The continuity of a surface as a function of its two fundamental forms

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Abstract

The fundamental theorem of surface theory asserts that, if a field of positive definite symmetric matrices of order two and a field of symmetric matrices of order two together satisfy the Gauß and Codazzi–Mainardi equations in a connected and simply connected open subset of $\mathbb{R}^2$, then there exists a surface in $\mathbb{R}^3$ with these fields as its first and second fundamental forms and this surface is unique up to isometries in $\mathbb{R}^3$. We establish here that a surface defined in this fashion varies continuously as a function of its two fundamental forms, for certain natural topologies.

Sur la continuité d'une surface en fonction de ses deux formes fondamentales

Résumé

Le théorème fondamental de la théorie des surfaces affirme que, si un champ de matrices symétriques définies positives d'ordre deux et un champ de matrices symétriques d'ordre deux vérifient ensemble les équations de Gauß et de Codazzi–Mainardi dans un ouvert connexe et simplement connexe de $\mathbb{R}^2$, alors il existe une surface dans $\mathbb{R}^3$ dont ces champs sont les première et deuxième formes fondamentales et cette surface est unique aux isométries de $\mathbb{R}^3$ près. On établit ici qu'une surface définie de cette façon varie continûment en fonction de ses deux formes fondamentales, pour des topologies convenables.

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Introduction

In two–dimensional nonlinear shell theories, the stored energy functions are often functions of the *first and second fundamental forms of the unknown deformed middle surface.* For instance, the well–known stored energy function w_K proposed by Koiter [13, Equations (4.2), (8.1), and (8.3)] for modeling shells made with a homogeneous and isotropic elastic material takes the form:

$$w_K = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} (\widetilde{a}_{\sigma\tau} - a_{\sigma\tau})(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} (\widetilde{b}_{\sigma\tau} - b_{\sigma\tau})(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta}),$$

where 2ε is the thickness of the shell,

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

$\lambda > 0$ and $\mu > 0$ are the two *Lamé constants* of the constituting material, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the given undeformed middle surface, $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$, and finally $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the unknown deformed middle surface under the action of given applied forces. Naturally, appropriate boundary conditions should also be specified along the boundary of the middle surface.

The stored energy function w_K was derived by W.T. Koiter from the three–dimensional one on the basis of various *a priori* assumptions of mechanical and geometrical nature. It comprises the *"membrane" part*

$$w_M = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} (\widetilde{a}_{\sigma\tau} - a_{\sigma\tau})(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta})$$

and the *"flexural" part*

$$w_F = \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} (\widetilde{b}_{\sigma\tau} - b_{\sigma\tau})(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta}).$$

The long–standing question of how to rigorously identify two–dimensional equations of nonlinearly elastic shells from three–dimensional elasticity was finally settled in two key contributions, one by Le Dret & Raoult [14] and one by Friesecke, James, Mora & Müller [11], who respectively justified the equations of a *membrane shell* and those of a *flexural shell* by means of Γ–*convergence theory* (a shell is a membrane one if there are no nonzero admissible displacements of its middle surface S that preserve the metric of S; it is a flexural one otherwise).

The stored energy function of a membrane shell is an *ad hoc* quasiconvex envelope, which turns out to be only a function of the covariant components $\widetilde{a}_{\alpha\beta}$ of the first fundamental form of the unknown deformed middle surface. It reduces to the above "membrane" part w_M in Koiter's stored energy function w_K for a specific class of displacement fields of the middle surface. By contrast, the stored energy function of a flexural shell is always equal to the above "flexural" part w_F in Koiter's stored energy function w_K.

Interestingly, a *formal* asymptotic analysis of the three–dimensional equations is only capable of delivering the above "restricted" expression w_M, but otherwise fails to provide the general expression, i.e., valid for all types of displacements, found by Le Dret & Raoult [14]; by contrast, the same formal approach yields the correct expression w_F. For details, see Miara [16], Lods & Miara [15], and Ciarlet [6].

An inspection of the above stored energy functions thus suggests a tempting approach to shell theory, where the functions $\widetilde{a}_{\alpha\beta}$ *and* $\widetilde{b}_{\alpha\beta}$ *would be regarded as the primary unknowns* in lieu of the customary (Cartesian or curvilinear) components of the displacement. In such an approach, the unknown components $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ must naturally satisfy the classical *Gauß* and *Codazzi–Mainardi equations* in order that they actually define a surface. Likewise, the force terms in the energy and the boundary conditions on the displacements must be adequately expressed in terms of these new unknowns.

The present paper, whose results have been announced in Ciarlet [7], constitutes one step in this direction: Its purpose is to establish that *there exist metrizable topologies for which a surface in* $\mathbb{R}^3$ *is a continuous function of its two fundamental forms.* A welcome, but certainly challenging, extension would be to obtain a similar result for Sobolev–type norms, more likely to be encountered in, e.g., an analysis of existence theory undertaken from this perspective.

1 Formulation of the problem

To begin with, we list some notations and conventions that will be consistently used throughout the article.

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^d$, $\mathbb{O}^d$, $\mathbb{S}^d$, and $\mathbb{S}^d_>$ respectively designate the sets of all square matrices of order d, of all orthogonal matrices of order d, of all symmetric matrices of order d, and of all symmetric and positive definite matrices of order d.

Latin indices and exponents vary in the set $\{1, 2, 3\}$ except when they are used for indexing sequences or when otherwise indicated, Greek indices and exponents vary in the set $\{1, 2\}$ except when otherwise indicated, and the summation convention with respect to repeated indices or exponents is used in conjunction with these rules. Kronecker's symbols are designated by δ_{ij} or δ_i^j according to the context.

Let $\boldsymbol{E}^3$ denote a three–dimensional Euclidean space, let $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$ denote the Euclidean inner product and exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{E}^3$, and let $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a} \in \boldsymbol{E}^3$.

Let there be given a *two–dimensional vector space*, identified with $\mathbb{R}^2$. Let y_α denote the coordinates of a point $y \in \mathbb{R}^2$ and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \boldsymbol{E}^3)$ be an *immersion*, i.e., a mapping such that the two vectors $\partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \omega$. The image $\boldsymbol{\theta}(\omega)$ is a *surface* in $\boldsymbol{E}^3$.

The *first fundamental form of the surface* $\boldsymbol{\theta}(\omega)$ is defined by means of its *covariant components*

$$a_{\alpha\beta}(y) := \partial_\alpha \boldsymbol{\theta}(y) \cdot \partial_\beta \boldsymbol{\theta}(y),\ y \in \omega,$$

which are used in particular for computing *lengths of curves on the surface* $\boldsymbol{\theta}(\omega)$, considered as being *isometrically imbedded in* $\boldsymbol{E}^3$.

The *second fundamental form of the surface* $\boldsymbol{\theta}(\omega)$ is defined by means of its *covariant components*

$$b_{\alpha\beta}(y) := \partial_{\alpha\beta}\boldsymbol{\theta}(y) \cdot \left\{ \frac{\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)|} \right\},\ y \in \omega,$$

which, together with those of the first fundamental form, are used for computing *curvatures of curves on the surface* $\boldsymbol{\theta}(\omega)$.

It is well known that the matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$ defined in this fashion cannot be arbitrary. More specifically, given an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \boldsymbol{E}^3)$, let the functions $C_{\alpha\beta\tau} \in \mathcal{C}^1(\omega)$ and $C^\sigma_{\alpha\beta} \in \mathcal{C}^1(\omega)$ be defined by

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \text{ and } C^\sigma_{\alpha\beta} := a^{\sigma\tau} C_{\alpha\beta\tau},\ \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ and some of their partial derivatives must satisfy the following relations (according to our rule governing Greek indices, these relations are meant to hold for all $\alpha, \beta, \sigma, \tau \in \{1, 2\}$):

$$\begin{aligned}\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} &= b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,\\ \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} &= 0 \text{ in } \omega,\end{aligned}$$

which respectively constitute the *Gauß*, and *Codazzi–Mainardi, equations*.

To see this, let $\boldsymbol{a}_\alpha := \partial_\alpha\boldsymbol{\theta}$ and $\boldsymbol{a}_3 := \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$. As is easily verified, the Gauß and Codazzi–Mainardi equations simply amount to re–writing the relations $\partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma$ in the form of the equivalent relations

$$\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_\tau \text{ and } \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3.$$

The vectors $\boldsymbol{a}_\alpha$ introduced above form the *covariant basis of the tangent plane* to the surface $\boldsymbol{\theta}(\omega)$, while the unit vector $\boldsymbol{a}_3$ is *normal* to the surface. The functions $a^{\alpha\beta}$ are the *contravariant* components of the metric tensor, the functions and $C_{\alpha\beta\tau}$ $C^\sigma_{\alpha\beta}$ are the *Christoffel symbols of the first* and *second kind*, and finally, the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu}$$

are the covariant components of the *Riemann–Christoffel curvature tensor* of the surface $\boldsymbol{\theta}(\omega)$.

Remark. The notations $C_{\alpha\beta\tau}$ and $C^\sigma_{\alpha\beta}$ are intended to avoid confusions with the "three–dimensional" Christoffel symbols Γ_{ijq} and Γ^p_{ij} introduced in Section 2. The notations $S_{\tau\alpha\beta\sigma}$ are likewise intended to avoid confusions with the components R_{qijk} of the "three–dimensional" Riemann Christoffel curvature tensor introduced in the same section. □

It is remarkable that, *conversely*, given two smooth enough matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$ under the additional assumptions that ω is connected and simply connected, the Gauß and Codazzi–Mainardi equations are also *sufficient* for the *existence* of an immersion $\boldsymbol{\theta} : \omega \to \boldsymbol{E}^3$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Besides, this immersion is *unique up to isometries in* $\boldsymbol{E}^3$.

A self–contained, complete, and essentially elementary, proof of this well–known result, often referred to as the "*fundamental theorem of surface theory*", is found in Ciarlet & Larsonneur [8]. This proof consists in showing that it can be established as a simple corollary to another well–known result of differential geometry, which asserts that, if the Riemann–Christoffel tensor associated with a field of positive definite symmetric matrices of order three vanishes in a connected and simply connected open subset of $\mathbb{R}^3$, then this field is the metric tensor field of an open set that can be isometrically imbedded in $\mathbb{R}^3$ and this open set is unique up to isometries in $\boldsymbol{E}^3$ (see Theorems 3 and 4 in Section 2). A direct proof of the fundamental theorem of surface theory is given in Klingenberg [12, Theorem 3.8.8]. Its "local" version, which constitutes *Bonnet's theorem*, is proved in, e.g., do Carmo [3].

This result comprises two essentially distinct parts, a *global existence result* (Theorem 1) and a *uniqueness result* (Theorem 2), the latter being called a *rigidity theorem.*

Theorem 1 (global existence theorem) *Let* ω *be a connected and simply connected open subset of* $\mathbb{R}^2$ *and let* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ *be two matrix fields that satisfy the Gauß and Codazzi–Mainardi equations, viz.,*

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,$$
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega,$$

where

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \text{ and } C^\sigma_{\alpha\beta} := a^{\sigma\tau} C_{\alpha\beta\tau}, \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \boldsymbol{E}^3)$ *such that*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

□

Theorem 2 (rigidity theorem) *Let* ω *be a connected open subset of* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \boldsymbol{E}^3)$ *and* $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^2(\omega; \boldsymbol{E}^3)$ *be two immersions such that their associated first and second fundamental forms satisfy* (with self–explanatory notations)

$$a_{\alpha\beta} = \widetilde{a}_{\alpha\beta} \text{ and } b_{\alpha\beta} = \widetilde{b}_{\alpha\beta} \text{ in } \omega.$$

Then there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ *and an orthogonal matrix* $\boldsymbol{Q} \in \mathbb{O}^3$ *such that*

$$\boldsymbol{\theta}(y) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\theta}}(y) \text{ for all } y \in \omega.$$

□

Together, Theorems 1 and 2 establish the existence of a mapping F that associates to any pair of matrix fields $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ satisfying the Gauß and Codazzi–Mainardi equations in ω a well–defined element $F((a_{\alpha\beta}), (b_{\alpha\beta}))$ in the quotient set $\mathcal{C}^3(\omega; \boldsymbol{E}^3)/R$, where $(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}) \in R$ means that there exists a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\theta}(y) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y \in \omega$.

A natural question thus arises as to whether there exist *ad hoc* topologies on the set $\mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^2(\omega; \mathbb{S}^2)$ and on the quotient set $\mathcal{C}^3(\omega; \boldsymbol{E}^3)/R$ such that the mapping F defined in this fashion is *continuous*.

2 A brief review of an analogous problem in dimension three

The purpose of this paper is to provide an affirmative answer to the above question through a proof that relies in an essential way on the solution to an *analogous problem in dimension three*. In this section, we accordingly formulate this analog problem. We also sketch its solution, as given by Ciarlet & Laurent [9, 10], so as to make the present paper self–contained.

To begin with, we introduce some notations specific to the three–dimensional case. Let $\rho(\boldsymbol{A})$ denote the spectral radius and let $|\boldsymbol{A}| := \{\rho(\boldsymbol{A}^T\boldsymbol{A})\}^{1/2}$ denote the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^3$.

Let there be given a three–dimensional vector space, identified with $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$.

Let Ω be an open subset of $\mathbb{R}^3$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If $f \in \mathcal{C}^\ell(\Omega; \mathbb{R})$, $\ell \geq 0$, and $K \Subset \Omega$, we let

$$\|f\|_{\ell,K} := \sup_{\begin{cases} x \in K \\ |\alpha| \leq \ell \end{cases}} |\partial^\alpha f(x)|,$$

where ∂^α stands for the standard multi–index notation for partial derivatives. If $\boldsymbol{\Theta} \in \mathcal{C}^\ell(\Omega; \boldsymbol{E}^3)$ or $\boldsymbol{A} \in \mathcal{C}^\ell(\Omega; \mathbb{M}^3)$ and $K \Subset \Omega$, we let (recall that $|\cdot|$ denotes both the Euclidean and the spectral norm):

$$|\boldsymbol{\Theta}|_{\ell,K} := \sup_{\begin{cases} x \in K \\ |\alpha| = \ell \end{cases}} |\partial^\alpha \boldsymbol{\Theta}(x)| \quad \text{and} \quad \|\boldsymbol{\Theta}\|_{\ell,K} := \sup_{\begin{cases} x \in K \\ |\alpha| \leq \ell \end{cases}} |\partial^\alpha \boldsymbol{\Theta}(x)|,$$

$$\|\boldsymbol{A}\|_{\ell,K} := \sup_{\begin{cases} x \in K \\ |\alpha| \leq \ell \end{cases}} |\partial^\alpha \boldsymbol{A}(x)|.$$

Let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ be an *immersion*, i.e., a mapping such that the three vectors $\partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all points $x \in \Omega$. Then the *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{S}^3_>)$ of the set $\boldsymbol{\Theta}(\Omega)$ (which is open in $\boldsymbol{E}^3$ since $\boldsymbol{\Theta}$ is an immersion) is defined by means of its *covariant components*

$$g_{ij}(x) := \partial_i \boldsymbol{\Theta}(x) \cdot \partial_j \boldsymbol{\Theta}(x),\ x \in \Omega,$$

which are used in particular for computing *lengths of curves inside the set* $\boldsymbol{\Theta}(\Omega)$, considered as being *isometrically imbedded in* $\boldsymbol{E}^3$.

When $\mathbb{R}^3$ is identified with $\boldsymbol{E}^3$, immersions such as $\boldsymbol{\Theta} = (\Theta_i) \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ may be thought of as *deformations* of the set Ω viewed as a *reference configuration*, in the sense of *geometrically exact three–dimensional elasticity* (although they should then be in addition injective and orientation–preserving in order to qualify for this definition; for details, see, e.g., Ciarlet [5, Section 1.4] or Antman [1, Chapter XII, Section 1]). In this context, the matrix $(g_{ij}(x))$ is usually denoted $\boldsymbol{C}(x) := (g_{ij}(x))$, and is called the (right) *Cauchy–Green tensor at* x. Note that one also has

$$(g_{ij}(x)) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x),$$

where $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) := (\partial_j \Theta_i(x)) \in \mathbb{M}^3$ denotes the *deformation gradient* at x (j denotes the column index).

We now recall two classical results from three–dimensional differential geometry, which are essential to the ensuing analysis. Theorem 3 provides sufficient conditions guaranteeing that, given a smooth enough matrix field $\boldsymbol{C} = (g_{ij}) : \Omega \to \mathbb{S}^3_>$, there exists an immersion $\boldsymbol{\Theta} : \Omega \to \boldsymbol{E}^3$ such that $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω, i.e., such that $\boldsymbol{C}$ is the metric tensor field of the set $\boldsymbol{\Theta}(\Omega)$, while Theorem 4 specifies how two such immersions differ (a self–contained, complete, and essentially elementary, proof of these well–known results, whose outline follows with some modifications and simplifications that of Blume [2], is found in Ciarlet & Larsonneur [8]).

Notice the analogies with Theorems 1 and 2.

Theorem 3 (global existence theorem) *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega,$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}, \text{ where } (g^{pq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ such that

$$\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} \text{ in } \Omega.$$

$\square$

Theorem 4 (rigidity theorem) *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ and $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ be two immersions whose associated metric tensors $\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ and $\widetilde{\boldsymbol{C}} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ satisfy*

$$\boldsymbol{C} = \widetilde{\boldsymbol{C}} \text{ in } \Omega.$$

Then there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

$\square$

The functions g^{ij} are the *contravariant* components of the metric tensor, the functions Γ^p_{ij} and Γ_{ijq} are the *Christoffel symbols of the first*, and *second, kind* and finally, the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$$

are the covariant components of the *Riemann–Christoffel curvature tensor*, of the set $\boldsymbol{\Theta}(\Omega)$. The relations $R_{qijk} = 0$ thus express that the *Riemann–Christoffel tensor of the set* $\boldsymbol{\Theta}(\Omega)$ (equipped with the metric tensor with covariant components g_{ij}) *vanishes.* For details, see, e.g., Choquet–Bruhat, Dewitt–Morette & Dillard–Bleick [4, p. 303].

Together, Theorems 3 and 4 establish the existence of a mapping $\mathcal{F}$ that associates to any matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfying $R_{qijk} = 0$ in Ω (the functions R_{qijk} being defined in terms of the functions g_{ij} as in Theorem 3) a well–defined element $\mathcal{F}(\boldsymbol{C})$ in the quotient set $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exists a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$.

As shown by Ciarlet & Laurent [10], there exist topologies on the space $\mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and on the quotient set $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$ such that the mapping $\mathcal{F}$ defined in this fashion is *continuous*. More specifically, the continuity of $\mathcal{F}$ is established as a consequence of the following crucial result, which will likewise play later on a key role (see Part (v) of the proof of Theorem 6).

Theorem 5 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, and $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields respectively satisfying $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$* (with self–explanatory notations), *such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{C}$ in Ω (such mappings exist by Theorem 1). *Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

□

The proof of Theorem 5 is broken into those of three lemmas. Lemma 1 deals with the special case where $\boldsymbol{C} = \boldsymbol{I}$; Lemma 2 deals with the special case where the mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ is injective; finally, Lemma 3 deals with the general case. For conciseness, the proofs of the next lemmas are only sketched. Complete proofs are found in Ciarlet & Laurent [10].

Lemma 1 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{I}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{id}\|_{3,K} = 0 \text{ for all } K \Subset \Omega,$$

where $\boldsymbol{id}$ denotes the identity mapping of $\mathbb{R}^3$, identified here with $\boldsymbol{E}^3$.

Sketch of proof. (i) *Let there be given mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, $n \geq 0$, that satisfy $(\boldsymbol{\nabla\Theta}^n)^T \boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ in Ω* (such mappings exist by Theorem 3). *Then* $\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{\ell,K} = 0$ *for all $K \Subset \Omega$ and for $\ell = 2, 3$.*

Given any immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, let $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$ and let the vectors $\boldsymbol{g}^q$ be defined by means of the relations $\boldsymbol{g}_i \cdot \boldsymbol{g}^q = \delta_i^q$. Then proving (i) relies on the relation

$$\partial_{ij}\boldsymbol{\Theta} = \partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})\boldsymbol{g}^q$$

applied to the mappings $\boldsymbol{\Theta}^n$ and on the uniform boundedness with respect to n of the norms $|(g_{ij}^n)^{-1}|_{0,K}$ on any $K \Subset \Omega$.

(ii) *There exist mappings $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ that satisfy $(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, and $\lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{id}|_{1,K} = 0$ for all $K \Subset \Omega$.*

Let $\boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be mappings that satisfy $(\boldsymbol{\nabla\psi}^n)^T \boldsymbol{\nabla\psi}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$ (such mappings exist by Theorem 3) and let x_0 denote a point in the set Ω. Since $\lim_{n\to\infty} \boldsymbol{\nabla\psi}^n(x_0)^T \boldsymbol{\nabla\psi}^n(x_0) = \boldsymbol{I}$ by assumption, Part (i) implies that there exist orthogonal matrices $\boldsymbol{Q}^n$, $n \geq 0$, such that

$$\lim_{n\to\infty} \boldsymbol{Q}^n \boldsymbol{\nabla\psi}^n(x_0) = \boldsymbol{I}.$$

Then the mappings

$$\widetilde{\boldsymbol{\Theta}}^n := \boldsymbol{Q}^n \boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3),\ n \geq 0,$$

satisfy $(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{C}^n$ in Ω, so that their gradients $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfy

$$\lim_{n\to\infty} \|\partial_i \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n\|_{0,K} = 0 \text{ for all } K \Subset \Omega,$$

by Part (i). In addition, $\lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x_0) = \lim_{n\to\infty} \boldsymbol{Q}^n \boldsymbol{\nabla\psi}^n(x_0) = \boldsymbol{I}$.

Hence a classical theorem about the differentiability of the limit of a sequence of mappings that are continuously differentiable on a connected open set and that take their values in a Banach space (see, e.g., Schwartz [17, Theorem 3.5.12]) shows that *the mappings $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n$ uniformly converge on every compact subset of Ω toward a limit $\boldsymbol{R} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ that satisfies $\partial_i \boldsymbol{R}(x) = \boldsymbol{0}$* for all $x \in \Omega$. This shows that $\boldsymbol{R}$ is a constant mapping since Ω is connected. Consequently, $\boldsymbol{R} = \boldsymbol{I}$ since in particular $\boldsymbol{R}(x_0) = \lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n(x_0) = \boldsymbol{I}$.

(iii) *There exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T \boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, and $\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = 0$ for all $K \Subset \Omega$ and for $\ell = 0, 1$.*

To see this, apply again the theorem about the differentiability of the limit of a sequence of mappings used in Part (ii) to the mappings

$$\boldsymbol{\Theta}^n := (\widetilde{\boldsymbol{\Theta}}^n - \{\widetilde{\boldsymbol{\Theta}}^n(x_0) - x_0\}) \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3),\ n \geq 0.$$

Lemma 2 *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying respectively $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Assume that there exists an injective mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ such that $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \boldsymbol{C}$ in Ω. Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T\boldsymbol{\nabla\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Sketch of proof. Let $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ and define the matrix fields $(\widehat{g}^n_{ij}) \in \mathcal{C}^2(\widehat{\Omega}; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$(\widehat{g}^n_{ij}(\widehat{x})) := \boldsymbol{\nabla\Theta}(x)^{-T}(g^n_{ij}(x))\boldsymbol{\nabla\Theta}(x)^{-1} \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}.$$

Then the assumptions of Lemma 2 imply that $\lim_{n\to\infty} \|\widehat{g}^n_{ij} - \delta_{ij}\|_{2,\widehat{K}} = 0$.

Given $\widehat{x} = (\widehat{x}_i) \in \widehat{\Omega}$, let $\widehat{\partial}_i = \dfrac{\partial}{\partial \widehat{x}_i}$. By Lemma 1 applied over the set $\widehat{\Omega}$, there exist mappings $\widehat{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\widehat{\Omega}; \boldsymbol{E}^3)$ satisfying $\widehat{\partial}_i\widehat{\boldsymbol{\Theta}}^n \cdot \widehat{\partial}_j\widehat{\boldsymbol{\Theta}}^n = \widehat{g}^n_{ij}$ in $\widehat{\Omega}$, $n \geq 0$, such that $\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0$ for all $\widehat{K} \Subset \widehat{\Omega}$. Then the mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, defined by letting $\boldsymbol{\Theta}^n(x) := \widehat{\boldsymbol{\Theta}}^n(\widehat{x})$ for all $x = \widehat{\boldsymbol{\Theta}}(\widehat{x}) \in \Omega$, satisfy $\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0$.

Lemma 3 *The assumption that the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \boldsymbol{E}^3$ is injective is superfluous in* Lemma 2, *all its other assumptions holding* verbatim. *In other words,* Theorem 5 *holds.*

Sketch of proof. (i) *Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \boldsymbol{C}$ in Ω. Then there exists a countable number of open balls $B_r \subset \Omega$, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^\infty B_r$ and such that, for each $r \geq 1$, the set $\bigcup_{s=1}^r B_s$ is connected and the restriction of $\boldsymbol{\Theta}$ to B_r is injective.*

These assertions, which essentially rely on the assumed connectedness of the set Ω, are established by means of an iterative procedure.

(ii) *By* Lemma 2, *there exist mappings $\boldsymbol{\Theta}^n_1 \in \mathcal{C}^3(B_1; \boldsymbol{E}^3)$ and $\widetilde{\boldsymbol{\Theta}}^n_2 \in \mathcal{C}^3(B_2; \boldsymbol{E}^3)$, $n \geq 0$, that satisfy*

$$\begin{aligned}
&(\boldsymbol{\nabla\Theta}^n_1)^T\boldsymbol{\nabla\Theta}^n_1 = \boldsymbol{C}^n \text{ in } B_1 \quad \text{and} \quad \lim_{n\to\infty} \|\boldsymbol{\Theta}^n_1 - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1,\\
&(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n_2)^T\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n_2 = \boldsymbol{C}^n \text{ in } B_2 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}^n_2 - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_2,
\end{aligned}$$

and by Theorem 4, *there exist vectors $\boldsymbol{a}^n \in \boldsymbol{E}^3$ and matrices $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that $\widetilde{\boldsymbol{\Theta}}^n_2(x) = \boldsymbol{a}^n + \boldsymbol{Q}^n\boldsymbol{\Theta}^n_1(x)$ for all $x \in B_1 \cap B_2$. Then $\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0}$ and $\lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}$.*

The proof hinges on the relations

$$\boldsymbol{\Theta}(x) = \lim_{p\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x) = \lim_{p\to\infty} (\boldsymbol{a}^p + \boldsymbol{Q}^p \boldsymbol{\Theta}_1^p(x)) \text{ for all } x \in B_1 \cap B_2.$$

(iii) *Let the mappings* $\boldsymbol{\Theta}_2^n \in \mathcal{C}^3(B_1 \cup B_2; \boldsymbol{E}^3)$, $n \geq 0$, *be defined by* $\boldsymbol{\Theta}_2^n(x) := \boldsymbol{\Theta}_1^n(x)$ *for all* $x \in B_1$, *and* $\boldsymbol{\Theta}_2^n(x) := (\boldsymbol{Q}^n)^T(\widetilde{\boldsymbol{\Theta}}_2^n(x) - \boldsymbol{a}^n)$ *for all* $x \in B_2$. *Then* $\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0$ *for all* $K \Subset B_1 \cup B_2$.

The plane containing the intersection of the boundaries of the open balls B_1 and B_2 is the common boundary of two closed half–spaces in $\mathbb{R}^3$, H_1 containing the center of B_1, and H_2 containing that of B_2 (by construction, the set $B_1 \cup B_2$ is connected; see Part (i)). Any compact subset K of $B_1 \cup B_2$ may thus be written as $K = K_1 \cup K_2$, where $K_1 := (K \cap H_1) \subset B_1$ and $K_2 := (K \cap H_2) \subset B_2$. Hence $\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_1} = 0$ and $\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$, the second relation following from the definition of the mapping $\boldsymbol{\Theta}_2^n$ on $B_2 \supset K_2$ and on the relations $\lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$ (Part (ii)), and $\lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}$ and $\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0}$ (Part (iii)).

(iv) *It remains to iterate the procedure described in* Parts (ii) *and* (iii).

Assume that, for some $r \geq 2$, mappings $\boldsymbol{\Theta}_r^n \in \mathcal{C}^3(\bigcup_{s=1}^r B_s; \boldsymbol{E}^3)$, $n \geq 0$, have been found that satisfy

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_r^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_r^n = \boldsymbol{C}^n \text{ in } \bigcup_{s=1}^r B_s \text{and } \lim_{n\to\infty} \|\boldsymbol{\Theta}_r^n - \boldsymbol{\Theta}\|_{2,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^r B_s.$$

Since the restriction of $\boldsymbol{\Theta}$ to B_{r+1} is injective (Part (i)), Lemma 2 shows that there exist mappings $\widetilde{\boldsymbol{\Theta}}_{r+1}^n \in \mathcal{C}^3(B_{r+1}; \boldsymbol{E}^3)$, $n \geq 0$, that satisfy

$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{r+1}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_{r+1}^n = \boldsymbol{C}^n \text{ in } B_{r+1}, \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_{r+1}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_{r+1},$$

and since the set $\bigcup_{s=1}^{r+1} B_s$ is connected (Part (i)), Theorem 4 shows that there exist vectors $\boldsymbol{c}^n \in \boldsymbol{E}^3$ and matrices $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$\widetilde{\boldsymbol{\Theta}}_{r+1}^n(x) = \boldsymbol{a}^n + \boldsymbol{Q}^n \boldsymbol{\Theta}_r^n(x) \text{ for all } x \in \left(\bigcup_{s=1}^r B_s\right) \cap B_{r+1}.$$

Then an argument similar to that used in Parts (ii) and (iii) shows that the mappings $\boldsymbol{\Theta}_{r+1}^n \in \mathcal{C}^3(\bigcup_{s=1}^r B_s; \boldsymbol{E}^3)$, $n \geq 0$, defined by

$$\boldsymbol{\Theta}_{r+1}^n(x) := \boldsymbol{\Theta}_r^n(x) \text{ for all } x \in \bigcup_{s=1}^r B_s,$$
$$\boldsymbol{\Theta}_{r+1}^n(x) := (\boldsymbol{Q}^n)^T(\widetilde{\boldsymbol{\Theta}}_r^n(x) - \boldsymbol{a}^n) \text{ for all } x \in B_{r+1},$$

satisfy

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_{r+1}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^r B_s.$$

It is then easily seen that the mappings $\mathbf{\Theta}^n : \Omega \to \boldsymbol{E}^3$, $n \geq 0$, defined by

$$\mathbf{\Theta}^n(x) := \mathbf{\Theta}^n_r(x) \text{ for all } x \in \bigcup_{s=1}^{r} B_s,\ r \geq 1,$$

possess all the properties announced in Lemma 3. $\square$

3 A key preliminary result

Let us first introduce the following two–dimensional analogs to the notations used in Section 2. Let ω be an open subset of $\mathbb{R}^3$. The notation $\kappa \Subset \omega$ means that κ is a compact subset of ω. If $f \in \mathcal{C}^\ell(\omega;\mathbb{R})$ or $\boldsymbol{\theta} \in \mathcal{C}^\ell(\omega;\boldsymbol{E}^3), \ell \geq 0$, and $\kappa \Subset \omega$, we let

$$\|f\|_{\ell,\kappa} := \sup_{\begin{cases} y\in\kappa \\ |\alpha|\leq\ell \end{cases}} |\partial^\alpha f(y)| \quad , \quad \|\boldsymbol{\theta}\|_{\ell,\kappa} := \sup_{\begin{cases} y\in\kappa \\ |\alpha|\leq\ell \end{cases}} |\partial^\alpha \boldsymbol{\theta}(y)|,$$

where ∂^α stands for the standard multi–index notation for partial derivatives and $|\cdot|$ denotes the Euclidean norm in the latter definition. If $\boldsymbol{A} \in \mathcal{C}^\ell(\omega;\mathbb{M}^3), \ell \geq 0$, and $\kappa \Subset \omega$, we likewise let

$$\|\boldsymbol{A}\|_{\ell,\kappa} = \sup_{\begin{cases} y\in\kappa \\ |\alpha|\leq\ell \end{cases}} |\partial^\alpha \boldsymbol{A}(y)|,$$

where $|\cdot|$ denotes the matrix spectral norm.

The next theorem constitutes the key step towards establishing the continuity of a surface as a function of its two fundamental forms (see Theorem 7 in Section 4).

Theorem 6 *Let ω be a connected and simply connected open subset of $\mathbb{R}^2$. Let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ be matrix fields satisfying the Gauß and Codazzi–Mainardi equations in ω and let $(a^n_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b^n_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ be matrix fields satisfying for each $n \geq 0$ the Gauß and Codazzi–Mainardi equations in ω. Assume that these matrix fields satisfy*

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\boldsymbol{E}^3)$ be any mapping that satisfies

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \right\} \text{ in } \omega$$

(such mappings exist by Theorem 1). *Then there exist mappings $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega;\boldsymbol{E}^3)$ satisfying*

$$a^n_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta}^n \cdot \partial_\beta\boldsymbol{\theta}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1\boldsymbol{\theta}^n \wedge \partial_2\boldsymbol{\theta}^n}{|\partial_1\boldsymbol{\theta}^n \wedge \partial_2\boldsymbol{\theta}^n|} \right\} \text{ in } \omega,\ n \geq 0,$$

such that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Proof. For clarity, the proof is broken into five parts.

(i) *Let the matrix fields* $(g_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3)$ *and* $(g^n_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3), n \geq 0$, *be defined by*

$$\begin{aligned} g_{\alpha\beta} &:= a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{and} \quad g_{i3} := \delta_{i3}, \\ g^n_{\alpha\beta} &:= a^n_{\alpha\beta} - 2x_3 b^n_{\alpha\beta} + x_3^2 c^n_{\alpha\beta} \quad \text{and} \quad g^n_{i3} := \delta_{i3}, n \geq 0 \end{aligned}$$

(the variable $y \in \omega$ is omitted, x_3 designates the variable in $\mathbb{R}$), *where*

$$\begin{aligned} c_{\alpha\beta} &:= b^\tau_\alpha b_{\beta\tau}, \quad b^\tau_\alpha := a^{\sigma\tau} b_{\alpha\sigma}, \quad (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}, \\ c^n_{\alpha\beta} &:= b^{\tau,n}_\alpha b^n_{\beta\tau}, \quad b^{\tau,n}_\alpha := a^{\sigma\tau,n} b^n_{\alpha\sigma}, \quad (a^{\sigma\tau,n}) := (a^n_{\alpha\beta})^{-1}, n \geq 0. \end{aligned}$$

Let ω_0 *be an open subset of* $\mathbb{R}^2$ *such that* $\overline{\omega}_0 \Subset \omega$. *Then there exists* $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ *such that the symmetric matrices*

$$\boldsymbol{C}(y, x_3) := (g_{ij}(y, x_3)) \text{ and } \boldsymbol{C}^n(y, x_3) := (g^n_{ij}(y, x_3)),\ n \geq 0,$$

are positive definite at all points $(y, x_3) \in \overline{\Omega}_0$, *where*

$$\Omega_0 := \omega_0 \times \left]-\varepsilon_0, \varepsilon_0\right[.$$

The matrices $\boldsymbol{C}(y, x_3) \in \mathbb{S}^3$ and $\boldsymbol{C}^n(y, x_3) \in \mathbb{S}^3$ are of the form (the notations are self–explanatory):

$$\begin{aligned} \boldsymbol{C}(y, x_3) &= \boldsymbol{C}_0(y) + x_3 \boldsymbol{C}_1(y) + x_3^2 \boldsymbol{C}_2(y), \\ \boldsymbol{C}^n(y, x_3) &= \boldsymbol{C}^n_0(y) + x_3 \boldsymbol{C}^n_1(y) + x_3^2 \boldsymbol{C}^n_2(y),\ n \geq 0. \end{aligned}$$

First, it is easily deduced from the matrix dentity $\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{I} + \boldsymbol{A}^{-1}(\boldsymbol{B} - \boldsymbol{A}))$ and the assumptions $\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{0,\overline{\omega}_0} = 0$ and $\lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{0,\overline{\omega}_0} = 0$ that there exists a constant M such that

$$\|(\boldsymbol{C}^n_0)^{-1}\|_{0,\overline{\omega}_0} + \|\boldsymbol{C}^n_1\|_{0,\overline{\omega}_0} + \|\boldsymbol{C}^n_2\|_{0,\overline{\omega}_0} \leq M \text{ for all } n \geq 0.$$

This uniform bound and the relations

$$\begin{aligned} \boldsymbol{C}(y, x_3) &= \boldsymbol{C}_0(y)\{\boldsymbol{I} + (\boldsymbol{C}_0(y))^{-1}(-2x_3 \boldsymbol{C}_1(y) + x_3^2 \boldsymbol{C}_2(y))\}, \\ \boldsymbol{C}^n(y, x_3) &= \boldsymbol{C}^n_0(y)\{\boldsymbol{I} + (\boldsymbol{C}^n_0(y))^{-1}(-2x_3 \boldsymbol{C}^n_1(y) + x_3^2 \boldsymbol{C}^n_2(y))\},\ n \geq 0, \end{aligned}$$

together imply that there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the matrices $\boldsymbol{C}(y, x_3)$ and $\boldsymbol{C}^n(y, x_3)$, $n \geq 0$, are invertible for all $(y, x_3) \in \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$.

These matrices are positive definite for $x_3 = 0$ by assumption. Hence they remain so for all $x_3 \in [-\varepsilon_0, \varepsilon_0]$ since they are invertible.

(ii) *Let* ω_ℓ, $\ell \geq 0$, *be open subsets of* $\mathbb{R}^2$ *such that* $\overline{\omega}_\ell \Subset \omega$ *for each* ℓ *and* $\omega = \bigcup_{\ell \geq 0} \omega_\ell$. *By* (i), *there exist numbers* $\varepsilon_\ell = \varepsilon_\ell(\omega_\ell) > 0$, $\ell \geq 0$, *such that the symmetric matrices* $\boldsymbol{C}(x) = (g_{ij}(x))$ *and* $\boldsymbol{C}^n(x) = (g^n_{ij}(x))$, $n \geq 0$, *defined for all* $x = (y, x_3) \in \omega \times \mathbb{R}$ *as* *in* (i), *are positive definite at all points* $x = (y, x_3) \in \overline{\Omega}_\ell$, *where* $\Omega_\ell := \omega_\ell \times]-\varepsilon_\ell, \varepsilon_\ell[$, *hence at all points* $x = (y, x_3)$ *of the open set*

$$\Omega := \bigcup_{\ell \geq 0} \Omega_\ell,$$

which is connected and simply connected.

The set Ω is connected since it is clearly arcwise connected. To show that Ω is simply connected, let $\boldsymbol{\gamma}$ be a loop in Ω, i.e., a mapping $\boldsymbol{\gamma} \in \mathcal{C}^0([0,1]; \mathbb{R}^3)$ that satisfies

$$\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1) \text{ and } \boldsymbol{\gamma}(t) \in \Omega \text{ for all } 0 \leq t \leq 1.$$

Let the projection operator $\boldsymbol{\pi} : \Omega \to \omega$ be defined by $\boldsymbol{\pi}(y, x_3) = y$ for all $(y, x_3) \in \Omega$, and let the mapping $\boldsymbol{\varphi}_0 : [0,1] \times [0,1] \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\varphi}_0(t, \lambda) := (1-\lambda)\boldsymbol{\gamma}(t) + \lambda \boldsymbol{\pi}(\boldsymbol{\gamma}(t)) \text{ for all } 0 \leq t \leq 1,\ 0 \leq \lambda \leq 1.$$

Then $\boldsymbol{\varphi}_0$ is a continuous mapping such that $\boldsymbol{\varphi}_0([0,1] \times [0,1]) \subset \Omega$; furthermore, $\boldsymbol{\varphi}_0(t, 0) = \boldsymbol{\gamma}(t)$ and $\boldsymbol{\varphi}_0(t, 1) = \boldsymbol{\pi}(\boldsymbol{\gamma}(t))$ for all $t \in [0,1]$. The mapping

$$\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\pi} \circ \boldsymbol{\gamma} \in \mathcal{C}^0([0,1]; \mathbb{R}^2)$$

is a loop in ω since $\widetilde{\boldsymbol{\gamma}}(0) = \boldsymbol{\pi}(\boldsymbol{\gamma}(0)) = \boldsymbol{\pi}(\boldsymbol{\gamma}(1)) = \widetilde{\boldsymbol{\gamma}}(1)$. Since ω is simply connected, there exist a mapping $\boldsymbol{\varphi}_1 \in \mathcal{C}^0([0,1] \times [0,1]; \mathbb{R}^2)$ and a point $y^0 \in \omega$ such that

$$\boldsymbol{\varphi}_1(t, 1) = \widetilde{\boldsymbol{\gamma}}(t) \text{ and } \boldsymbol{\varphi}_1(t, 2) = y^0 \text{ for all } 0 \leq t \leq 1,$$

and

$$\boldsymbol{\varphi}_1(t, \lambda) \in \omega \text{ for all } 0 \leq t \leq 1,\ 1 \leq \lambda \leq 2.$$

Then the mapping $\boldsymbol{\varphi} \in \mathcal{C}^0([0,1] \times [0,2]; \mathbb{R}^3)$ defined by

$$\begin{aligned} \boldsymbol{\varphi}(t, \lambda) &= \boldsymbol{\varphi}_0(t, \lambda) \quad \text{for all} \quad 0 \leq t \leq 1,\ 0 \leq \lambda \leq 1, \\ \boldsymbol{\varphi}(t, \lambda) &= \boldsymbol{\varphi}_1(t, \lambda) \quad \text{for all} \quad 0 \leq t \leq 1,\ 1 \leq \lambda \leq 2, \end{aligned}$$

is a homotopy in Ω that reduces the loop $\boldsymbol{\gamma}$ to the point $(y^0, 0) \in \Omega$. Hence the set Ω is simply connected.

(iii) *The set* Ω *being defined as in* (ii), *let the functions* $R_{qijk} \in \mathcal{C}^0(\Omega)$ *and* $R^n_{qijk} \in \mathcal{C}^0(\Omega)$, $n \geq 0$, *be constructed as in* Theorem 3 *from the matrix fields* $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *and* $(g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$. *Then*

$$R_{qijk} = 0 \text{ in } \Omega \text{ and } R^n_{qijk} = 0 \text{ in } \Omega \text{ for all } n \geq 0.$$

We simply indicate here the flavor of the proof of this crucial result. Its detailed proof is provided in Ciarlet & Larsonneur [8], where it was also used in an essential way.

First, one shows that at any point in the set $\overline{\Omega}_0 = \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is determined as in (i), the matrix $(g^{pq}) := (g_{ij})^{-1}$ is given by

$$g^{\alpha\beta} = \sum_{n \geq 0} (n+1) x_3^n a^{\alpha\sigma} (\boldsymbol{B}^n)^\beta_\sigma \text{ and } g^{i3} = \delta^{i3},$$

where

$$b_\sigma^\tau := a^{\alpha\tau} b_{\alpha\sigma}, \quad (\boldsymbol{B})_\sigma^\beta := b_\sigma^\beta \text{ and } (\boldsymbol{B}^n)_\sigma^\beta := b_\sigma^{\sigma_1} \cdots b_{\sigma_{n-1}}^\beta \text{ for } n \geq 2,$$

i.e., $(\boldsymbol{B}^n)_\sigma^\beta$ designates for any $n \geq 0$ the element at the α–*th* row and β–*th* column of the matrix $\boldsymbol{B}^n$. Each one of the above series is absolutely convergent in the space $\mathcal{C}^2(\overline{\Omega}_0)$.

Straightforward computations then show that the functions $\Gamma_{ijq} = \Gamma_{jiq} \in \mathcal{C}^1(\overline{\Omega}_0)$ and $\Gamma_{ij}^p = \Gamma_{ji}^p \in \mathcal{C}^1(\overline{\Omega}_0)$ defined by

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma_{ij}^p := g^{pq}\Gamma_{ijq}$$

have the following expressions:

$$\begin{aligned}
&\Gamma_{\alpha\beta\sigma} = C_{\alpha\beta\sigma} - x_3(b_\alpha^\tau|_\beta a_{\tau\sigma} + 2C_{\alpha\beta}^\tau b_{\tau\sigma}) + x_3^2(b_\alpha^\tau|_\beta b_{\tau\sigma} + C_{\alpha\beta}^\tau c_{\tau\sigma}),\\
&\Gamma_{\alpha\beta 3} = -\Gamma_{\alpha 3\beta} = b_{\alpha\beta} - x_3 c_{\alpha\beta},\\
&\Gamma_{\alpha 33} = \Gamma_{3\beta 3} = \Gamma_{33q} = 0,\\
&\Gamma_{\alpha\beta}^\sigma = C_{\alpha\beta}^\sigma - \sum\nolimits_{n\geq 0} x_3^{n+1} b_\alpha^\tau|_\beta (\boldsymbol{B}^n)_\tau^\sigma,\\
&\Gamma_{\alpha\beta}^3 = b_{\alpha\beta} - x_3 c_{\alpha\beta},\\
&\Gamma_{\alpha 3}^\beta = -\sum\nolimits_{n\geq 0} x_3^n (\boldsymbol{B}^{n+1})_\alpha^\beta,\\
&\Gamma_{3\beta}^3 = \Gamma_{33}^p = 0,
\end{aligned}$$

where

$$b_\alpha^\tau|_\beta := \partial_\beta b_\alpha^\tau + C_{\beta\mu}^\tau b_\alpha^\mu - C_{\alpha\beta}^\mu b_\mu^\tau,$$

and the Christoffel symbols $C_{\alpha\beta\tau}$ and $C_{\alpha\beta}^\sigma$ are defined from the functions $a_{\alpha\beta}$ as in Theorem 1. We simply point out that the assumed *Codazzi–Mainardi equations* are needed to conclude that the factor of x_3 in the function $\Gamma_{\alpha\beta\sigma}$ is indeed that announced. We also note that the computation of the factor of x_3^2 in $\Gamma_{\alpha\beta\sigma}$ relies in particular on the relations

$$\partial_\alpha c_{\beta\sigma} = b_\beta^\tau|_\alpha b_{\sigma\tau} + b_\sigma^\mu|_\alpha b_{\mu\beta} + C_{\alpha\beta}^\mu c_{\sigma\mu} + C_{\alpha\sigma}^\mu c_{\beta\mu}.$$

Define next the functions $R_{qijk} \in \mathcal{C}^0(\overline{\Omega}_0)$ by

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}.$$

Observing that, in order that the relations

$$R_{qijk} = 0 \text{ in } \overline{\Omega}_0$$

hold, it is sufficient that

$$R_{1212} = 0, \quad R_{\alpha 2\beta 3} = 0, \quad R_{\alpha 3\beta 3} = 0 \text{ in } \overline{\Omega}_0,$$

it is then established that these last relations indeed hold, by means of a series of elementary, but lengthy and sometimes delicate, computations. Note that while neither the assumed *Gauß* nor the assumed *Codazzi–Mainardi* equations are needed for establishing the relations $R_{\alpha 3\beta 3} = 0$ in $\overline{\Omega}_0$, the latter are needed for establishing $R_{\alpha 2\beta 3} = 0$ in $\overline{\Omega}$ and the former are needed for establishing $R_{1212} = 0$ in $\overline{\Omega}_0$.

By repeating the same computations over each one of the sets $\overline{\Omega}_\ell = \overline{\omega}_\ell \times [-\varepsilon_\ell, \varepsilon_\ell]$, $\ell \geq 0$, found in Part (ii), we conclude that the functions R_{qijk} vanish in Ω. The same argument shows that the functions R^n_{qijk} vanish in Ω for all $n \geq 0$.

(iv) *The matrix fields* $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *and* $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *defined in* (ii) *satisfy*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Given any compact subset K of Ω, there exists a finite set Λ_K of integers such that $K \subset \bigcup_{\ell\in\Lambda_K} \Omega_\ell$. Since by assumption,

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0,\ \ell \in \Lambda_K,$$

it follows that

$$\lim_{n\to\infty} \|\boldsymbol{C}^n_p - \boldsymbol{C}_p\|_{2,\overline{\omega}_\ell} = 0,\ \ell \in \Lambda_k,\ p = 0, 1, 2,$$

where the matrices $\boldsymbol{C}_p$ and $\boldsymbol{C}^n_p$, $n \geq 0$, $p = 0, 1, 2$, are those defined in the proof of Part (i). The definition of the norm $\|\cdot\|_{2,\overline{\Omega}_\ell}$ then implies that

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,\overline{\Omega}_\ell} = 0,\ \ell \in \Lambda_K.$$

The conclusion follows from the finiteness of the set Λ_K.

(v) *Conclusion of the proof.*

Given any mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \boldsymbol{E}^3)$ that satisfies

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \right\} \text{ in } \omega,$$

let the mapping $\boldsymbol{\Theta} : \Omega \to \boldsymbol{E}^3$ be defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \Omega,$$

where $\boldsymbol{a}_3 := \dfrac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}$, and let

$$g_{ij} := \partial_i\boldsymbol{\Theta} \cdot \partial_j\boldsymbol{\Theta}.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \Omega,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $c_{\alpha\beta} = a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

In other words, *the matrices* (g_{ij}) *constructed in this fashion coincide over the set* Ω *with those defined in* (i). Since Parts (ii), (iii), and (iv) of the above proof together show that all the assumptions of Theorem 5 are satisfied by the fields $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \textit{ for all } K \Subset \Omega.$$

We now show that the mappings

$$\boldsymbol{\theta}^n(\cdot) := \boldsymbol{\Theta}^n(\cdot, 0) \in \mathcal{C}^3(\omega; \boldsymbol{E}^3)$$

indeed satisfy

$$a^n_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta}^n \cdot \partial_\beta \boldsymbol{\theta}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n}{|\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n|} \right\} \text{ in } \omega.$$

Dropping the exponent n for notational convenience in this part of the proof, let $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$. Then $\partial_{33}\boldsymbol{\Theta} = \partial_3 \boldsymbol{g}_3 = \Gamma^p_{33}\boldsymbol{g}_p = \boldsymbol{0}$ (see Part (iii)). Hence there exists a mapping $\boldsymbol{\theta}^1 \in \mathcal{C}^3(\omega; \boldsymbol{E}^3)$ such that

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{\theta}^1(y) \text{ for all } (y, x_3) \in \Omega.$$

Consequently, $\boldsymbol{g}_\alpha = \partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1$ and $\boldsymbol{g}_3 = \boldsymbol{\theta}^1$. The relations $g_{i3} = \boldsymbol{g}_i \cdot \boldsymbol{g}_3 = \delta_{i3}$ then show that

$$(\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1) \cdot \boldsymbol{\theta}^1 = 0 \text{ and } \boldsymbol{\theta}^1 \cdot \boldsymbol{\theta}^1 = 1.$$

These relations imply that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{\theta}^1 = 0$. Hence either $\boldsymbol{\theta}^1 = \boldsymbol{a}_3$ or $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ in ω. But $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ is ruled out since we must have

$$\{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}\} \cdot \boldsymbol{\theta}^1 = \det(g_{ij})|_{x_3=0} > 0.$$

Noting that

$$\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \text{ implies } \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = -\partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3,$$

we obtain, on the one hand,

$$\begin{aligned} g_{\alpha\beta} &= (\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{a}_3) \cdot (\partial_\beta \boldsymbol{\theta} + x_3 \partial_\beta \boldsymbol{a}_3) \\ &= \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} - 2x_3 \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3 + x_3^2 \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \text{ in } \Omega. \end{aligned}$$

Since, on the other hand,

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ in } \Omega,$$

we conclude that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3 \text{ in } \omega,$$

as desired.

It remains to verify that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

But these relations immediately follow from the relations

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega,$$

combined with the observations that a compact subset of ω is also one of Ω, that $\boldsymbol{\Theta}(\cdot, 0) = \boldsymbol{\theta}$ and $\boldsymbol{\Theta}^n(\cdot, 0) = \boldsymbol{\theta}^n$, and finally, that

$$\|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} \le \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,\kappa}.$$

□

Remark. At first glance, it seems that Theorem 6 could be established by a proof similar to that of its "three–dimensional counterpart", viz. Theorem 5. A quick inspection reveals, however, that the proof of Lemma 2 does not carry over to the situation covered by the former. □

4 Continuity in metric spaces

Let ω be an open subset of $\mathbb{R}^3$. For any integers $\ell \ge 0$ and $d \ge 1$, the space $\mathcal{C}^\ell(\omega; \mathbb{R}^d)$ becomes a *locally convex topological space* when its topology is defined by the family of semi–norms $\|\cdot\|_{\ell,\kappa}$, $\kappa \Subset \omega$, and a sequence $(\boldsymbol{\theta}^n)_{n\ge 0}$ converges to $\boldsymbol{\theta}$ with respect to this topology if and only if

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{\ell,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Furthermore, this topology is *metrizable*: Let $(\kappa_i)_{i\ge 0}$ be any sequence of subsets of ω that satisfy

$$\kappa_i \Subset \omega \text{ and } \kappa_i \subset \operatorname{int} \kappa_{i+1} \text{ for all } i \ge 0, \text{ and } \omega = \bigcup_{i=0}^{\infty} \kappa_i.$$

Then

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{\ell,\kappa} = 0 \text{ for all } \kappa \Subset \omega \iff \lim_{n\to\infty} d_\ell(\boldsymbol{\theta}^n, \boldsymbol{\theta}) = 0,$$

where

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\theta}\|_{\ell,\kappa_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\theta}\|_{\ell,\kappa_i}}.$$

For details, see, e.g., Yosida [18, Chapter 1].

Let $\dot{\mathcal{C}}^3(\omega; \boldsymbol{E}^3) := \mathcal{C}^3(\omega; \boldsymbol{E}^3)/R$ denote the quotient set of $\mathcal{C}^3(\omega; \boldsymbol{E}^3)$ by the equivalence relation R, where $(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}) \in R$ means that there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\theta}(y) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y \in \omega$. Then it is easily verified that the set $\dot{\mathcal{C}}^3(\omega; \boldsymbol{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\psi}}) := \inf_{\begin{cases} \boldsymbol{\kappa} \in \dot{\boldsymbol{\theta}} \\ \boldsymbol{\chi} \in \dot{\boldsymbol{\psi}} \end{cases}} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}) = \inf_{\begin{cases} \boldsymbol{a} \in \boldsymbol{E}^3 \\ \boldsymbol{Q} \in \mathbb{O}^3 \end{cases}} d_3(\boldsymbol{\theta}, \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\theta}}$ denotes the equivalence class of $\boldsymbol{\theta}$ *modulo* R.

The announced continuity of a surface as a function of its two fundamental forms is then a corollary to Theorem 6. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 7 *Let ω be connected and simply connected open subset of $\mathbb{R}^2$. Let*

$$\begin{aligned}\mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2) := \{((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>) \times \mathcal{C}^2(\omega; \mathbb{S}^2);\\ \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} \;&=\; b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,\\ \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} \;&=\; 0 \text{ in } \omega\}.\end{aligned}$$

Given any element $((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2)$, let $F(((a_{\alpha\beta}), (b_{\alpha\beta}))) \in \dot{\mathcal{C}}^3(\omega; \boldsymbol{E}^3)$ denote the equivalence class modulo R *of any $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \boldsymbol{E}^3)$ that satisfies*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Then the mapping

$$F : \{\mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2); d_2\} \to \{\dot{\mathcal{C}}^3(\omega; \boldsymbol{E}^3); \dot{d}^3\}$$

defined in this fashion is continuous.

Proof. Since $\{\mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}); d_2\}$ and $\{\dot{\mathcal{C}}^3(\omega; \boldsymbol{E}^3); \dot{d}^3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through F into convergent sequences.

Let then $((a_{\alpha\beta}), (b_{\alpha\beta})) \in \mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2)$ and $((a^n_{\alpha\beta}), (b^n_{\alpha\beta})) \in \mathcal{C}_0^2(\omega; \mathbb{S}^2_> \times \mathbb{S}^2)$, $n \geq 0$, be such that

$$\lim_{n \to \infty} d_2(((a^n_{\alpha\beta}), (b^n_{\alpha\beta})), ((a_{\alpha\beta}), (b_{\alpha\beta}))) = 0,$$

i.e., such that

$$\lim_{n \to \infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \textit{ and } \lim_{n \to \infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Let there be given any $\boldsymbol{\theta} \in F(((a_{\alpha\beta}), (b_{\alpha\beta})))$. Then Theorem 6 shows that there exist $\boldsymbol{\theta}^n \in F(((a^n_{\alpha\beta}), (b^n_{\alpha\beta})))$, $n \geq 0$, such that

$$\lim_{n \to \infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega,$$

i.e., such that

$$\lim_{n \to \infty} d_3(\boldsymbol{\theta}^n, \boldsymbol{\theta}) = 0.$$

Consequently,

$$\lim_{n \to \infty} \dot{d}_3(F(((a^n_{\alpha\beta}), (b^n_{\alpha\beta}))), F(((a_{\alpha\beta}), (b_{\alpha\beta})))) = 0,$$

and the proof is complete. $\square$

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