On Rigid and Infinitesimal Rigid Displacements in Shell Theory

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Abstract

Let ω be an open connected subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}$ be an immersion from ω into $\mathbb{R}^3$. It is first established that the set formed by all rigid displacements, i.e., that preserve the metric and the curvature, of the surface $\boldsymbol{\theta}(\omega)$ is a submanifold of dimension 6 and of class $\mathcal{C}^\infty$ of the space $\boldsymbol{H}^1(\omega)$. It is then shown that the vector space formed by all the infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\omega)$ is nothing but the tangent space at the origin to this submanifold. In this fashion, the "infinitesimal rigid displacement lemma on a surface", which plays a key rôle in shell theory, is put in its proper perspective.

Résumé

Soit ω un ouvert connexe de $\mathbb{R}^2$ et soit $\boldsymbol{\theta}$ une immersion de ω dans $\mathbb{R}^3$. On établit d'abord que l'ensemble formé par tous les déplacements rigides, c'est-à-dire ceux qui préservent la métrique et la courbure, de la surface $\boldsymbol{\theta}(\omega)$ est une sous-variété de dimension 6 et de classe $\mathcal{C}^\infty$ de l'espace $\boldsymbol{H}^1(\omega)$. On établit ensuite que l'espace vectoriel formé par tous les déplacements rigides infinitésimaux de la surface $\boldsymbol{\theta}(\omega)$ n'est autre que l'espace tangent à cette sous-variété à l'origine. De cette façon, le "lemme du déplacement rigide infinitésimal sur une surface", qui joue un rôle important en théorie des coques, est placé en perspective.

Keywords: Submanifold; Rigidity theorem; Infinitesimal rigid displacement lemma; Shell theory

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Introduction

Further details about the various notions and notations used here are provided in the next sections.

The following *"infinitesimal rigid displacement lemma on a surface"* plays a crucial rôle in *linearized shell theory*: Let ω be an open connected subset of $\mathbb{R}^2$, let $\boldsymbol{\theta}$ be a smooth enough immersion from ω into a three-dimensional Euclidean space $\mathbb{E}^3$, and let $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ be a vector field that satisfies

$$\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0 \text{ a.e. in } \omega \text{ and } \rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0 \text{ in } H^{-1}(\omega),$$

where

$$\begin{aligned}\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) &= \frac{1}{2}(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha),\\ \rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) &= \partial_{\alpha\beta}(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \partial_\beta \boldsymbol{a}_3 - \partial_\beta(\tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3) - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3,\end{aligned}$$

the vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are tangent to the surface $\boldsymbol{\theta}(\omega)$, the unit vector $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$ is normal to $\boldsymbol{\theta}(\omega)$, and the functions $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols. *Then there exist vectors $\boldsymbol{c} \in \mathbb{E}^3$ and $\boldsymbol{d} \in \mathbb{E}^3$ such that*

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{\theta}(y) \text{ for almost all } y \in \omega.$$

The infinitesimal rigid displacement lemma on a surface was first established in [4, Theorem 5.1-1] for vector fields $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ such that $\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in \boldsymbol{H}^2(\omega)$, under the assumptions that ω is bounded with a Lipschitz-continuous boundary and that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega})$. See also [5], or [7, Theorem 2.6-2], [2], [11] for simpler proofs, or [3], [6] for generalizations under substantially weaker regularity assumptions on the mapping $\boldsymbol{\theta}$.

In shell theory, the set $\boldsymbol{\theta}(\omega) \subset \mathbb{E}^3$ is viewed as the *reference configuration* of the middle surface of an elastic shell and the field $\tilde{\boldsymbol{\eta}}$ is viewed as a *displacement field* of the surface $\boldsymbol{\theta}(\omega)$.

The functions $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ and $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ are the covariant components of the *linearized change of metric tensor* and of the *linearized change of curvature tensor* associated with a displacement field $\tilde{\boldsymbol{\eta}}$ and a displacement field of the above form $\tilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{\theta}$ is called an *infinitesimal rigid displacement* of the surface $\boldsymbol{\theta}(\omega)$.

The infinitesimal rigid displacement lemma on a surface plays a crucial rôle for establishing the *uniqueness* (possibly in a quotient space) and, in conjunction with an *inequality of Korn's type on a surface*, the *existence* of solutions to the well-known *Koiter's equations*, proposed in 1970 by W.T. Koiter [13] for modeling *linearly elastic shells*. For details, see [7, Chapter 2].

One objective of this paper is *to put this lemma in its proper perspective*, as the linearized counterpart of the familiar *rigidity theorem* of surface theory, once this theorem has been properly extended to the Sobolev space $\boldsymbol{H}^1(\omega)$.

This extension, which is carried out in Theorem 3, itself relies on an extension, due to [9], to a Sobolev space setting of the familiar rigidity theorem for open sets in three-dimensional differential geometry. For convenience this extension is first reviewed in Section 1 (see Theorem 1).

It is then shown in Theorem 4 and its corollary that the set $\boldsymbol{M}_{\text{rig}}$ formed by all the *rigid displacements* of the surface $\boldsymbol{\theta}(\omega)$, i.e., those that satisfy the assumptions of the extended rigidity theorem, is a *submanifold of dimension* 6 *and of class* $\mathcal{C}^\infty$ *of the space* $\boldsymbol{H}^1(\omega)$.

It is finally established in Theorem 5 that the vector space spanned by the infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\omega)$ is nothing but the *tangent space at the origin to the manifold* $\boldsymbol{M}_{\text{rig}}$. This result hinges on the well-known characterization of the tangent space at $\boldsymbol{I}$ to the special orthogonal group and on an extension, also due to [9], of the three-dimensional infinitesimal rigid displacement lemma in curvilinear coordinates. For convenience, this extension is also reviewed in Section 1 (see Theorem 2).

The results of this paper have been announced in [10].

1 Preliminaries

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^3, \mathbb{O}^3, \mathbb{O}^3_+$, and $\mathbb{A}^3$ respectively designate the sets of all square matrices of order 3, of all orthogonal matrices of order 3, of all matrices $\boldsymbol{Q} \in \mathbb{O}^3$ with $\det \boldsymbol{Q} = 1$, and of all antisymmetric matrices of order 3.

Latin indices range over the set $\{1, 2, 3\}$ except when they are used for indexing sequences, and the summation convention with respect to repeated indices is used in conjunction with this rule.

The notation $\mathbb{E}^3$ designates a three-dimensional Euclidean space and $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{a} \wedge \boldsymbol{b}$, and $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ respectively designate the Euclidean inner product and the exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3$, and the Euclidean norm of $\boldsymbol{a} \in \mathbb{E}^3$.

Let Ω be an open subset of $\mathbb{R}^3$, let x_i denote the coordinates of a point $x \in \mathbb{R}^3$, and let $\partial_i := \partial/\partial x_i$. Let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{E}^3)$ be an *immersion*, i.e., a mapping such that the three vectors $\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all points $x \in \Omega$. The *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ of the set $\boldsymbol{\Theta}(\Omega)$ (which is open in $\mathbb{E}^3$ since $\boldsymbol{\Theta}$ is an immersion; see, e.g., [15, Theorem 3.8.10]) is then defined by means of its covariant components

$$g_{ij}(x) := \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x), \quad x \in \Omega.$$

The classical *rigidity theorem for open sets* asserts that, if two immersions $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{\mathcal{C}}^1(\Omega) := \mathcal{C}^1(\Omega; \mathbb{E}^3)$ and $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega)$ have the same metric tensor fields, i.e., if $\widetilde{g}_{ij} = g_{ij}$ in Ω (with self-explanatory notations), and Ω is connected, then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \text{ for all } x \in \Omega.$$

For a proof, see, e.g., [8, Theorem 3].

The following result, established in [9], shows that *a similar result holds under the assumption that* $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega) := H^1(\Omega, \mathbb{E}^3)$. Note that this extension itself relies on a crucial extension of the classical Liouville theorem, originally due to Reshetnyak [14] and recently given a particularly concise and elegant proof by Friesecke, James and Müller [12]. The notation $\boldsymbol{\nabla\Theta}(x)$ designates the matrix whose columns are the vectors $\boldsymbol{g}_i(x)$, $x \in \Omega$.

Theorem 1 *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega)$ be a mapping that satisfies* $\det \boldsymbol{\nabla\Theta} > 0$ *in Ω. Assume that there exists a vector field $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$ that satisfies*

$$\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \text{ a.e. in } \Omega \quad \text{and} \quad \widetilde{g}_{ij} = g_{ij} \text{ a.e. in } \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

□

In three-dimensional elasticity, the set $\boldsymbol{\Theta}(\Omega)$ is viewed as the *reference configuration* of a three-dimensional elastic body (under the additional, but irrelevant here, assumption that the immersion $\boldsymbol{\Theta}$ is injective) and a field $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ is viewed as a *displacement field* of the reference configuration $\boldsymbol{\Theta}(\Omega)$, the set $\widetilde{\boldsymbol{\Theta}}(\Omega)$, where $\widetilde{\boldsymbol{\Theta}} := \boldsymbol{\Theta} + \widetilde{\boldsymbol{v}}$, being its associated *deformed configuration.*

The covariant components of the *linearized change of metric tensor* associated with a displacement field $\widetilde{\boldsymbol{v}}$ of the set $\boldsymbol{\Theta}(\Omega)$ are then defined by

$$e_{i\|j}(\widetilde{\boldsymbol{v}}) := \frac{1}{2}[\widetilde{g}_{ij} - g_{ij}]^{\text{lin}},$$

where $\widetilde{g}_{ij}$ and g_{ij} are the covariant components of the metric tensors of the sets $\widetilde{\boldsymbol{\Theta}}(\Omega)$ and $\boldsymbol{\Theta}(\Omega)$ and $[\cdots]^{\text{lin}}$ denotes the linear part with respect to $\widetilde{\boldsymbol{v}}$ in the expression $[\cdots]$. An immediate computation then shows that

$$e_{i\|j}(\widetilde{\boldsymbol{v}}) = \frac{1}{2}(\partial_i \widetilde{\boldsymbol{v}} \cdot \boldsymbol{g}_j + \partial_j \widetilde{\boldsymbol{v}} \cdot \boldsymbol{g}_i), \text{ where } \boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}.$$

A displacement field $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ that satisfies $e_{i\|j}(\widetilde{\boldsymbol{v}}) = 0$ a.e. in Ω is called an *infinitesimal rigid displacement.* The next theorem, due to [9], is an extension of the *infinitesimal rigid displacement lemma in curvilinear coordinates* found in [7, Theorem 1.7-3].

Theorem 2 *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega) \cap \boldsymbol{H}^1(\Omega)$ be a mapping that satisfies* $\det \boldsymbol{\nabla\Theta} > 0$ *in Ω. Then a vector field $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ satisfies $e_{i\|j}(\widetilde{\boldsymbol{v}}) = 0$ a.e. in Ω if and only if there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{A} \in \mathbb{A}^3$ such that*

$$\widetilde{\boldsymbol{v}}(x) = \boldsymbol{c} + \boldsymbol{A\Theta}(x) \text{ for almost all } x \in \Omega.$$

□

2 The classical rigidity theorem on a surface and its extension to Sobolev spaces

Greek indices range over the set $\{1, 2\}$ and the summation convention for Latin indices also applies to these. Let ω be an open subset of $\mathbb{R}^2$, let y_α denote the coordinates of a point $y \in \mathbb{R}^2$, and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. Let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega) := \mathcal{C}^1(\omega; \mathbb{E}^3)$ be an *immersion*, i.e., a mapping such that the two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$$

are linearly independent at all points $y \in \omega$. The image $\boldsymbol{\theta}(\omega)$ is a *surface in* $\mathbb{E}^3$. Note that the vectors $\boldsymbol{a}_\alpha(y)$ span the tangent plane to the surface $\boldsymbol{\theta}(\omega)$ at the point $\boldsymbol{\theta}(y)$.

The *first fundamental form* of the surface $\boldsymbol{\theta}(\omega)$ is defined by means of its covariant components

$$a_{\alpha\beta}(y) := \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y), \quad y \in \omega,$$

used in particular for computing *lengths of curves* on the surface $\boldsymbol{\theta}(\omega)$, considered as being isometrically imbedded in $\mathbb{E}^3$.

Let

$$\boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}, \quad y \in \omega,$$

so that $\boldsymbol{a}_3(y)$ is a unit vector, normal to the surface $\boldsymbol{\theta}(\omega)$ at $\boldsymbol{\theta}(y)$. If $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$, the *second fundamental form* of the surface is defined by means of its covariant components

$$b_{\alpha\beta}(y) := -\boldsymbol{a}_\alpha(y) \cdot \partial_\beta \boldsymbol{a}_3(y), \quad y \in \omega,$$

which, together with those of the first fundamental form, are used for computing *curvatures of curves* on the surface $\boldsymbol{\theta}(\omega)$.

The classical *rigidity theorem on a surface* asserts that, if two immersions $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{\mathcal{C}}^2(\omega) := \mathcal{C}^2(\omega; \mathbb{E}^3)$ and $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^2(\omega)$ have the same first and second fundamental forms, i.e., if $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ in ω (with self-explanatory notations) and ω is connected, then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y) \text{ for all } y \in \omega.$$

For a proof, see, e.g., [8, Theorem 6].

We now show that *a similar result holds under the assumptions that* $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega) := H^1(\omega; \mathbb{E}^3)$ *and* $\widetilde{\boldsymbol{a}}_3 := \dfrac{\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2}{|\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2|} \in \boldsymbol{H}^1(\omega)$ (again with self-explanatory notations). Naturally, our first task will be to verify that the vector field $\widetilde{\boldsymbol{a}}_3$, which is not necessarily well defined a.e. in ω for an arbitrary mapping $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$, is nevertheless well defined a.e. in ω for those mappings $\widetilde{\boldsymbol{\theta}}$ that satisfy the assumptions of the next theorem. This observation will in turn imply that the functions $\widetilde{b}_{\alpha\beta} = -\widetilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3$ are likewise well defined a.e. in ω.

Theorem 3 (rigidity theorem) *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$. Assume that there exists a vector field $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$ that satisfies*

$$\widetilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega, \quad \widetilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega), \quad \text{and} \quad \widetilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y) \text{ for almost all } y \in \omega.$$

PROOF. (i) To begin with, we record several *technical preliminaries.*

First, we observe that the relations $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω and the assumption that $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$ is an immersion together imply that

$$|\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2| = \sqrt{\det(\widetilde{a}_{\alpha\beta})} = \sqrt{\det(a_{\alpha\beta})} > 0 \text{ a.e. in } \omega.$$

Consequently, *the vector field $\widetilde{\boldsymbol{a}}_3$, and thus the functions $\widetilde{b}_{\alpha\beta}$, are well defined a.e. in ω.*

Second, we establish that

$$b_{\alpha\beta} = b_{\beta\alpha} \text{ in } \omega \quad \text{and} \quad \widetilde{b}_{\alpha\beta} = \widetilde{b}_{\beta\alpha} \text{ a.e. in } \omega,$$

i.e., that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \boldsymbol{a}_\beta \cdot \partial_\alpha \boldsymbol{a}_3$ in ω and $\widetilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3 = \widetilde{\boldsymbol{a}}_\beta \cdot \partial_\alpha \widetilde{\boldsymbol{a}}_3$ a.e. in ω. To this end, we note that either the assumptions $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$, or the assumptions $\boldsymbol{\theta} \in \boldsymbol{H}^1(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{H}^1(\omega)$, imply that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in L^1_{\text{loc}}(\omega)$, hence that $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in \mathcal{D}'(\omega)$.

Given any $\varphi \in \mathcal{D}(\omega)$, let U denote an open subset of $\mathbb{R}^2$ such that $\operatorname{supp} \varphi \subset U$ and $\overline{U}$ is a compact subset of ω. Denoting by ${}_{X'}\langle \cdot, \cdot \rangle_X$ the duality pairing between a topological vector space X and its dual X', we have

$$\begin{aligned} {}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3, \varphi \rangle_{\mathcal{D}(\omega)} &= \int_\omega \varphi \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \, \mathrm{d}y \\ &= \int_\omega \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y - \int_\omega (\partial_\beta \varphi) \partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \, \mathrm{d}y. \end{aligned}$$

Observing that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0$ a.e. in ω and that

$$\begin{aligned} -\int_\omega \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y &= -\int_U \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y \\ &= {}_{H^{-1}(U;\mathbb{E}^3)}\langle \partial_\beta (\partial_\alpha \boldsymbol{\theta}), \, \varphi \boldsymbol{a}_3 \rangle_{H^1_0(U;\mathbb{E}^3)}, \end{aligned}$$

we reach the conclusion that the expression ${}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3, \, \varphi \rangle_{\mathcal{D}(\omega)}$ is symmetric with respect to α and β since $\partial_{\alpha\beta} \boldsymbol{\theta} = \partial_{\beta\alpha} \boldsymbol{\theta}$ in $\mathcal{D}'(U)$. Hence $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3$ in $L^1_{\text{loc}}(\omega)$, and the announced symmetries are established.

Third, let

$$\widetilde{c}_{\alpha\beta} := \partial_\alpha \widetilde{\boldsymbol{a}}_3 \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3 \quad \text{and} \quad c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3.$$

Then we claim that $\widetilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω. To see this, we note that the matrix fields $(\widetilde{a}^{\alpha\beta}) := (\widetilde{a}_{\alpha\beta})^{-1}$ and $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$ are well defined and equal a.e. in ω since $\boldsymbol{\theta}$ is an immersion and $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω. The formula of Weingarten can thus be applied a.e. in ω, showing that $\widetilde{c}_{\alpha\beta} = \widetilde{a}^{\sigma\tau} \widetilde{b}_{\sigma\alpha} \widetilde{b}_{\tau\beta}$ a.e. in ω.

The assertion then follows from the assumptions $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω.

(ii) *Starting from the set* ω *and the mapping* $\boldsymbol{\theta}$ (as given in the statement of Theorem 3), *we next construct a set* Ω *and a mapping* $\boldsymbol{\Theta}$ *that satisfy the assumptions of Theorem 1.* More precisely, let

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \quad \text{for all} \quad (y, x_3) \in \omega \times \mathbb{R}\,.$$

Then the mapping $\boldsymbol{\Theta} := \omega \times \mathbb{R} \to \mathbb{E}^3$ defined in this fashion is clearly continuously differentiable on $\omega \times \mathbb{R}$ and

$$\det \boldsymbol{\nabla\Theta}(y, x_3) = \sqrt{\det(a_{\alpha\beta}(y))}\{1 - x_3(b_1^1 + b_2^2)(y) + x_3^2(b_1^1 b_2^2 - b_1^2 b_2^1)(y)\}$$

for all $(y, x_3) \in \omega \times \mathbb{R}$, where

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y), \quad y \in \omega.$$

Let ω_n, $n \geq 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_n$ is a compact subset of ω and $\omega = \bigcup_{n\geq 0} \omega_n$. Then the continuity of the functions $a_{\alpha\beta}, a^{\alpha\beta}, b_{\alpha\beta}$ and the assumption that $\boldsymbol{\theta}$ is an immersion together imply that, for each $n \geq 0$, there exists $\varepsilon_n > 0$ such that

$$\det \boldsymbol{\nabla\Theta}(y, x_3) > 0 \text{ for all } (y, x_3) \in \overline{\omega}_n \times [-\varepsilon_n, \varepsilon_n].$$

Besides, there is no loss of generality in assuming that $\varepsilon_n \leq 1$ (this property will be used in part (iii)).

Let then

$$\Omega := \bigcup_{n\geq 0} (\omega_n \times]-\varepsilon_n, \varepsilon_n[).$$

Then it is clear that Ω *is a connected open subset of* $\mathbb{R}^3$ and that *the mapping* $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega)$ *satisfies* $\det \boldsymbol{\nabla\Theta} > 0$ *in* Ω.

Finally, note that the covariant components $g_{ij} \in \mathcal{C}^0(\Omega)$ of the metric tensor field associated with the mapping $\boldsymbol{\Theta}$ are given by (the symmetries $b_{\alpha\beta} = b_{\beta\alpha}$ established in (i) are used here)

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1.$$

(iii) Starting with the mapping $\widetilde{\boldsymbol{\theta}}$ (as given in the statement of Theorem 3), we *construct a mapping* $\widetilde{\boldsymbol{\Theta}}$ *that satisfies the assumptions of Theorem 1.* To this end, we define a mapping $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbb{E}^3$ by letting

$$\widetilde{\boldsymbol{\Theta}}(y, x_3) := \widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y) \text{ for all } (y, x_3) \in \Omega,$$

where the set Ω is defined as in (ii). Hence $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$, since $\Omega \subset \omega \times]-1, 1[$. Besides, $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \det \boldsymbol{\nabla\Theta}$ a.e. in Ω since the functions $\widetilde{b}_\alpha^\beta := \widetilde{a}^{\beta\sigma}\widetilde{b}_{\alpha\sigma}$, which are well defined a.e. in ω, are equal, again a.e. in ω, to the functions b_α^β. Likewise, the components $\widetilde{g}_{ij} \in L^1(\Omega)$ of the metric tensor field associated with the mapping $\widetilde{\boldsymbol{\Theta}}$ satisfy $\widetilde{g}_{ij} = g_{ij}$ a.e. in Ω since $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω by assumption and $\widetilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω by part (i).

(iv) By Theorem 1, there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}_+^3$ such that

$$\widetilde{\boldsymbol{\theta}}(y) + x_3\widetilde{\boldsymbol{a}}_3(y) = \boldsymbol{c} + \boldsymbol{Q}(\boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y)) \text{ for almost all } (y, x_3) \in \Omega.$$

Differentiating with respect to x_3 in this equality between functions in $\boldsymbol{H}^1(\Omega)$ shows that $\widetilde{\boldsymbol{a}}_3(y) = \boldsymbol{Q}\boldsymbol{a}_3(y)$ for almost all $y \in \omega$. Hence $\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y)$ for almost all $y \in \omega$ as announced. □

Remarks. (1) The existence of $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$ satisfying the assumptions of Theorem 3 implies that $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{\mathcal{C}}^1(\omega)$ and $\widetilde{\boldsymbol{a}}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$, and that $\boldsymbol{\theta} \in \boldsymbol{H}^1(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{H}^1(\omega)$.

(2) It is easily seen that the conclusion of Theorem 3 is still valid if the assumptions $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$ and $\widetilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega)$ are replaced by the weaker assumptions $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1_{\text{loc}}(\omega)$ and $\widetilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1_{\text{loc}}(\omega)$.

3 The submanifold of rigid displacements on a surface

All the results needed below about *submanifolds in infinite-dimensional Banach spaces* are found in [1]. The *tangent space* at a point m of a submanifold $\mathcal{M}$ of a Banach space X is denoted $T_m\mathcal{M}$. If $f : X \to Y$ is a Fréchet-differentiable mapping into a Banach space Y, the *tangent map* at m, i.e., the restriction to $T_m\mathcal{M}$ of the Fréchet derivative of f at m, is denoted $T_m f$.

We now establish that the set $\boldsymbol{M}$ formed by all the mappings $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$ that satisfy the assumptions of the rigidity theorem on a surface (Theorem 3) is a *finite-dimensional submanifold of the space* $\boldsymbol{H}^1(\omega)$. Note that the assumptions $\boldsymbol{\theta} \in \boldsymbol{H}^1(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{H}^1(\omega)$ have been added to those of Theorem 3, simply to guarantee that the set $\boldsymbol{M}$ is non-empty.

We also characterize the *tangent space to* $\boldsymbol{M}$ *at* $\boldsymbol{\theta}$. Another equally important characterization of the same tangent space, in terms of the linearized change of metric and linearized change of curvature tensors, will be given in Theorem 5.

The notations used here are the same as in Theorem 3. In particular, $\boldsymbol{H}^1(\omega) = H^1(\omega; \mathbb{E}^3)$, $\boldsymbol{\mathcal{C}}^1(\omega) = \mathcal{C}^1(\omega; \mathbb{E}^3)$, and the covariant components of the first, and second, fundamental forms of the surfaces $\widetilde{\boldsymbol{\theta}}(\omega)$ and $\boldsymbol{\theta}(\omega)$ are respectively designated by $\widetilde{a}_{\alpha\beta}$ and $a_{\alpha\beta}$, and $\widetilde{b}_{\alpha\beta}$ and $b_{\alpha\beta}$.

Theorem 4 *Let* ω *be a connected open subset of* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$

be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$. *Then the set*

$$\boldsymbol{M} := \{\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega);\ \widetilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega,\ \widetilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega),\ \widetilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega\}$$

is a submanifold of class $\mathcal{C}^\infty$ *and of dimension* 6 *of the space* $\boldsymbol{H}^1(\omega)$ *and its tangent space at* $\boldsymbol{\theta}$ *is given by*

$$T_{\boldsymbol{\theta}}\boldsymbol{M} = \{\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega); \exists\, \boldsymbol{c} \in \mathbb{E}^3,\ \exists\, \boldsymbol{A} \in \mathbb{A}^3, \widetilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\theta} \text{ a.e. in } \omega\}.$$

PROOF. (i) Define the linear mapping

$$\boldsymbol{f} : (\boldsymbol{c}, \boldsymbol{F}) \in \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{F}) = \boldsymbol{c} + \boldsymbol{F}\boldsymbol{\theta} \in \boldsymbol{H}^1(\omega).$$

By the rigidity theorem (Theorem 3), the above set $\boldsymbol{M}$ may be equivalently defined as

$$\boldsymbol{M} = \boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+).$$

Since the mapping $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ need not be injective, some care has to be exercised for proving that the image $\boldsymbol{M}$ of the manifold $\mathbb{E}^3 \times \mathbb{O}^3_+$ through $\boldsymbol{f}$ is a submanifold of $\boldsymbol{H}^1(\omega)$. To this end, we need to prove that the restriction $\boldsymbol{f}^\sharp$ of the mapping $\boldsymbol{f}$ to the set $\mathbb{E}^3 \times \mathbb{O}^3_+$ is an *embedding*, in the sense that it satisfies the properties established in (ii) and (iii) below.

(ii) First, we show that, *for each* $(\boldsymbol{c}, \boldsymbol{Q}) \in \mathbb{E}^3 \times \mathbb{O}^3_+$, *the tangent map* $T_{(\boldsymbol{c},\boldsymbol{Q})}\boldsymbol{f}$ *is injective, with a closed range having a closed complement in* $\boldsymbol{H}^1(\omega)$.

Since $\boldsymbol{f}$ is linear, the tangent map $T_{(\boldsymbol{c},\boldsymbol{Q})}\boldsymbol{f}$ is simply the restriction of $\boldsymbol{f}$ to

$$T_{(\boldsymbol{c},\boldsymbol{Q})}(\mathbb{E}^3 \times \mathbb{O}^3_+) = \mathbb{E}^3 \times \boldsymbol{Q}\mathbb{A}^3.$$

So, given any $\boldsymbol{Q} \in \mathbb{O}^3_+$, let $\boldsymbol{d} \in \mathbb{E}^3$ and $\boldsymbol{A} \in \mathbb{A}^3$ be such that

$$\boldsymbol{d} + \boldsymbol{Q}\boldsymbol{A}\boldsymbol{\theta}(y) = \boldsymbol{0} \text{ for all } y \in \omega.$$

Multiplying on the left by $\boldsymbol{Q}^T$ and differentiating with respect to y_α yield $\boldsymbol{A}\boldsymbol{a}_\alpha(y) = \boldsymbol{0}$ for all $y \in \omega$. Fix $y_0 \in \omega$; then the relation $\boldsymbol{A}\boldsymbol{a}_\alpha(y_0) = \boldsymbol{0}$ shows that there exist $\alpha_i \in \mathbb{R}$ such that $\boldsymbol{A} = (\alpha_i\beta_j)$, where β_j denotes the j-th Cartesian component of the vector $\boldsymbol{a}_3(y_0)$. The relation $\boldsymbol{A} + \boldsymbol{A}^T = \boldsymbol{0}$ then implies that $\boldsymbol{A} = \boldsymbol{0}$, hence that $\boldsymbol{d} = \boldsymbol{0}$. Consequently, the tangent map $T_{(\boldsymbol{c},\boldsymbol{Q})}\boldsymbol{f}$ is injective for each $(\boldsymbol{c}, \boldsymbol{Q})$.

That $T_{(\boldsymbol{c},\boldsymbol{Q})}\boldsymbol{f}$ has a closed range is clear, since $\boldsymbol{f}(\mathbb{E}^3 \times \boldsymbol{Q}\mathbb{A}^3)$ is the image by a linear mapping of a finite-dimensional space. That $\boldsymbol{f}(\mathbb{E}^3 \times \boldsymbol{Q}\mathbb{A}^3)$ has a closed complement in $\boldsymbol{H}^1(\omega)$ is equally clear, since $\boldsymbol{H}^1(\omega)$ is a Hilbert space.

(iii) Second, we show that *the restriction* $\boldsymbol{f}^\sharp$ *of the mapping* $\boldsymbol{f}$ *to the submanifold* $\mathbb{E}^3 \times \mathbb{O}^3_+$ *is a homeomorphism, hence a* $\mathcal{C}^\infty$*-diffeomorphism since* $\boldsymbol{f}$

is linear, from $\mathbb{E}^3 \times \mathbb{O}^3_+$ *onto the image* $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+)$ *equipped with the relative topology induced by that of* $\boldsymbol{H}^1(\omega)$.

To begin with, let us establish that $\boldsymbol{f}^\sharp$ is injective. So, let $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathbb{E}^3$ and $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathbb{O}^3_+$ be such that

$$\boldsymbol{c}_1 - \boldsymbol{c}_2 + (\boldsymbol{Q}_1 - \boldsymbol{Q}_2)\boldsymbol{\theta}(y) = 0 \text{ for all } y \in \omega.$$

Differentiating with respect to y_α and fixing $y_0 \in \omega$ show that there exist $\alpha_i \in \mathbb{R}$ such that the i-th row vector of the matrix $(\boldsymbol{Q}_1 - \boldsymbol{Q}_2)$ is of the form $\alpha_i \boldsymbol{b}_3^T$, where $\boldsymbol{b}_3 := \boldsymbol{a}_3(y_0)$. Let $\boldsymbol{b}_1$ be any vector that satisfies $|\boldsymbol{b}_1| = 1$ and $\boldsymbol{b}_1 \cdot \boldsymbol{b}_3 = 0$, let $\boldsymbol{b}_2 := \boldsymbol{b}_3 \wedge \boldsymbol{b}_1$, and let $\boldsymbol{B}$ be the matrix with $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ as its column vectors. Then $\boldsymbol{B} \in \mathbb{O}^3_+$ and

$$(\boldsymbol{Q}_1 - \boldsymbol{Q}_2)\boldsymbol{B} = \begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

Hence the first and second column vectors of the matrices $\boldsymbol{Q}_1\boldsymbol{B}$ and $\boldsymbol{Q}_2\boldsymbol{B}$ are identical. Since both matrices $\boldsymbol{Q}_1\boldsymbol{B}$ and $\boldsymbol{Q}_2\boldsymbol{B}$ belong to $\mathbb{O}^3_+$, they are thus equal. Therefore $\boldsymbol{Q}_1 = \boldsymbol{Q}_2$ since $\boldsymbol{B}$ is invertible. This equality in turn implies that $\boldsymbol{c}_1 = \boldsymbol{c}_2$.

Since $\mathbb{E}^3 \times \mathbb{M}^3$ is a finite-dimensional space, the linear mapping $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ is continuous and so is its restriction $\boldsymbol{f}^\sharp : \mathbb{E}^3 \times \mathbb{O}^3_+ \to \boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+)$. It thus remains to establish that the inverse mapping of $\boldsymbol{f}^\sharp$ is also continuous. So, let $(\boldsymbol{c}_n, \boldsymbol{Q}_n) \in \mathbb{E}^3 \times \mathbb{O}^3_+$, $n \geq 1$, and $(\boldsymbol{d}, \boldsymbol{R}) \in \mathbb{E}^3 \times \mathbb{O}^3_+$ be such that

$$\boldsymbol{f}^\sharp(\boldsymbol{c}_n, \boldsymbol{Q}_n) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\chi} := \boldsymbol{f}^\sharp(\boldsymbol{d}, \boldsymbol{R}) \text{ in } \boldsymbol{H}^1(\omega).$$

Since $\boldsymbol{Q}_n \in \mathbb{O}^3_+$, $n \geq 1$, there exist a subsequence $(\boldsymbol{Q}_m)_{m\geq 1}$ and $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that $\boldsymbol{Q}_m \underset{m\to\infty}{\longrightarrow} \boldsymbol{Q}$, which in turn implies that

$$\boldsymbol{c}_m = \boldsymbol{f}^\sharp(\boldsymbol{c}_m, \boldsymbol{Q}_m) - \boldsymbol{Q}_m\boldsymbol{\theta} \underset{m\to\infty}{\longrightarrow} \boldsymbol{c} := \boldsymbol{\chi} - \boldsymbol{Q}\boldsymbol{\theta}.$$

It thus follows that $\boldsymbol{f}^\sharp(\boldsymbol{c}_m, \boldsymbol{Q}_m) \to \boldsymbol{f}^\sharp(\boldsymbol{c}, \boldsymbol{Q})$ since $\boldsymbol{f}^\sharp$ is continuous, hence that $\boldsymbol{c} = \boldsymbol{d}$ and $\boldsymbol{Q} = \boldsymbol{R}$ since $\boldsymbol{f}^\sharp$ is injective. The whole sequence $(\boldsymbol{c}_n, \boldsymbol{Q}_n)_{n\geq 1}$ thus converges to $(\boldsymbol{d}, \boldsymbol{R})$ since the limit is unique. This shows that the inverse mapping of $\boldsymbol{f}^\sharp$ is continuous.

(iv) By (ii) and (iii), *the mapping* $\boldsymbol{f}^\sharp : \mathbb{E}^3 \times \mathbb{O}^3_+ \to \boldsymbol{H}^1(\omega)$ *is an embedding.* Since $\mathbb{E}^3 \times \mathbb{O}^3_+$ is a submanifold of dimension 6 of $\mathbb{E}^3 \times \mathbb{M}^3$ (the special orthogonal group $\mathbb{O}^3_+$ is a submanifold of dimension 3 of $\mathbb{M}^3$), *the set* $\boldsymbol{M} = \boldsymbol{f}^\sharp(\mathbb{E}^3 \times \mathbb{O}^3_+)$ *is thus a submanifold of dimension* 6 *of* $\boldsymbol{H}^1(\omega)$ (see [1, Section 3.5]).

Since the manifolds $\mathbb{E}^3 \times \mathbb{O}^3_+$ and $\boldsymbol{H}^1(\omega)$ are of class $\mathcal{C}^\infty$ (the special orthogonal group is a submanifold of class $\mathcal{C}^\infty$ of $\mathbb{M}^3$) and the mapping $\boldsymbol{f}^\sharp$ is of class $\mathcal{C}^\infty$, *the submanifold* $\boldsymbol{M}$ *is also of class* $\mathcal{C}^\infty$.

(v) Since $\boldsymbol{f}$ is linear and $T_{\boldsymbol{I}}\mathbb{O}^3_+ = \mathbb{A}^3$, the tangent space to $\boldsymbol{M}$ at $\boldsymbol{\theta}$ is given by

$$\begin{aligned} T_{\boldsymbol{\theta}}\boldsymbol{M} &= T_{\boldsymbol{f}(\boldsymbol{0},\boldsymbol{I})}\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+) = \boldsymbol{f}(T_{(\boldsymbol{0},\boldsymbol{I})}(\mathbb{E}^3 \times \mathbb{O}^3_+)) = \boldsymbol{f}(\mathbb{E}^3 \times \mathbb{A}^3) \\ &= \{\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \exists\, \boldsymbol{c} \in \mathbb{E}^3,\ \exists\, \boldsymbol{A} \in \mathbb{A}^3,\ \widetilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\theta} \text{ a.e. in } \omega\}, \end{aligned}$$

and the proof is complete. □

If the mapping $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ is *injective*, in which case $\boldsymbol{f}$ is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{E}^3 \times \mathbb{M}^3$ onto $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$, the above proof that $\boldsymbol{M} = \boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+)$ is a submanifold of $\boldsymbol{H}^1(\omega)$ can be substantially simplified: Since submanifolds of class $\mathcal{C}^\infty$ are preserved by $\mathcal{C}^\infty$-diffeomorphisms, $\boldsymbol{M}$ is a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$. As a closed subspace of the Hilbert space $\boldsymbol{H}^1(\omega)$, the image $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$ has a closed complement, i.e., $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$ is "split" in $\boldsymbol{H}^1(\omega)$. The set $\boldsymbol{M}$ is thus also a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of $\boldsymbol{H}^1(\omega)$ (this conclusion follows from the definition of a submanifold; see [1, Definition 3.2.1]).

Interestingly, one can establish that *the mapping* $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ *is injective if and only if the surface* $\boldsymbol{\theta}(\omega)$ *is not contained in a plane.* To see this, let $\boldsymbol{c} \in \mathbb{E}^3$ and $\boldsymbol{F} \in \mathbb{M}^3$ be such that

$$\boldsymbol{c} + \boldsymbol{F}\boldsymbol{\theta}(y) = 0 \text{ for all } y \in \omega.$$

Differentiating with respect to y_α yields $\boldsymbol{F}a_\alpha(y) = 0$ for all $y \in \omega$. This means that the rows of the matrix $\boldsymbol{F}$, which is independent of $y \in \omega$, are at each $y \in \omega$ proportional to the vector $\boldsymbol{a}_3^T(y)$. Hence $\boldsymbol{F} \neq \boldsymbol{0}$ implies that $\boldsymbol{a}_3(y)$ is the same vector for all $y \in \omega$; hence the surface $\boldsymbol{\theta}(y)$ is contained in a plane. If, conversely, $\boldsymbol{\theta}(y)$ is contained in a plane, then $\boldsymbol{a}_3(y) = \boldsymbol{a}_3$ for all $y \in \omega$, the matrix $\boldsymbol{F} \in \mathbb{M}^3$ with all row vectors equal to $\boldsymbol{a}_3$ does not vanish, yet $\boldsymbol{F}\boldsymbol{a}_\alpha(y) = 0$ for all $y \in \omega$.

In shell theory, the surface $\boldsymbol{\theta}(\omega)$ is the *reference configuration* of the middle surface of an elastic shell (under the additional assumption that the immersion $\boldsymbol{\theta}$ is injective, but this assumption is irrelevant for our present purposes). Then, for each $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$, the surface $\widetilde{\boldsymbol{\theta}}(\omega)$ is a *deformed configuration* of the middle surface and the field $\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ defined by

$$\widetilde{\boldsymbol{\theta}} = \boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}$$

is a *displacement field* of the reference configuration $\boldsymbol{\theta}(\omega)$. If in particular $\widetilde{\boldsymbol{\theta}} \in \boldsymbol{M}$, the field $\widetilde{\boldsymbol{\eta}}$ defined in this fashion is called a *rigid displacement*, and the subset $\boldsymbol{M}_{\mathrm{rig}}$ of $\boldsymbol{H}^1(\omega)$ defined by

$$\boldsymbol{M} = \boldsymbol{\theta} + \boldsymbol{M}_{\mathrm{rig}}$$

is accordingly called the *manifold of rigid displacements* (of the reference configuration $\boldsymbol{\theta}(\omega)$). We now reformulate Theorem 4 in terms of the manifold $\boldsymbol{M}_{\mathrm{rig}}$.

Corollary to Theorem 4 *Let ω be a connected open subset of $\mathbb{R}^2$, and let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$. Then the manifold of rigid displacements of the surface $\boldsymbol{\theta}(\omega)$, viz.,*

$$\boldsymbol{M}_{\mathrm{rig}} := \{\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \widetilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega,\ \widetilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega),\ \widetilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega\},$$

is a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of the space $\boldsymbol{H}^1(\omega)$ and its tangent space at $\mathbf{0}$ is given by

$$\begin{aligned} T_{\mathbf{0}}\boldsymbol{M}_{\mathrm{rig}} &= T_{\boldsymbol{\theta}}\boldsymbol{M} \\ &= \{\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \exists\, \boldsymbol{c} \in \mathbb{E}^3,\ \exists\, \boldsymbol{A} \in \mathbb{A}^3,\ \widetilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\theta} \text{ a.e. in } \omega\}. \end{aligned}$$

□

4 The infinitesimal rigid displacement lemma on a surface revisited

The covariant components of the *linearized change of metric tensor* and *linearized change of curvature tensor* associated with a smooth enough displacement field $\widetilde{\boldsymbol{\eta}}$ of the surface $\boldsymbol{\theta}(\omega)$, viewed as above as a reference configuration, are defined by

$$\gamma_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) := \frac{1}{2}[\widetilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\mathrm{lin}} \quad \text{and} \quad \rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) := [\widetilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\mathrm{lin}},$$

where $a_{\alpha\beta}$ and $\widetilde{a}_{\alpha\beta}$, and $b_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$, respectively designate the covariant components of the first, and second, fundamental forms of the surfaces $\boldsymbol{\theta}(\omega)$ and $\widetilde{\boldsymbol{\theta}}(\omega)$ where $\widetilde{\boldsymbol{\theta}} := \boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}$, and $[\cdots]^{\mathrm{lin}}$ denotes the linear part with respect to $\widetilde{\boldsymbol{\eta}}$ in the expression $[\cdots]$. A formal computation immediately gives

$$\gamma_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = \frac{1}{2}(\partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha), \text{ where } \boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}.$$

This expression thus shows that

$$\gamma_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in L^2_{\mathrm{loc}}(\omega) \text{ if } \widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega) \text{ and } \boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega).$$

Another formal, but substantially less immediate, computation shows that

$$\rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = \partial_{\alpha\beta}(\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \partial_\beta \boldsymbol{a}_3 - \partial_\beta(\widetilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3) - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3,$$

where the functions $\Gamma^\sigma_{\alpha\beta} := a^{\sigma\tau} \boldsymbol{a}_\tau \cdot \partial_\alpha \boldsymbol{a}_\beta$ are the Christoffel symbols of the surface $\boldsymbol{\theta}(\omega)$. See, e.g., [7, Theorem 2.5-1] for the effective computation that leads to the above expression of the functions $\rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$, noting in this respect that the functions $\widetilde{b}_{\alpha\beta}$ are well defined a.e. in ω when $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ (see Theorem 3) or when the $\boldsymbol{W}^{1,\infty}(\omega)$-norm of the field $\widetilde{\boldsymbol{\eta}}$ is small enough. The above expression thus shows that

$$\rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in H^{-1}(\omega) \text{ if } \widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega) \text{ and } \boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^2(\omega) \text{ and } \boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^2(\omega).$$

Under these assumptions on the mapping $\boldsymbol{\theta}$ and the field $\boldsymbol{a}_3$, a displacement field $\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ that satisfies $\gamma_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = 0$ a.e. in ω and $\rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = 0$ in $H^{-1}(\omega)$ is called an *infinitesimal rigid displacement* of the surface $\boldsymbol{\theta}(\omega)$. Accordingly, the *infinitesimal rigid displacement lemma on a surface* stated in the Introduction consists in identifying the vector space $\boldsymbol{V}^{\text{lin}}_{\text{rig}}$ formed by such displacements.

The next theorem shows that this lemma has also a *remarkably simple interpretation* in terms of the manifold $\boldsymbol{M}_{\text{rig}}$ of rigid displacements introduced at the end of Section 3. The proof is reminiscent of that used in [2] or [11] for establishing the Korn inequality on a surface as a consequence of its three-dimensional counterpart in curvilinear coordinates.

Theorem 5 *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^2(\omega) \cap \boldsymbol{H}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^2(\omega) \cap \boldsymbol{H}^1(\omega)$. Then the space of infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\omega)$, viz.,*

$$\boldsymbol{V}^{\text{lin}}_{\text{rig}} := \{\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \gamma_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = 0 \text{ a.e. in } \omega \text{ and } \rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = 0 \text{ in } H^{-1}(\omega)\},$$

is given by

$$\boldsymbol{V}^{\text{lin}}_{\text{rig}} = T_{\boldsymbol{0}} \boldsymbol{M}_{\text{rig}},$$

where the tangent space $T_0 \boldsymbol{M}_{\text{rig}}$ has been identified in the Corollary to Theorem 4.

PROOF. (i) Since the set ω is open and connected, there exist open and connected subsets ω_n, $n \geq 0$, of $\mathbb{R}^2$ such that $\overline{\omega}_n$ is a compact subset of ω and $\omega_n \subset \omega_{n+1}$ for any $n \geq 0$, and $\omega = \bigcup_{n \geq 0} \omega_n$. Let the mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\omega \times \mathbb{R})$ be defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \omega \in \mathbb{R}.$$

As shown in part (ii) of the proof of Theorem 3, there exist $0 < \varepsilon_n \leq 1$ such that $\det \boldsymbol{\nabla\Theta}(y, x_3) > 0$ for all $(y, x_3) \in \Omega$, where

$$\Omega := \bigcup_{n \geq 0} \Omega_n \quad \text{and} \quad \Omega_n := \omega_n \times]-\varepsilon_n, \varepsilon_n[.$$

(ii) Given any displacement field $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ that satisfies $\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in H^2_{\text{loc}}(\omega)$, let

$$\tilde{\boldsymbol{v}}(y, x_3) := \tilde{\boldsymbol{\eta}}(y) - x_3(\{\partial_\alpha(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3\}\boldsymbol{a}^\alpha)(y)$$

for almost all $(y, x_3) \in \Omega$, where Ω is defined as in part (i) and $\boldsymbol{a}^\alpha := a^{\alpha\beta}\boldsymbol{a}_\beta$. The vector field $\tilde{\boldsymbol{v}}$ defined in this fashion satisfies $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1_{\text{loc}}(\Omega)$. Hence $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega_n)$ for all $n \geq 0$.

A careful computation then shows that, for any $n \geq 0$, the covariant components $e_{i\|j}(\tilde{\boldsymbol{v}}) \in L^2(\Omega_n)$ of the linearized change of metric tensor (see Section 1) associated with the above displacement field $\tilde{\boldsymbol{v}}$ are related in Ω_n to the functions $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) \in L^2(\omega_n)$ and $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) \in L^2(\omega_n)$ by means of the relations

$$\begin{aligned} e_{\alpha\|\beta}(\tilde{\boldsymbol{v}}) &= \gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) - x_3\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) + \frac{x_3^2}{2}\{b^\sigma_\alpha \rho_{\beta\sigma}(\tilde{\boldsymbol{\eta}}) + b^\tau_\beta \rho_{\alpha\tau}(\tilde{\boldsymbol{\eta}}) - 2b^\sigma_\alpha b^\tau_\beta \gamma_{\sigma\tau}(\tilde{\boldsymbol{\eta}})\}, \\ e_{i\|3}(\tilde{\boldsymbol{v}}) &= 0, \end{aligned}$$

where $b^\sigma_\alpha := a^{\sigma\beta}b_{\alpha\beta}$.

(iii) Let a displacement field $\tilde{\boldsymbol{\eta}} \in \boldsymbol{V}^{\text{lin}}_{\text{rig}}$ be given. The definition of the distributions $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ and the assumptions $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0$ in $H^{-1}(\omega)$ together imply that $\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in H^2_{\text{loc}}(\omega)$, thus allowing to conclude from part (ii) that, for each $n \geq 0$, $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega_n)$ and $e_{i\|j}(\tilde{\boldsymbol{v}}) = 0$ a.e. in Ω_n.

Theorem 2 can thus be applied (each open set $\Omega_n = \omega_n \times]-\varepsilon_n, \varepsilon_n[$ is connected since ω_n is connected), showing that, for each $n \geq 0$, there exist a vector $\boldsymbol{c}_n \in \mathbb{E}^3$ and a matrix $\boldsymbol{A}_n \in \mathbb{A}^3$ such that $\tilde{\boldsymbol{v}}(x) = \boldsymbol{c}_n + \boldsymbol{A}_n\boldsymbol{\Theta}(x)$ for almost all $x \in \Omega_n$, i.e., such that

$$\tilde{\boldsymbol{\eta}}(y) - x_3(\{\partial_\alpha(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3\}\boldsymbol{a}^\alpha)(y) = \boldsymbol{c}_n + \boldsymbol{A}_n\{\boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y)\}$$

for almost all $(y, x_3) \in \Omega_n$. Differentiating with respect to x_3 this equality between functions in $\boldsymbol{H}^1(\Omega_n)$, we conclude that

$$(\{\partial_\alpha(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3\}\boldsymbol{a}^\alpha)(y) = \boldsymbol{A}_n\boldsymbol{a}_3(y)$$

for almost all $y \in \omega_n$. Hence

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{c}_n + \boldsymbol{A}_n\boldsymbol{\theta}(y)$$

for almost all $y \in \omega_n$.

That the vectors $\boldsymbol{c}_n$ and $\boldsymbol{A}_n$ are in fact independent of $n \geq 0$ is a consequence of the inclusions $\omega_n \subset \omega_{n+1}$, $n \geq 0$. For, if $\boldsymbol{d} \in \mathbb{E}^3$ and $\boldsymbol{A} \in \mathbb{A}^3$ are such that $\boldsymbol{d} + \boldsymbol{A\theta}(y) = 0$ for all $y \in \omega_n$ for some $n \geq 0$, then $\boldsymbol{d} = \mathbf{0}$ and $\boldsymbol{A} = \mathbf{0}$ (see part (ii) of the proof of Theorem 4). □

By Theorem 5, the infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\omega)$ thus span the tangent space at the origin to the manifold formed by the rigid displacements of $\boldsymbol{\theta}(\omega)$. This is the essence of the "infinitesimal rigid displacement lemma on a surface".

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