}(\Omega) \subset L^{2s}(\Omega)$, this implies that $\dfrac{\partial^2\boldsymbol{\varphi}}{\partial x_i\partial x_j} \in L^{2s}(\Omega)$. Since the set Ω is bounded and has a sufficiently smooth boundary, this implies in turn that $\boldsymbol{\varphi} \in W^{2,2s}(\Omega;\mathbb{R}^3)$. Using now the equations

$$\frac{\partial^3\boldsymbol{\varphi}}{\partial x_i\partial x_j\partial x_\ell} = \frac{\partial\Gamma^k_{ij}}{\partial x_\ell}\frac{\partial\boldsymbol{\varphi}}{\partial x_k} + \Gamma^k_{ij}\frac{\partial^2\boldsymbol{\varphi}}{\partial x_k\partial x_\ell}$$

and the relations $\Gamma^k_{ij} \in W^{1,s}(\Omega) \subset L^{2s}(\Omega)$, we infer that $\dfrac{\partial^3\boldsymbol{\varphi}}{\partial x_i\partial x_j\partial x_\ell} \in L^s(\Omega;\mathbb{R}^3)$; hence $\boldsymbol{\varphi} \in W^{3,s}(\Omega;\mathbb{R}^3)$.

Since $\boldsymbol{\varphi} \in C^1(\overline{\Omega};\mathbb{R}^3)$ (by the Sobolev embedding theorem) and

$$(\boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi})(x) = \boldsymbol{C}(x) \in \mathbb{S}^3_> \text{ for all } x \in \overline{\Omega},$$

the vector field $\boldsymbol{\varphi}$ satisfies either $\inf_{x\in\Omega}\boldsymbol{\nabla\varphi}(x) > 0$ for all $x \in \overline{\Omega}$, or $\inf_{x\in\Omega}\boldsymbol{\nabla\varphi}(x) < 0$ for all $x \in \overline{\Omega}$. Since $\boldsymbol{\varphi}$ is defined up to a rigid body motion in $\mathbb{R}^3$, we may choose $\boldsymbol{\varphi}$ to satisfy the condition

$$\inf_{x\in\Omega}\boldsymbol{\nabla\varphi}(x) > 0 \text{ for all } x \in \overline{\Omega}.$$

The restriction $\boldsymbol{\varphi}|_{\Gamma_0}$ belongs to the space $W^{3-1/s,s}(\Gamma_0;\mathbb{R}^3)$. Hence the Sobolev embedding theorem implies that $\boldsymbol{\varphi}|_{\Gamma_0} \in W^{2,p}(\Gamma_0;\mathbb{R}^3)$ for some $p > 2$. The relations $\boldsymbol{A}_x(\boldsymbol{C}) = \boldsymbol{A}_x(\boldsymbol{I})$ and

$\boldsymbol{B}_x(\boldsymbol{C}) = \boldsymbol{B}_x(\boldsymbol{I})$ on $T_x\Gamma_0 \times T_x\Gamma_0$ for all $x \in \Gamma_0$ satisfied by $\boldsymbol{C}$ means that the first and second fundamental forms associated with the immersion $\boldsymbol{\varphi}|_{\Gamma_0}$ coincide respectively with the first and second fundamental forms associated with the immersion $\boldsymbol{id}|_{\Gamma_0}$. Since both immersions belong to the space $W^{2,p}(\Gamma_0;\mathbb{R}^3)$ and since Γ_0 is connected, the uniqueness part of the fundamental theorem of surface theory in its generalized form due to S. Mardare [14, Theorem 9] shows that there exist a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a proper orthogonal matrix $\boldsymbol{Q}$ such that $x = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\varphi}(x)$ at all $x \in \Gamma_0$.

The above arguments show that the vector field $\tilde{\boldsymbol{\varphi}} : \Omega \to \mathbb{R}^3$ defined by $\tilde{\boldsymbol{\varphi}}(x) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\varphi}(x)$ at all $x \in \Omega$ belongs to the set $\boldsymbol{D}(\Omega)$ and $\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}} = \boldsymbol{C}$ in Ω.

(ii) *The mapping $\mathcal{F}$ is a homeomorphism from $\boldsymbol{D}(\Omega)$ onto its image $\mathbb{T}(\Omega) = \mathcal{F}(\boldsymbol{D}(\Omega))$.* We need to prove that the mapping $\mathcal{F}|_{\boldsymbol{D}(\Omega)}$ is injective, continuous, and that its inverse $\mathcal{G} : \mathbb{T}(\Omega) \to W^{3,s}(\Omega;\mathbb{R}^3)$ is also continuous.

If two fields $\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}} \in \boldsymbol{D}(\Omega)$ satisfy $\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}^T\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}} = \boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi}$, then there exist a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a proper orthogonal matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that $\tilde{\boldsymbol{\varphi}}(x) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\varphi}(x)$ for all $x \in \Omega$ (cf. [13, 15]). Then the boundary conditions $\tilde{\boldsymbol{\varphi}}(x) = \boldsymbol{\varphi}(x) = x$ for all $x \in \Gamma_0$ imply that $\boldsymbol{a} = \boldsymbol{0}$ and $\boldsymbol{Q} = \boldsymbol{0}$. Hence the mapping $\mathcal{F}|_{\boldsymbol{D}(\Omega)}$ is *injective*.

The mapping $\mathcal{F}|_{\boldsymbol{D}(\Omega)}$ is *continuous* thanks to the Sobolev embedding theorem.

It remains to prove that its inverse is also continuous. First, an argument similar to that used in Ciarlet & C. Mardare [9] (where different function spaces were used) shows that, given any tensor field $\boldsymbol{C} = \mathcal{F}(\boldsymbol{\varphi}) \in \mathbb{T}(\Omega)$, there exist constants $c_0, \delta > 0$ such that

$$\inf_{\boldsymbol{a}\in\mathbb{R}^3, \boldsymbol{Q}\in\mathbb{O}^3_+} \|\tilde{\boldsymbol{\varphi}} - (\boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\varphi})\|_{W^{3,s}(\Omega)} \leq c_0\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}$$

for all $\tilde{\boldsymbol{C}} = \mathcal{F}(\tilde{\boldsymbol{\varphi}}) \in \mathbb{T}(\Omega)$ satisfying $\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)} < \delta$. The set $\mathbb{O}^3_+$ being compact and the space $\mathbb{R}^3$ finite-dimensional, the infimum is attained at some vector $\tilde{\boldsymbol{a}} \in \mathbb{R}^3$ and matrix $\tilde{\boldsymbol{Q}} \in \mathbb{O}^3_+$ that depend on $\tilde{\boldsymbol{\varphi}}$. Combined with the above inequality and with the Sobolev embedding theorem, this implies that

$$\|\tilde{\boldsymbol{\varphi}} - (\tilde{\boldsymbol{a}} + \tilde{\boldsymbol{Q}}\boldsymbol{\varphi})\|_{C^0(\overline{\Omega})} \leq c_0\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}.$$

Using now the boundary condition $\tilde{\boldsymbol{\varphi}}(x) = \boldsymbol{\varphi}(x) = x$ for all $x \in \Gamma_0$, we infer from the above inequality that, in particular,

$$|\tilde{\boldsymbol{a}} + (\tilde{\boldsymbol{Q}} - \boldsymbol{I})x| \leq c_0\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)} \text{ for all } x \in \Gamma_0,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. Let three points x_0, x_1, x_2 in Γ_0 be such that the vectors $(x_1 - x_0)$ and $(x_2 - x_0)$ are linearly independent. Then the previous inequality shows that

$$|\tilde{\boldsymbol{a}} + (\tilde{\boldsymbol{Q}} - \boldsymbol{I})x_0| \leq c_0\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}$$

and

$$|(\tilde{\boldsymbol{Q}} - \boldsymbol{I})(x_1 - x_0)| + |(\tilde{\boldsymbol{Q}} - \boldsymbol{I})(x_2 - x_0)| \leq 4c_0\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}.$$

The last inequality implies that there exists a constant c_1, depending only on the constant c_0 and on the points x_0, x_1, x_2, such that

$$|(\tilde{\boldsymbol{Q}} - \boldsymbol{I})\boldsymbol{b}| \leq c_1\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}$$

for all unit vector $\boldsymbol{b}$ in the plane $\mathcal{H}$ spanned by the vectors $(x_1 - x_0)$ and $(x_2 - x_0)$. Let $\boldsymbol{c} \in \mathbb{R}^3$ be a unit vector normal to $\mathcal{H}$. Since the matrix $\tilde{\boldsymbol{Q}}$ is proper orthogonal, the vector $\tilde{\boldsymbol{Q}}\boldsymbol{c}$ is unit and normal to the plane $\tilde{\boldsymbol{Q}}(\mathcal{H})$. It follows that the angle between the vectors $\boldsymbol{c}$ and $\tilde{\boldsymbol{Q}}\boldsymbol{c}$ is equal to the angle between the planes $\mathcal{H}$ and $\tilde{\boldsymbol{Q}}(\mathcal{H})$, so that

$$|(\tilde{\boldsymbol{Q}} - \boldsymbol{I})\boldsymbol{c}| = \sup_{\boldsymbol{b}\in\mathcal{H},|\boldsymbol{b}|=1} |(\tilde{\boldsymbol{Q}} - \boldsymbol{I})\boldsymbol{b}|.$$

Therefore,

$$|\tilde{\boldsymbol{Q}} - \boldsymbol{I}| = \sup_{\boldsymbol{v}\in\mathbb{R}^3,|\boldsymbol{v}|=1} |(\tilde{\boldsymbol{Q}} - \boldsymbol{I})\boldsymbol{v}| \leq 2c_1\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}.$$

There thus exists a constant c_2 such that

$$|\tilde{\boldsymbol{a}}| + |\tilde{\boldsymbol{Q}} - \boldsymbol{I}| \leq c_2\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}.$$

Therefore, there exists a constant c depending only on $\boldsymbol{\varphi}$ such that

$$\begin{aligned}\|\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}\|_{W^{3,s}(\Omega)} &\leq \|\tilde{\boldsymbol{\varphi}} - (\tilde{\boldsymbol{a}} + \tilde{\boldsymbol{Q}}\boldsymbol{\varphi})\|_{W^{3,s}(\Omega)} + \|\tilde{\boldsymbol{a}}\|_{W^{3,s}(\Omega)} + \|(\tilde{\boldsymbol{Q}} - \boldsymbol{I})\boldsymbol{\varphi}\|_{W^{3,s}(\Omega)} \\ &\leq c\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{W^{2,s}(\Omega)}.\end{aligned}$$

This inequality shows that *the inverse of the mapping $\mathcal{F}|_{\boldsymbol{D}(\Omega)}$ is continuous*. □

3. Intrinsic formulations of the displacement-traction problem of nonlinear elasticity

For details about the modeling of three-dimensional nonlinear elasticity, either as a boundary value problem or as a minimization problem, see, e.g., Ciarlet [6, Chapters 1-5]. The assumptions on the sets Ω and Γ_0 and on the number s are the same as in Section 2.

Consider an elastic body with reference configuration $\overline{\Omega}$, assumed to be held fixed on the portion Γ_0 of the boundary $\Gamma := \partial\Omega$, and let $\Gamma_1 := \Gamma \setminus \Gamma_0$.

The main objective of elasticity theory is to determine the deformation $\boldsymbol{\varphi} : \overline{\Omega} \to \mathbb{R}^3$ undergone by the elastic body in the presence of *applied body and surface forces*, given by their densities $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ and $\boldsymbol{h} : \Gamma_1 \to \mathbb{R}^3$ per unit volume and per unit area, respectively; for simplicity, we assume here that the applied forces are *dead loads*, i.e., that they do not depend on the unknown deformation $\boldsymbol{\varphi}$.

This objective is achieved in two stages. First, thanks to the *stress principle of Euler and Cauchy* and to *Cauchy's theorem*, this amounts to finding the *second Piola-Kirchhoff stress tensor* $\boldsymbol{\Sigma} : \overline{\Omega} \to \mathbb{S}^3$ that, together with the deformation $\boldsymbol{\varphi}$, satisfy the following *equations of equilibrium in the reference configuration*:

$$\begin{aligned} -\mathbf{div}\,(\boldsymbol{\nabla}\boldsymbol{\varphi}\boldsymbol{\Sigma}) = \boldsymbol{f} \quad & \text{in } \Omega, \\ \boldsymbol{\varphi} = \boldsymbol{id} \quad & \text{on } \Gamma_0, \\ (\boldsymbol{\nabla}\boldsymbol{\varphi}\boldsymbol{\Sigma})\boldsymbol{n} = \boldsymbol{h} \quad & \text{on } \Gamma_1, \end{aligned} \tag{3}$$

where $\boldsymbol{n}$ denotes the unit outer normal vector field along Γ_1.

Second, the above equations of equilibrium must be supplemented by the *constitutive equation* of the elastic material, which relates the stress tensor field $\boldsymbol{\Sigma}$ and the deformation $\boldsymbol{\varphi}$ by means of a given function $\hat{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{S}^3$, called the *response function* of the elastic material under consideration, as

$$\boldsymbol{\Sigma}(x) = \hat{\boldsymbol{\Sigma}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) \ \text{ for all } x \in \overline{\Omega}. \tag{4}$$

The system formed by the equations (3) and (4) constitute the *displacement-traction problem of nonlinear elasticity*. In the particular cases where $\Gamma_0 = \Gamma$, or $\Gamma_0 = \emptyset$, this problem is respectively called a *pure displacement problem*, or a *pure traction problem*.

The *classical formulation* of the displacement-traction problem consists in replacing the unknown $\boldsymbol{\Sigma}$ in the equations (3) by its expression given by the constitutive equation (4), so that the deformation $\boldsymbol{\varphi}$ becomes in effect the sole *primary unknown*.

The *intrinsic formulation* of the displacement-traction problem, which we will now introduce, consists in replacing both unknowns $\boldsymbol{\Sigma}$ and $\boldsymbol{\varphi}$ in terms of the corresponding *Cauchy-Green tensor* $\boldsymbol{C}$, so that this tensor becomes in effect the sole *primary unknown*.

On the one hand, the *principle of material frame-indifference* implies that there exists a function $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}^3_> \to \mathbb{S}^3$ such that

$$\boldsymbol{\Sigma}(x) = \tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{C}(x)) \text{ for all } x \in \overline{\Omega}.$$

(compare this equation with (4)), so that the stress tensor $\boldsymbol{\Sigma}$ is a function of the matrix field $\boldsymbol{C}$.

On the other hand, we characterized in Section 2 the set $\mathbb{T}(\Omega)$ of all admissible Cauchy-Green tensor fields as a subset of the Banach space $W^{2,s}(\Omega; \mathbb{S}^3)$ and we showed that any deformation $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ can be reconstructed from the associated Cauchy-Green tensor field $\boldsymbol{C}$ via the mapping $\mathcal{G} := \mathcal{F}^{-1}$ (Theorem 2.2).

These two observations combined show that *the matrix field* $\boldsymbol{C}$ *can be considered as the primary unknown in the displacement-traction problem of nonlinear elasticity*, since both the deformation $\boldsymbol{\varphi}$ and the stress $\boldsymbol{\Sigma}$ in the elastic body are functions of $\boldsymbol{C}$. More specifically, the following result holds, as a simple consequence of Theorem 2.2:

Theorem 3.1. *Assume that the inverse* $\mathcal{G} : \mathbb{T}(\Omega) \to \boldsymbol{D}(\Omega)$ *of the mapping* $\mathcal{F} : \boldsymbol{D}(\Omega) \to \mathbb{T}(\Omega)$ *is differentiable (this is the case if* $\Gamma_0 = \Gamma$*; cf.* Theorem 2.2*). Then a deformation field* $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ *satisfies the classical formulation of the displacement-traction problem, viz.,*

$$\begin{aligned} -\mathbf{div}\,\{\boldsymbol{\nabla\varphi}\hat{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{\nabla\varphi})\} = \boldsymbol{f} &\quad \text{in } \Omega, \\ \boldsymbol{\varphi} = \boldsymbol{id} &\quad \text{on } \Gamma_0, \\ \{\boldsymbol{\nabla\varphi}\hat{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{\nabla\varphi})\}\boldsymbol{n} = \boldsymbol{h} &\quad \text{on } \Gamma_1, \end{aligned}$$

if and only if the associated Cauchy-Green tensor field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ *satisfies the following intrinsic formulation of the same problem:*

$$\begin{aligned} -\mathbf{div}\,\{\boldsymbol{\nabla}(\mathcal{G}(\boldsymbol{C}))\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\} = \boldsymbol{f} &\quad \text{in } \Omega, \\ \{\boldsymbol{\nabla}(\mathcal{G}(\boldsymbol{C}))\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\}\boldsymbol{n} = \boldsymbol{h} &\quad \text{on } \Gamma_1. \end{aligned}$$

The above displacement-traction problem can also be formulated as a minimization problem if the elastic material constituting the body is *hyperelastic*. This means that there exists a function $\hat{W} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$, called the *stored energy function* of the elastic material under consideration, such that

$$\boldsymbol{F}\hat{\boldsymbol{\Sigma}}(x, \boldsymbol{F}) = \frac{\partial \hat{W}}{\partial \boldsymbol{F}}(x, \boldsymbol{F}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+.$$

For such a material, the equations (3)-(4) *formally* constitute the Euler equations associated with the critical points of the *total energy*

$$J(\boldsymbol{\varphi}) := \int_\Omega \hat{W}(x, \boldsymbol{\nabla\varphi}(x))\, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx - \int_{\Gamma_1} \boldsymbol{h} \cdot \boldsymbol{\varphi}\, d\Gamma,$$

which is thus defined over a set of deformations $\boldsymbol{\varphi} : \overline{\Omega} \to \mathbb{R}^3$ of suitable regularity that satisfy $\boldsymbol{\varphi} = \boldsymbol{id}$ on Γ_0. Finding the minimizers of this functional constitutes the *classical formulation* of the minimization problem associated with the displacement-traction problem of nonlinear elasticity.

We now describe the *intrinsic formulation* of the same minimization problem. For a hyperelastic material, the *principle of material frame-indifference* implies that there exists a function $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}^3_> \to \mathbb{R}$ such that

$$\hat{W}(x, \boldsymbol{F}) = \tilde{W}(x, \boldsymbol{F}^T \boldsymbol{F}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+.$$

Thanks again to the homeomorphism $\mathcal{F} : \boldsymbol{\varphi} \in \boldsymbol{D}(\Omega) \mapsto \boldsymbol{\nabla\varphi}^T \boldsymbol{\nabla\varphi} \in \mathbb{T}(\Omega)$ (Theorem 2.2), the total energy $J(\boldsymbol{\varphi})$ can therefore be expressed for all $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ as

$$J(\boldsymbol{\varphi}) = \mathcal{I}(\boldsymbol{C}) \text{ for all } \boldsymbol{C} = \mathcal{F}(\boldsymbol{\varphi}), \boldsymbol{\varphi} \in \boldsymbol{D}(\Omega),$$

where

$$\mathcal{I}(\boldsymbol{C}) := \int_\Omega \tilde{W}(\cdot, \boldsymbol{C})\, dx - \int_\Omega \boldsymbol{f} \cdot \mathcal{G}(\boldsymbol{C})\, dx - \int_{\Gamma_1} \boldsymbol{h} \cdot \mathcal{G}(\boldsymbol{C})\, d\Gamma \text{ for all } \boldsymbol{C} \in \mathbb{T}(\Omega) \tag{5}$$

and $\mathcal{G} = \mathcal{F}^{-1}$. In this way, *the matrix field $\boldsymbol{C}$ can be considered as the primary unknown in the minimization problem associated with the displacement-traction problem of nonlinear elasticity*. More specifically, the following result holds, again as a simple consequence of Theorems 2.2:

Theorem 3.2. *The deformation field $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ minimizes the total energy J over $\boldsymbol{D}(\Omega)$ if and only if the corresponding Cauchy-Green tensor field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ minimizes the functional $\mathcal{I}$ over $\mathbb{T}(\Omega)$.*

4. The manifold of admissible Cauchy-Green tensor fields, when $\boldsymbol{\Gamma_0 = \Gamma}$

Our proofs of the existence theorems in Sections 6 and 7 crucially hinge on the next theorem, which shows that, *when* $\Gamma_0 = \Gamma$, the set $\mathbb{T}(\Omega)$ of admissible Cauchy-Green tensors (as defined and characterized in Theorem 2.2) becomes a *Banach manifold*.

Theorem 4.1. *Assume that $\Gamma_0 = \Gamma$. Then the set $\mathbb{T}(\Omega)$ is a manifold of class C^∞ in the Banach space $W^{2,s}(\Omega; \mathbb{S}^3)$, and the mapping*

$$\mathcal{F} : \boldsymbol{\varphi} \in W^{3,s}(\Omega; \mathbb{R}^3) \to \boldsymbol{\nabla\varphi}^T \boldsymbol{\nabla\varphi} \in W^{2,s}(\Omega; \mathbb{S}^3)$$

is a C^∞ diffeomorphism from $\boldsymbol{D}(\Omega)$ onto $\mathbb{T}(\Omega)$.

Proof. It suffices to prove that the set $\boldsymbol{D}(\Omega)$ is a C^∞-manifold in the Banach space $W^{3,s}(\Omega; \mathbb{R}^3)$ and that the mapping $\mathcal{F}|_{\boldsymbol{D}(\Omega)}$ is an embedding of class C^∞. For convenience, the proof of these two assertions, which follows the same lines as the proof of Theorem 3.5 in C. Mardare [12], is broken into four stages.

(i) *The set $\boldsymbol{D}(\Omega)$ is a C^∞-manifold in the Banach space $W^{3,s}(\Omega; \mathbb{R}^3)$*. The set

$$\boldsymbol{V}(\Omega) := W^{3,s}(\Omega; \mathbb{R}^3) \cap W^{1,s}_0(\Omega; \mathbb{R}^3)$$

is a closed subspace of the Banach space $W^{3,s}(\Omega;\mathbb{R}^3)$. As an open subset of the closed affine subspace $\boldsymbol{id}+\boldsymbol{V}(\Omega)$, the set $\boldsymbol{D}(\Omega)$ is a submanifold of class C^∞ of the Banach space $W^{3,s}(\Omega;\mathbb{R}^3)$. Besides, the tangent space to $\boldsymbol{D}(\Omega)$ at any $\boldsymbol{\varphi}\in\boldsymbol{D}(\Omega)$ is the space $\boldsymbol{V}(\Omega)$.

(ii) *At every $\boldsymbol{\varphi}\in\boldsymbol{D}(\Omega)$, the tangent mapping*

$$T_{\boldsymbol{\varphi}}\mathcal{F}:\boldsymbol{V}(\Omega)\to W^{2,s}(\Omega;\mathbb{S}^3)$$

is injective. Since $\mathcal{F}$ is a bilinear mapping, it is easily seen that its tangent map at $\boldsymbol{\varphi}$ is defined by

$$T_{\boldsymbol{\varphi}}\mathcal{F}(\boldsymbol{u})=2\boldsymbol{e}(\boldsymbol{u}):=\boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}^T\boldsymbol{\nabla}\boldsymbol{\varphi}\text{ for all }\boldsymbol{u}\in\boldsymbol{V}(\Omega).$$

Since $\boldsymbol{\varphi}\in W^{3,s}(\Omega;\mathbb{R}^3)$ and $\inf_{x\in\Omega}\det\boldsymbol{\nabla}\boldsymbol{\varphi}(x)>0$, the mapping $\boldsymbol{\varphi}$ is locally a C^1-diffeomorphism. Thus the Korn inequality in the curvilinear coordinates defined by $\boldsymbol{\varphi}$ (see [7]) shows that there exists a constant C such that

$$\|\boldsymbol{u}\|_{H^1(\Omega)}\le C\|\boldsymbol{e}(\boldsymbol{u})\|_{L^2(\Omega)}\text{ for all }\boldsymbol{u}\in\boldsymbol{V}(\Omega).\tag{6}$$

This implies that $T_{\boldsymbol{\varphi}}\mathcal{F}$ is injective.

(iii) *The tangent mapping $T_{\boldsymbol{\varphi}}\mathcal{F}$ has a closed split range in* $W^{2,s}(\Omega;\mathbb{S}^3)$. We have to prove that the image of $T_{\boldsymbol{\varphi}}\mathcal{F}$, defined by

$$\mathcal{A}:=\{\boldsymbol{e}(\boldsymbol{u})\in W^{2,s}(\Omega;\mathbb{S}^3);\ \boldsymbol{u}\in\boldsymbol{V}(\Omega)\},$$

is closed in the space $W^{2,s}(\Omega;\mathbb{S}^3)$ and that there exists a closed subspace $\mathcal{B}$ of the same space such that $W^{2,s}(\Omega;\mathbb{S}^3)=\mathcal{A}\oplus\mathcal{B}$; cf. Abraham, Marsden & Ratiu [1, Definition 2.1.14]. The fact that $\mathcal{A}$ is closed is a consequence of the following Korn inequality in curvilinear coordinates:

$$\|\boldsymbol{u}\|_{W^{3,s}(\Omega)}\le C\|\boldsymbol{e}(\boldsymbol{u})\|_{W^{2,s}(\Omega)}\text{ for all }\boldsymbol{u}\in\boldsymbol{V}(\Omega).$$

The proof of this inequality is similar to that of the Korn inequality (6) as given in Duvaut & Lions [11] and for this reason is not given here.

Let

$$\mathcal{B}:=\{\boldsymbol{C}\in W^{2,s}(\Omega;\mathbb{S}^3);\int_\Omega\boldsymbol{C}:\boldsymbol{e}(\boldsymbol{v})dx=0\text{ for all }\boldsymbol{v}\in\boldsymbol{V}(\Omega)\},$$

where : denotes the usual matrix inner product. It is clear that the set $\mathcal{B}$ is closed in $W^{2,s}(\Omega;\mathbb{S}^3)$ and that $\mathcal{A}\cap\mathcal{B}=\{\boldsymbol{0}\}$. It remains to show that $\mathcal{A}+\mathcal{B}=W^{2,s}(\Omega;\mathbb{S}^3)$. Let $\boldsymbol{C}\in W^{2,s}(\Omega;\mathbb{S}^3)$ be fixed, but otherwise arbitrary, and define $\boldsymbol{u}\in H^1_0(\Omega;\mathbb{R}^3)$ as the unique solution of the variational equations

$$\int_\Omega\boldsymbol{e}(\boldsymbol{u}):\boldsymbol{e}(\boldsymbol{v})dx=\int_\Omega\boldsymbol{C}:\boldsymbol{e}(\boldsymbol{v})dx\ \text{ for all }\boldsymbol{v}\in H^1_0(\Omega;\mathbb{R}^3).$$

By using the regularity of the solution to these variational equations (which holds since the boundary Γ is smooth enough and $\Gamma_0=\Gamma$; cf., e.g., Ciarlet [6, Theorem 6.3-6]), one deduces that $\boldsymbol{u}\in W^{3,s}(\Omega;\mathbb{R}^3)$. Hence $\boldsymbol{e}(\boldsymbol{u})\in\mathcal{A}$. The definition of $\boldsymbol{u}$ then implies that $(\boldsymbol{C}-\boldsymbol{e}(\boldsymbol{u}))$ belongs to the set $\mathcal{B}$, so that $\boldsymbol{C}=\boldsymbol{e}(\boldsymbol{u})+(\boldsymbol{C}-\boldsymbol{e}(\boldsymbol{u}))$ belongs to the set $(\mathcal{A}+\mathcal{B})$.

(iv) *Conclusion.* The tangent mapping $T_{\boldsymbol{\varphi}}\mathcal{F}$ being injective and having a closed split range at every $\boldsymbol{\varphi}\in\boldsymbol{D}(\Omega)$, the mapping $\mathcal{F}$ is an immersion, according to [1, Definition 3.5.6]. Since it is also a homeomorphism onto its image (Theorem 2.2), $\mathcal{F}$ is in fact en embedding; cf. [1, Definition 3.5.9]. Hence its image $\mathbb{T}(\Omega)$ is a manifold in the Banach space $W^{2,s}(\Omega;\mathbb{S}^3)$; cf. [1, p. 201]. This manifold is of class C^∞ since $\mathcal{F}$ is of class C^∞. That the mapping $\mathcal{F}$ is a diffeomorphism of class C^∞ is a consequence of the inverse function theorem of [1, Theorem 3.5.1]). □

5. Some theorems from the classical approach

This section gathers other preliminaries needed for our existence theorems, viz., those results from the classical approach to nonlinear elasticity that will be used in the rest of this paper.

The first theorem establishes the existence of solutions to the classical formulation of the pure displacement problem of nonlinear elasticity by means of the implicit function theorem, as revisited by Zhang [16, Theorem 2.6]:

Theorem 5.1. *Assume that Ω is a bounded, open subset of $\mathbb{R}^3$, with a boundary Γ of class C^3 and that the response function $\hat{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{S}^3$ satisfies the following three assumptions: $\hat{\boldsymbol{\Sigma}}$ is of class C^3, $\hat{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{I}) = \boldsymbol{0}$, and there exists a constant $C > 0$ such that*

$$\int_\Omega \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \boldsymbol{F}}(x, \boldsymbol{I})\boldsymbol{\nabla v}(x) : \boldsymbol{\nabla v}(x) dx \geq C\|\boldsymbol{v}\|^2_{H^1(\Omega)} \text{ for all } \boldsymbol{v} \in \mathcal{D}(\Omega; \mathbb{R}^3).$$

Let $s > 3/2$. Then there exist two constants $\varepsilon > 0$ and $\delta > 0$ such that, for each $\boldsymbol{f} \in W^{1,s}(\Omega; \mathbb{R}^3)$ such that $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon$, there exists a unique vector field $\boldsymbol{\varphi} \in W^{3,s}(\Omega; \mathbb{R}^3)$ that satisfies:

$$\begin{aligned} &\|\boldsymbol{\varphi} - \boldsymbol{id}\|_{W^{3,s}(\Omega)} < \delta, \\ &-\mathbf{div}\,\{\boldsymbol{\nabla\varphi}\hat{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{\nabla\varphi})\} = \boldsymbol{f} \text{ in } \Omega \text{ and } \boldsymbol{\varphi} = \boldsymbol{id} \text{ on } \Gamma. \end{aligned}$$

The second theorem establishes the existence of minimizers to the classical formulation of the minimization problem associated with the displacement-traction problem by means of J. Ball's theory of polyconvexity; cf. [3, 4, 6, 16]. We recall that a stored energy function $\hat{W} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$ is *polyconvex* if, for each $x \in \overline{\Omega}$, there exists a *convex* function $\mathcal{W}(x, \cdot) : \mathbb{M}^3 \times \mathbb{M}^3 \times (0, \infty) \to \mathbb{R}$ such that

$$\hat{W}(x, \boldsymbol{F}) = \mathcal{W}(x, \boldsymbol{F}, \mathbf{Cof}\boldsymbol{F}, \det \boldsymbol{F}) \text{ for all } \boldsymbol{F} \in \mathbb{M}^3_+,$$

where $\mathbf{Cof}\boldsymbol{F}$ designates the cofactor matrix of the matrix $\boldsymbol{F}$.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded and connected open subset of $\mathbb{R}^3$, with a Lipschitz-continuous boundary, the set Ω being locally on the same side of its boundary. Let Γ_0 be a non-empty, relatively open subset of the boundary of Ω.*

Consider a polyconvex function $\hat{W} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$ that satisfies the following properties: The function $\mathcal{W}(\cdot, \boldsymbol{F}, \boldsymbol{H}, \delta) : \Omega \to \mathbb{R}$ is measurable for all $(\boldsymbol{F}, \boldsymbol{H}, \delta) \in \mathbb{M}^3 \times \mathbb{M}^3 \times (0, \infty)$, there exist numbers $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$, $\alpha > 0$, and $\beta \in \mathbb{R}$ such that

$$\hat{W}(x, \boldsymbol{F}) \geq \alpha(\|\boldsymbol{F}\|^p + \|\mathbf{Cof}\boldsymbol{F}\|^q + (\det \boldsymbol{F})^r) - \beta \text{ for all } (x, \boldsymbol{F}) \in \Omega \times \mathbb{M}^3_+,$$

and, for almost all $x \in \Omega$, $\hat{W}(x, \boldsymbol{F}) \to +\infty$ if $\boldsymbol{F} \in \mathbb{M}^3_+$ is such that $\det \boldsymbol{F} \to 0^+$.

Let $\boldsymbol{f} \in L^{6/5}(\Omega; \mathbb{R}^3)$, let the set of admissible deformations be defined by

$$\begin{aligned} \mathcal{M} := \{\boldsymbol{\psi} \in W^{1,p}(\Omega; \mathbb{R}^3);\ &\mathbf{Cof}(\boldsymbol{\nabla\psi}) \in L^q(\Omega; \mathbb{M}^3), \det \boldsymbol{\nabla\psi} \in L^r(\Omega) \\ &\det(\boldsymbol{\nabla\psi}) > 0 \text{ a.e. in } \Omega,\ \boldsymbol{\psi} = \boldsymbol{id} \text{ on } \partial\Omega\}, \end{aligned}$$

and assume that $\inf_{\boldsymbol{\psi} \in \mathcal{M}} J(\boldsymbol{\psi}) < \infty$*, where the functional $J : \mathcal{M} \to \mathbb{R}$ is defined by*

$$J(\boldsymbol{\psi}) := \int_\Omega \hat{W}(x, \boldsymbol{\nabla\psi}(x))\, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\Psi}\, dx \text{ for all } \boldsymbol{\psi} \in \mathcal{M}.$$

Then there exists $\boldsymbol{\varphi} \in \mathcal{M}$ such that $J(\boldsymbol{\varphi}) = \inf_{\boldsymbol{\psi} \in \mathcal{M}} J(\boldsymbol{\psi})$.

The existence results of Theorems 5.1 and 5.2 hold under different sets of assumptions on the data, but the "intersection" of these assumptions is nonempty. A natural question therefore arises: Do the solutions given by Theorems 5.1 and 5.2 coincide when both theorems apply? The answer is affirmative, at least for a specific class of elastic materials, as shown by Zhang [16, Theorem 3.4]:

Theorem 5.3. *Let $p > 3$ and assume that Ω is a bounded, connected, open subset of $\mathbb{R}^3$, with a boundary of class C^3. Assume that the stored energy function of the material constituting the body is given by*

$$\hat{W}(x, \boldsymbol{F}) = a|\boldsymbol{F}|^p + b|\mathbf{Cof}\boldsymbol{F}|^q + G(\boldsymbol{F}, \mathbf{Cof}\boldsymbol{F}, \det \boldsymbol{F}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+,$$

where $p \geq 2$, $q \geq \frac{p}{p-1}$, $a > 0$, $b > 0$, and $G : \mathbb{M}^3 \times \mathbb{M}^3 \times (0, \infty) \to \mathbb{R}$ is a convex function of class C^4 that is bounded from below and satisfies $G(\boldsymbol{F}_n, \boldsymbol{H}_n, \delta_n) \to \infty$ whenever $(\boldsymbol{F}_n, \boldsymbol{H}_n, \delta_n) \to (\boldsymbol{F}, \boldsymbol{H}, 0^+)$ as $n \to \infty$.

Then there exists a constant $\varepsilon > 0$ such that the solutions given by Theorems 5.1 *and* 5.2 *coincide whenever $\|\boldsymbol{f}\|_{L^p(\Omega)} < \varepsilon$.*

6. Existence of solutions to the intrinsic formulation of the pure displacement problem

We assume in the rest of this paper that $\Gamma_0 = \Gamma$, which means that we restrict our study of the existence of solutions to the *pure displacement problem* of nonlinear elasticity. Otherwise, the set Ω satisfies the assumptions of Section 2, and $s > 3/2$. In addition, we assume that the boundary of Ω is *connected*.

The objective of this section is to show that the intrinsic formulation of the pure displacement problem given in Section 3 has solutions provided that $\boldsymbol{f} \in W^{1,s}(\Omega; \mathbb{R}^3)$ is sufficiently small in the corresponding norm and the response function $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}^3_> \to \mathbb{S}^3$ has a specific (but natural) behavior "when $\boldsymbol{C} \in \mathbb{S}^3_>$ is close to $\boldsymbol{I}$".

Recall that, when $\Gamma_0 = \Gamma$, $\mathbb{T}(\Omega)$ is a manifold in the Banach space $W^{2,s}(\Omega; \mathbb{S}^3)$ and that its tangent space at $\boldsymbol{I}$ is the image by the tangent mapping to $\mathcal{F}$ of the tangent space at $\boldsymbol{id}$ of the Banach manifold $\boldsymbol{D}(\Omega)$ (Theorem 4.1). This means that

$$T_{\boldsymbol{I}}(\mathbb{T}(\Omega)) = T_{\boldsymbol{id}}\mathcal{F}(\boldsymbol{V}(\Omega)),$$

where

$$\begin{aligned}
&\boldsymbol{V}(\Omega) = W^{3,s}(\Omega; \mathbb{R}^3) \cap W^{1,s}_0(\Omega; \mathbb{R}^3),\\
&\mathcal{F}(\boldsymbol{\varphi}) = \boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi} \in W^{2,s}(\Omega; \mathbb{S}^3) \text{ for all } \boldsymbol{\varphi} \in W^{3,s}(\Omega; \mathbb{R}^3);
\end{aligned}$$

cf. Section 4. We then have the following *existence result for the intrinsic formulation of the pure displacement problem*:

Theorem 6.1. *Assume that the mapping $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}^3_> \to \mathbb{S}^3$ satisfies the following three assumptions: $\tilde{\boldsymbol{\Sigma}}$ is of class C^3, $\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{I}) = \boldsymbol{0}$, and there exists a constant $c > 0$ such that*

$$\int_\Omega \frac{\partial \tilde{\boldsymbol{\Sigma}}}{\partial \boldsymbol{C}}(x, \boldsymbol{I})\boldsymbol{B}(x) : \boldsymbol{B}(x)dx \geq c\|\boldsymbol{B}\|_{L^2(\Omega;\mathbb{S}^3)} \text{ for all } \boldsymbol{B} \in W^{2,s}(\Omega; \mathbb{S}^3).$$

Then there exist two constants $\varepsilon > 0$ and $\delta > 0$ such that, for each $\boldsymbol{f} \in W^{1,s}(\Omega; \mathbb{R}^3)$ such that $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon$, there exists a unique tensor field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ that satisfies

$$\|\boldsymbol{C} - \boldsymbol{I}\|_{W^{2,s}(\Omega;\mathbb{S}^3)} < \delta \text{ and } -\mathbf{div}\,\{\boldsymbol{\nabla}(\mathcal{G}(\boldsymbol{C}))\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\} = \boldsymbol{f} \text{ in } \Omega.$$

Proof. The central idea is to apply the *implicit function theorem on Banach manifolds* (cf. Abraham, Marsden & Ratiu [1, Theorem 3.5.1]) to the mapping

$$\mathcal{H} : \boldsymbol{C} \in \mathbb{T}(\Omega) \mapsto -\mathbf{div}\,\{\boldsymbol{\nabla}(\mathcal{G}(\boldsymbol{C}))\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\} \in W^{1,s}(\Omega; \mathbb{R}^3)$$

in a neighborhood of $\boldsymbol{I} \in \mathbb{T}(\Omega)$. To this end, we need to prove that $\mathcal{H}$ is at least of class C^1 and that its tangent mapping at $\boldsymbol{I}$, which is given by

$$T_{\boldsymbol{I}}\mathcal{H} : \boldsymbol{B} \in T_{\boldsymbol{I}}(\mathbb{T}(\Omega)) \mapsto -\mathbf{div}\,\Big\{\frac{\partial \tilde{\boldsymbol{\Sigma}}}{\partial \boldsymbol{C}}(\cdot, \boldsymbol{I})\boldsymbol{B}\Big\} \in W^{1,s}(\Omega; \mathbb{R}^3),$$

is an isomorphism. To see this, we note that $\mathcal{H} = \mathcal{K} \circ \mathcal{G}$, where the mapping $\mathcal{K}$ is defined by

$$\mathcal{K} : \boldsymbol{\varphi} \in \boldsymbol{D}(\Omega) \mapsto \mathcal{K}(\boldsymbol{\varphi}) := -\mathbf{div}\,\{\boldsymbol{\nabla}\boldsymbol{\varphi}\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi})\} \in W^{1,s}(\Omega; \mathbb{R}^3),$$

and the mapping $\mathcal{G}$ is the inverse of the mapping $\mathcal{F} : \boldsymbol{D}(\Omega) \to \mathbb{T}(\Omega)$ (cf. Theorem 4.1).

But the mapping $\mathcal{K}$ is precisely the mapping that is defined by the left-hand side of the system of partial differential equations that constitute the pure displacement problem of nonlinear elasticity (Theorem 5.1), with a response function $\hat{\boldsymbol{\Sigma}}$ defined by

$$\hat{\boldsymbol{\Sigma}}(x, \boldsymbol{F}) = \tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{F}^T\boldsymbol{F}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+.$$

Thus the mapping $\mathcal{K}$ is of class C^1 and its tangent mapping at $\boldsymbol{id} \in \boldsymbol{D}(\Omega)$, which is defined by

$$T_{\boldsymbol{id}}\mathcal{K} : \boldsymbol{u} \in \boldsymbol{V}(\Omega) \mapsto -\mathbf{div}\,\Big\{\frac{\partial \tilde{\boldsymbol{\Sigma}}}{\partial \boldsymbol{C}}(\boldsymbol{\nabla}\boldsymbol{u}^T + \boldsymbol{\nabla}\boldsymbol{u})\Big\} \in W^{1,s}(\Omega; \mathbb{R}^3),$$

is an isomorphism.

The mapping $\mathcal{G}$ is the inverse of the C^∞-mapping $\mathcal{F}|_{\boldsymbol{D}(\Omega)}$, whose tangent map at $\boldsymbol{id}$, viz.,

$$T_{\boldsymbol{id}}\mathcal{F} : \boldsymbol{u} \in \boldsymbol{V}(\Omega) \mapsto (\boldsymbol{\nabla}\boldsymbol{u}^T + \boldsymbol{\nabla}\boldsymbol{u}) \in T_{\boldsymbol{I}}(\mathbb{T}(\Omega)),$$

is an isomorphism; cf. Section 4. Thus $\mathcal{G}$ is also of class C^∞ and its tangent mapping at $\boldsymbol{I}$, given by $T_{\boldsymbol{I}}\mathcal{G} = (T_{\boldsymbol{id}}\mathcal{F})^{-1}$, is an isomorphism.

We then infer from the above observations that the composite mapping $\mathcal{H} = \mathcal{K} \circ \mathcal{G}$ is at least of class C^1 and that its tangent mapping at $\boldsymbol{I}$, which is given by $T_{\boldsymbol{I}}\mathcal{H} = T_{\boldsymbol{id}}\mathcal{K} \circ T_{\boldsymbol{I}}\mathcal{G}$, is an isomorphism. □

Remark 6.2. Let $\boldsymbol{\varphi} := \mathcal{G}(\boldsymbol{C}) \in \boldsymbol{D}(\Omega)$, where the tensor field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ is the solution of the pure displacement problem solved in Theorem 6.1. Since $\boldsymbol{\varphi} \in W^{3,s}(\Omega; \mathbb{R}^3) \subset C^1(\overline{\Omega}; \mathbb{R}^3)$ and $\det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) > 0$ for all $x \in \overline{\Omega}$, Theorem 5.5.2 of [6] shows that $\boldsymbol{\varphi}(\Omega) = \Omega$, $\boldsymbol{\varphi}(\overline{\Omega}) = \overline{\Omega}$, and $\boldsymbol{\varphi} : \overline{\Omega} \to \overline{\Omega}$ is one-to-one. □

7. Existence of solutions to the intrinsic formulation of the minimization problem associated with the pure displacement problem

We now turn our attention to the intrinsic formulation given in Section 3 of the minimization problem associated with the pure displacement problem of nonlinear elasticity.

The assumptions are those of the previous section; in particular, $\Gamma_0 = \Gamma$ and $s > 3/2$. In addition, we assume that the material constituting the body is *hyperelastic*, with a *stored energy function* of the form proposed by Ciarlet and Geymonat [8], viz.,

$$\hat{W}(x, \boldsymbol{F}) := a|\boldsymbol{F}|^2 + b|\mathbf{Cof}\boldsymbol{F}|^2 + c(\det \boldsymbol{F})^2 - d\log(\det \boldsymbol{F}) - (3a + 3b + c) \tag{7}$$

for all $(x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+$, where $|\boldsymbol{A}|$ denotes the Frobenius norm of a matrix $\boldsymbol{A} \in \mathbb{M}^3$, and the constants $a > 0$, $b > 0$, $c > 0$ and $d > 0$ are so chosen that

$$\hat{W}(x, \boldsymbol{F}) = \frac{\lambda}{2}(\operatorname{tr} \boldsymbol{E})^2 + \mu \operatorname{tr}(\boldsymbol{E}^2) + o(|\boldsymbol{E}|^2) \text{ with } \boldsymbol{E} := \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) \tag{8}$$

for all $(x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+$, where $\lambda > 0$ and $\mu > 0$ are the *Lamé constants* of the elastic material under consideration.

Remark 7.1. One possible choice of the constants (they are not uniquely defined) appearing in the definition of the stored energy function of (7) in terms of the Lamé coefficients of the material is given by

$$a = \frac{\mu^2}{\lambda + 2\mu},\ b = \frac{\lambda\mu}{2(\lambda + 2\mu)},\ c = \frac{\lambda^2}{4(\lambda + 2\mu)},\ d = \frac{\lambda + 2\mu}{2}.$$

□

Note that the function $\hat{W}$ is independent of $x \in \overline{\Omega}$ and of course depends on $\boldsymbol{F} \in \mathbb{M}^3_+$ only via $\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}$. Indeed, a simple computation shows that

$$\hat{W}(x, \boldsymbol{F}) = \tilde{W}(\boldsymbol{C}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+,$$

where

$$\tilde{W}(\boldsymbol{C}) := a \operatorname{tr} \boldsymbol{C} + b \operatorname{tr} \mathbf{Cof}\boldsymbol{C} + c \det \boldsymbol{C} - \frac{d}{2} \log \det \boldsymbol{C} - (3a + 3b + c) \text{ for all } \boldsymbol{C} \in \mathbb{S}^3_>.$$

Remark 7.2. The stored energy function of (7) is chosen here essentially for simplicity; otherwise more general hyperelastic materials can be as well considered, for instance those with a stored energy function of the form:

$$\hat{W}(x, \boldsymbol{F}) := a|\boldsymbol{F}|^p + b|\mathbf{Cof}\boldsymbol{F}|^q + G(\det \boldsymbol{F}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+,$$

where $p \geq 2$, $q \geq \frac{p}{p-1}$, and the constants $a > 0$, $b > 0$ and the function G are so chosen that

(i) G is convex, bounded from below, and satisfies $\lim_{\delta\to 0^+} G(\delta) = +\infty$,
(ii) $G'(1) + q3^{q/2-1}b < 0$,
(iii) $G''(1) + G'(1) + 2q(2q - 1)3^{q/2-2}b + p(p - 2)3^{p/2-2}a \geq 0$.

□

We now show that the intrinsic formulation given in Section 3 of the minimization problem associated with the pure displacement problem of nonlinear elasticity has a solution provided that an ad hoc norm of the body force density is "small enough". The Banach manifold $\mathbb{T}(\Omega)$ is defined in Theorem 2.2; the functional $\mathcal{I}$ is defined in (5).

Theorem 7.3. *There exist two constants $\varepsilon > 0$ and $\delta > 0$ with the following properties: For each $\boldsymbol{f} \in W^{1,s}(\Omega;\mathbb{R}^3)$ such that $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon$, there exists a unique tensor field $\boldsymbol{C}_0 \in \mathbb{T}(\Omega)$ that satisfies*

$$\|\boldsymbol{C}_0 - \boldsymbol{I}\|_{W^{2,s}(\Omega)} < \delta,$$

$$\mathcal{I}(\boldsymbol{C}_0) = \inf_{\boldsymbol{C}\in\mathbb{T}(\Omega)} \mathcal{I}(\boldsymbol{C}), \text{ where } \mathcal{I}(\boldsymbol{C}) := \int_\Omega \tilde{W}(\cdot,\boldsymbol{C})dx - \int_\Omega \boldsymbol{f}\cdot\mathcal{G}(\boldsymbol{C})dx \text{ for all } \boldsymbol{C} \in \mathbb{T}(\Omega).$$

Proof. The functional $\mathcal{I}$ is *not* coercive over the Banach manifold $\mathbb{T}(\Omega)$, so that the direct methods of the Calculus of Variations do not apply. The idea is then to show instead that the Euler-Lagrange equation associated with this minimization problem possesses a unique solution and that this solution minimizes the functional $\mathcal{I}$. The minimizer is unique since any other minimizer must satisfy the same Euler-Lagrange equation.

Finding a solution to the Euler-Lagrange equation associated with the above minimization problem consists in finding a field $\boldsymbol{C} \in \mathbb{T}(\Omega)$ that satisfies the equation

$$-\mathbf{div}\left(2\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{C})\frac{\partial\tilde{W}}{\partial\boldsymbol{C}}(\boldsymbol{C})\right) = \boldsymbol{f} \text{ in } \Omega.$$

In order to prove the existence of such a field, we need to show that the response function $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega}\times\mathbb{S}^3_> \to \mathbb{S}^3$ defined by

$$\tilde{\boldsymbol{\Sigma}}(x,\boldsymbol{C}) = 2\frac{\partial\tilde{W}}{\partial\boldsymbol{C}}(\boldsymbol{C}) \text{ for all } (x,\boldsymbol{C}) \in \overline{\Omega}\times\mathbb{S}^3_>$$

satisfies all the hypotheses of Theorem 6.1.

First, it is clear that $\tilde{\boldsymbol{\Sigma}}$ is of class C^∞. Next, since the function $\hat{W}$ satisfies the relation (8), it follows that the function $\tilde{W}$, which is defined in terms of $\hat{W}$ by the relations $\tilde{W}(\boldsymbol{F}^T\boldsymbol{F}) = \hat{W}(x,\boldsymbol{F})$ for all $(x,\boldsymbol{F}) \in \overline{\Omega}\times\mathbb{M}^3_+$, satisfies the relations

$$\tilde{\boldsymbol{\Sigma}}(x,\boldsymbol{C}) = \frac{\lambda}{2}\operatorname{tr}(\boldsymbol{C}-\boldsymbol{I}) + \mu(\boldsymbol{C}-\boldsymbol{I}) + o(|\boldsymbol{C}-\boldsymbol{I}|) \text{ for all } (x,\boldsymbol{C}) \in \overline{\Omega}\times\mathbb{S}^3_>;$$

cf. Ciarlet [6, Theorem 4.2-2]. This next implies that $\tilde{\boldsymbol{\Sigma}}(\cdot,\boldsymbol{I}) = \boldsymbol{0}$ and that

$$\int_\Omega \frac{\partial\tilde{\boldsymbol{\Sigma}}}{\partial\boldsymbol{C}}(x,\boldsymbol{I})\boldsymbol{B}(x) : \boldsymbol{B}(x)dx = \int_\Omega \left\{\frac{\lambda}{2}(\operatorname{tr}\boldsymbol{B}(x))^2 + \mu|\boldsymbol{B}(x)|^2\right\}dx \geq \mu\|\boldsymbol{B}\|^2_{L^2(\Omega)}$$

for all $\boldsymbol{B} \in L^2(\Omega;\mathbb{S}^3)$, thus a fortiori for all $\boldsymbol{B} \in W^{2,s}(\Omega;\mathbb{S}^3)$. All the assumptions of Theorem 6.1 being thus satisfied, there exist two constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, if $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon_0$, there exists a unique tensor field $\boldsymbol{C}_0 \in \mathbb{T}(\Omega)$ that satisfies

$$\begin{aligned} &\|\boldsymbol{C}_0 - \boldsymbol{I}\|_{W^{1,s}(\Omega)} < \delta_0, \\ &-\mathbf{div}\,\{\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{C}_0)\tilde{\boldsymbol{\Sigma}}(\cdot,\boldsymbol{C}_0)\} = \boldsymbol{f} \text{ in } \Omega. \end{aligned} \tag{9}$$

It remains to prove that $\boldsymbol{C}_0$ minimizes the functional $\mathcal{I}$ provided $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon$ for some well chosen constant $0 < \varepsilon \leq \varepsilon_0$. To this end, define the set

$$\begin{aligned} \boldsymbol{D}_0(\Omega) := \{&\boldsymbol{\psi} \in H^1(\Omega;\mathbb{R}^3); \mathbf{Cof}\boldsymbol{\nabla}\boldsymbol{\psi} \in L^2(\Omega;\mathbb{M}^3), \det\boldsymbol{\nabla}\boldsymbol{\psi} \in L^2(\Omega), \\ &\det\boldsymbol{\nabla}\boldsymbol{\psi} > 0 \text{ a.e. in } \Omega,\ \boldsymbol{\psi} = \boldsymbol{id} \text{ on } \Gamma\}. \end{aligned}$$

On the one hand, since the function $\hat{W}$ is polyconvex and satisfies all the assumptions of the fundamental existence theorem of Ball [3] (see Theorem 5.2), there exists a vector field $\boldsymbol{\varphi}_0 \in \boldsymbol{D}_0(\Omega)$ that minimizes over the set $\boldsymbol{D}_0(\Omega)$ the functional J defined by

$$J(\boldsymbol{\psi}) = \int_\Omega \hat{W}(x, \boldsymbol{\nabla}\boldsymbol{\psi}(x))dx - \int_\Omega \boldsymbol{f}(x) \cdot \boldsymbol{\psi}(x)dx \text{ for all } \boldsymbol{\psi} \in \boldsymbol{D}_0(\Omega).$$

On the other hand, since the response function $\hat{W} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{S}^3$ defined by

$$\hat{\boldsymbol{\Sigma}}(x, \boldsymbol{F}) = \tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{F}^T\boldsymbol{F}) \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+$$

satisfies the assumptions of the implicit function theorem as revisited by Zhang [16, Theorem 2.6] (reproduced here in Theorem 5.1), there exist two constants $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, if $\boldsymbol{f} \in W^{1,s}(\Omega)$ satisfies $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon_1$, the boundary value problem

$$\begin{aligned} -\mathbf{div}\,\{\boldsymbol{\nabla}\boldsymbol{\varphi})\tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{\nabla}\boldsymbol{\varphi})\} &= \boldsymbol{f} \quad \text{in } \Omega, \\ \boldsymbol{\varphi} &= \boldsymbol{id} \quad \text{on } \Gamma, \end{aligned} \tag{10}$$

has a unique solution $\boldsymbol{\varphi}_1 \in W^{3,s}(\Omega; \mathbb{R}^3)$ satisfying $\|\boldsymbol{\varphi}_1 - \boldsymbol{id}\|_{W^{3,s}(\Omega)} < \delta_1$.

Thanks to [16, Theorem 3.4] (reproduced here in Theorem 5.3), there exists $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$ such that $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_1$ for all $\boldsymbol{f} \in W^{1,s}(\Omega; \mathbb{R}^3)$ satisfying $\|\boldsymbol{f}\|_{W^{1,s}(\Omega)} < \varepsilon$. Thus the vector field $\boldsymbol{\varphi}_0$ satisfies relations (10), from which it follows (cf. Theorem 3.1) that the matrix field $\boldsymbol{\nabla}\boldsymbol{\varphi}_0^T\boldsymbol{\nabla}\boldsymbol{\varphi}_0$ is a solution to problem (9). In addition,

$$\|\boldsymbol{\nabla}\boldsymbol{\varphi}_0^T\boldsymbol{\nabla}\boldsymbol{\varphi}_0 - \boldsymbol{I}\|_{W^{2,s}(\Omega)} \leq 2\|\boldsymbol{\nabla}\boldsymbol{\varphi}_0\|_{W^{2,s}(\Omega)}\|\boldsymbol{\nabla}\boldsymbol{\varphi}_0 - \boldsymbol{I}\|_{W^{2,s}(\Omega)} < 2(1+\delta_1)\delta_1 < \delta_0,$$

provided that ε is chosen sufficiently small. Then the uniqueness of the solution to problem (9) shows that $\boldsymbol{\nabla}\boldsymbol{\varphi}_0^T\boldsymbol{\nabla}\boldsymbol{\varphi}_0 = \boldsymbol{C}_0$.

Given any matrix field $\boldsymbol{C} \in \mathbb{T}(\Omega)$, there exists, by Theorem 2.2, a vector field $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ such that $\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi}$. Since then $\boldsymbol{\varphi} \in \boldsymbol{D}_0(\Omega)$, we have $J(\boldsymbol{\varphi}_0) \leq J(\boldsymbol{\varphi})$. Therefore,

$$\mathcal{I}(\boldsymbol{C}_0) = \mathcal{I}(\boldsymbol{\nabla}\boldsymbol{\varphi}_0^T\boldsymbol{\nabla}\boldsymbol{\varphi}_0) = J(\boldsymbol{\varphi}_0) \leq J(\boldsymbol{\varphi}) = \mathcal{I}(\boldsymbol{\nabla}\boldsymbol{\varphi}^T\boldsymbol{\nabla}\boldsymbol{\varphi}) = \mathcal{I}(\boldsymbol{C}).$$

This shows that the tensor field $\boldsymbol{C}_0$ is a minimizer of $\mathcal{I}$ over the set $\mathbb{T}(\Omega)$. □

Remark 7.4. Let $\boldsymbol{\varphi}_0 := \mathcal{G}(\boldsymbol{C}_0) \in \boldsymbol{D}(\Omega)$, where the tensor field $\boldsymbol{C}_0 \in \mathbb{T}(\Omega)$ is the solution to the minimization problem solved in Theorem 7.3. Since $\boldsymbol{\varphi}_0 \in W^{3,s}(\Omega; \mathbb{R}^3) \subset \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^3)$ and $\det \boldsymbol{\nabla}\boldsymbol{\varphi}_0(x) > 0$ for all $x \in \overline{\Omega}$, Theorem 5.5.2 of [6] shows that $\boldsymbol{\varphi}_0(\Omega) = \Omega$, $\boldsymbol{\varphi}_0(\overline{\Omega}) = \overline{\Omega}$, and $\boldsymbol{\varphi}_0 : \overline{\Omega} \to \overline{\Omega}$ is one-to-one. □

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References

[1] Abraham R., Marsden, J.E. and Ratiu, T.: *Manifolds, Tensor Analysis, and Applications*, Second Edition, Springer-Verlag, New York, 1988.

[2] Antman, S.S.: Ordinary differential equations of nonlinear elasticity I: Foundations of the theories of non-linearly elastic rods and shells, *Arch. Rational Mech. Anal*. 61 (1976), 307-351.

[3] Ball, J.: Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* 63, 337-403 (1977)

[4] Ball, J. and Murat, F.: $W^{1,p}$-Quasiconvexity and variational problems for multiple integrals, *J. Functional Anal.* 58 (1984), 225-253.

[5] Choquet-Bruhat Y., DeWitt-Morette C. and Dillard-Bleick M.: *Analysis, manifolds, and physics. Part I . Basics*, Second Edition, North-Holland, Amsterdam, 1982.

[6] Ciarlet, P.G. : *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*, North-Holland, Amsterdam, 1988.

[7] Ciarlet, P.G. : *Mathematical Elasticity, Volume III: Theory of Shells*, North-Holland, Amsterdam, 2003.

[8] Ciarlet, P.G. and Geymonat G.: On constitutive equations in compressible nonlinear elasticity, *C. R. Acad. Sci. Paris* Sér. II 295 (1982), 423-426

[9] Ciarlet, P.G. and Mardare, C.: Recovery of a manifold with boundary and its continuity as a function of its metric tensor, *J. Math. Pures Appl.* 83, 811-843 (2004)

[10] Ciarlet, P.G. and Mardare, C.: Boundary conditions in intrinsic elasticity, in preparation.

[11] G. Duvaut and J. L. Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972.

[12] Mardare, C.: C^∞-regularity of a manifold as a function of its metric tensor, *Analysis and Applications* 4, 19-30 (2006)

[13] Mardare, S.: On isometric immersions of a Riemannian space with little regularity, *Analysis and Applications* 2, 193-226 (2004)

[14] Mardare, S.: On Pfaff systems with L^p coefficients and their applications in differential geometry, *J. Math. Pures Appl.* 84 (2005), 1659–1692.

[15] Mardare, S.: On systems of first order linear partial differential equations with L^p coefficients. *Advances in Differential Equations* 12, 301-360 (2007)

[16] Zhang, K.: Energy minimizers in nonlinear elastostatics and the implicit function theorem, *Arch. Rational Mech. Anal.* 114, 95-117 (1991)